INVARIANT DIFFERENTIAL OPERATORS ON THE TANGENT SPACE OF SOME SYMMETRIC SPACES

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Abstract. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, with an involutive automorphism ϑ and set $\mathfrak{k} = \mathrm{Ker}(\vartheta - I)$, $\mathfrak{p} = \mathrm{Ker}(\vartheta + I)$. We consider the differential operators, $\mathcal{D}(\mathfrak{p})^K$, on $\mathfrak{p}$ that are invariant under the action of the adjoint group K of $\mathfrak{k}$. Write $\tau : \mathfrak{k} \to \mathrm{Der}\, \mathcal{O}(\mathfrak{p})$ for the differential of this action. Then we prove, for the class of symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ considered by Sekiguchi [33], that $\{d \in \mathcal{D}(\mathfrak{p}) : d(\mathcal{O}(\mathfrak{p})^K) = 0\} = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.

An immediate consequence of this equality is the following result of Sekiguchi: Let $(\mathfrak{g}_0, \mathfrak{k}_0)$ be a real form of one of these symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, and suppose that T is a K_0-invariant eigendistribution on $\mathfrak{p}_0$ that is supported on the singular set. Then, $T = 0$. In the diagonal case $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}' \oplus \mathfrak{g}', \mathfrak{g}')$ this is a well-known result due to Harish-Chandra.

1. Introduction

To begin with, assume that G is a connected, complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with Weyl group W. Thus, G acts on $\mathfrak{g}$ via the adjoint action and this induces an action of G on the ring of regular functions, $\mathcal{O}(\mathfrak{g}) \cong S(\mathfrak{g}^*)$, and hence on $\mathcal{D}(\mathfrak{g})$, the ring of differential operators with coefficients in $\mathcal{O}(\mathfrak{g})$. Let $\tau : \mathfrak{g} \to \mathrm{Der}\, \mathcal{O}(\mathfrak{g}) \subset \mathcal{D}(\mathfrak{g})$ denote the differential of the adjoint action of G on $\mathfrak{g}$. In [8] Harish-Chandra defines a ring homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$, with kernel $\mathcal{I}(\mathfrak{g}) = \{d \in \mathcal{D}(\mathfrak{g})^G : d(f) = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{g})^G\}$. The main results of [17, 18] show that δ is surjective, with kernel

$$\mathcal{I}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^G. \tag{1.1}$$

One significance of these results is that they easily imply two fundamental theorems of Harish-Chandra: Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, with adjoint group G_0 and write $\mathfrak{g}_0'$ for the regular semisimple elements of $\mathfrak{g}_0$. Then:

$$\text{If } T \text{ is } G_0\text{-invariant distribution on } \mathfrak{g}_0, \text{ then } \mathcal{I}(\mathfrak{g}) \cdot T = 0; \tag{1.2}$$

$$\text{The only } G_0\text{-invariant eigendistribution supported on } \mathfrak{g}_0 \setminus \mathfrak{g}_0' \text{ is } T = 0. \tag{1.3}$$

In [33], Sekiguchi generalized (1.3) to a class of "nice" symmetric spaces and it is therefore natural to ask whether the results from [17, 18] can also be generalized to these spaces. Despite the fact that there is no analogue of Harish-Chandra's map δ, these generalizations do exist, and the purpose of this paper is to describe them. Before we can state the results formally, we need some notation.

Fix a non-degenerate, G-invariant symmetric bilinear form κ on the reductive Lie algebra $\mathfrak{g}$ such that κ is the Killing form on the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Fix an

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involutive automorphism ϑ of $\mathfrak{g}$ preserving κ and set $\mathfrak{k} = \mathrm{Ker}(\vartheta - I)$, $\mathfrak{p} = \mathrm{Ker}(\vartheta + I)$. Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the pair $(\mathfrak{g}, \mathfrak{k})$, or $(\mathfrak{g}, \vartheta)$, is called a *symmetric pair*. Recall that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to κ and that $\mathfrak{k}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Denote by K the connected reductive subgroup of G with Lie algebra $\mathfrak{k}$. The group K acts on $\mathfrak{p}$ via the adjoint action and the differential of this action induces a Lie algebra homomorphism $\tau : \mathfrak{k} \to \mathrm{Der}\, S(\mathfrak{p}^*)$ defined by $(\tau(a).f)(v) = \frac{d}{dt}_{|t=0} f(e^{-ta}.v)$, for $a \in \mathfrak{k}$, $f \in S(\mathfrak{p}^*)$ and $v \in \mathfrak{p}$.

Let $\mathcal{D}(\mathfrak{p})$ denote the algebra of differential operators on $\mathfrak{p}$ with coefficients in $\mathcal{O}(\mathfrak{p}) = S(\mathfrak{p}^*)$. Notice that K has an induced action on $S(\mathfrak{p})$, $\mathcal{O}(\mathfrak{p})$ and $\mathcal{D}(\mathfrak{p})$. Set

$$\mathcal{K}(\mathfrak{p}) = \left\{ d \in \mathcal{D}(\mathfrak{p}) : d(f) = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{p})^K \right\}$$

and $\mathcal{I}(\mathfrak{p}) = \mathcal{K}(\mathfrak{p}) \cap \mathcal{D}(\mathfrak{p})^K$. Clearly, $\mathcal{K}(\mathfrak{p})$ is a K-stable left ideal of $\mathcal{D}(\mathfrak{p})$ containing $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.

Consider the special case when one is in the *diagonal case* where $G = G_1 \times G_1$ with $\vartheta(x, y) = (y, x)$ for some reductive group G_1; thus $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathfrak{g}_1)$. Then $K = G_1$ with its adjoint action on $\mathfrak{p} = \mathfrak{g}_1$, and a slightly stronger version of (1.1), see [18, Theorem 1.1], asserts that the equality $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ holds in this case. The first main aim of this paper is to generalize this result to a class of symmetric pairs introduced by J. Sekiguchi [33]. If $\mathfrak{g}$ is semisimple, these are defined as follows: Let Σ be the restricted root system associated to a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$. Then, $(\mathfrak{g}, \mathfrak{k})$ is *nice* if $\dim \mathfrak{g}^{\alpha} + \dim \mathfrak{g}^{2\alpha} \leqslant 2$ for all $\alpha \in \Sigma$. If $\mathfrak{g}$ is reductive, then $(\mathfrak{g}, \mathfrak{k})$ is *nice* provided that $([\mathfrak{g}, \mathfrak{g}], \mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}])$ is nice. The reader is referred to Section 2 and [33] for further details and for a classification of nice pairs. Observe that the diagonal case is obviously nice.

Theorem A. (See Theorem 4.2.) *Let $(\mathfrak{g}, \mathfrak{k})$ a nice symmetric pair. Then $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ and, therefore, $\mathcal{I}(\mathfrak{p}) = (\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))^K$.*

The key step in the proof of Theorem A is Theorem 3.8, which forms a mild generalization of [18, Theorem 5.2]. This, in turn, is an interpretation in terms of $\mathcal{D}$-modules of the theorem of Harish-Chandra that asserts that non-zero G_0-invariant eigendistributions on $\mathfrak{g}_0$ cannot have nilpotent support. We take this opportunity to provide a complete algebraic proof of our generalization of this result (rather than modifying Harish-Chandra's analytic proof) for two reasons: First, it applies to all modules rather than just distributions and secondly this different approach may prove useful in other, similar problems. The reason for restricting ourselves to nice symmetric pairs in Theorem A is that Theorem 3.8 does not hold in general; indeed in §6 we show that it fails when $(\mathfrak{g}, \mathfrak{k})$ has rank one and $\dim \mathfrak{p} = 2r \geqslant 4$.

As an immediate corollary of Theorem A one obtains a generalization of a fundamental result of Harish-Chandra (see [9, Theorem 4] or (1.2)) from the diagonal case to nice symmetric pairs:

Corollary B. (See Corollary 4.3.) *Assume that the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is nice and is the complexification of a real symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$. Write K_0 for the connected Lie group satisfying* $\mathrm{Lie}(K_0) = \mathfrak{k}_0$. *Let $U \subset \mathfrak{p}_0$ be a K_0-stable open subset and T be a K_0-invariant distribution on U. Then $\mathcal{I}(\mathfrak{p}) \cdot T = 0$.*

With a little extra work we are able to generalize a second result of Harish-Chandra (see [8, Theorem 3] or (1.3)) to nice symmetric pairs:

Corollary C. (See Corollary 5.8.) *Assume that the pair $(\mathfrak{g}, \mathfrak{k})$ is nice and is the complexification of a real symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$. Let U be an open subset of $\mathfrak{p}_0$ and* *write U' for the set of regular semisimple elements in U. Let T be a locally invariant eigendistribution on U such that T is supported on $U \setminus U'$. Then $T = 0$.*

Actually, this result was proved by Sekiguchi in [33] where he was even able to prove it for invariant eigenhyperfunctions T. Our (algebraic) proof of Corollary C does not permit us to prove this stronger result, but it is conjectured [33, Conjecture 7.1] that such a hyperfunction is already a distribution.

The proof that (1.1) implies (1.3) in [18] uses the fact that Harish-Chandra's map δ is surjective. Since there is no analogue of that map for general symmetric pairs, more work is needed to deduce Corollary C from Theorem A. The required extra facts are provided by the following result, in which the Gelfand-Kirillov dimension of a module M is denoted by $\operatorname{GKdim} M$. One should note that, in the diagonal case, $\operatorname{Im}\delta = \mathcal{D}(\mathfrak{h})^W$ is a fixed ring of the Weyl algebra by a finite group and so the conclusion of this theorem is well-known.

Theorem D. (See Theorems 5.5 and 5.6.) *Let $(\mathfrak{g}, \mathfrak{k})$ be a nice symmetric pair and set $R(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})^K/\mathcal{I}(\mathfrak{p})$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$. Then:*

(1) *$R(\mathfrak{p})$ is a simple ring with $\operatorname{GKdim} R(\mathfrak{p}) = 2\dim\mathfrak{a}$;*
(2) *$\operatorname{GKdim} M \geqslant \dim\mathfrak{a}$ for every non-zero finitely generated $R(\mathfrak{p})$-module M.*

The importance of the algebra $R(\mathfrak{p})$ also lies in the fact that it identifies with the algebra of radial components of K-invariant differential operators on $\mathfrak{p}$. Indeed, let $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group associated to $\mathfrak{a}$. By composing the restriction to K-invariant functions with the Chevalley isomorphism one gets a homomorphism, the *radial component map*,

$$rad : \mathcal{D}(\mathfrak{p})^K \longrightarrow \mathcal{D}(\mathfrak{p}/\!\!/K) \xrightarrow{\sim} \mathcal{D}(\mathfrak{a}/W)$$

such that $\operatorname{Ker} rad = \mathcal{I}(\mathfrak{p})$. Thus $R(\mathfrak{p}) \cong \operatorname{Im} rad$ although, except in some trivial cases, this does not equal $\mathcal{D}(\mathfrak{a}/W)$. Once again, Theorem D does not hold for general symmetric pairs; indeed, in Section 6, we give an example where $R(\mathfrak{p})$ has non-zero finite dimensional modules.

The examples mentioned above are consequences of a detailed study of invariant differential operators of rank one symmetric pairs given in Section 6. One further consequence of this study is that there are some feasible ways in which one may be able to generalize the results of this paper to arbitrary symmetric pairs, and to more general representations of K. The precise conjectures are given in Section 7.

2. Orbit theory

We continue with the notation of §1. We collect here various definitions and results needed in the subsequent sections.

If $x, y \in \mathfrak{g}$ and $g \in G$, set $\operatorname{ad}(x).y = [x, y]$ and $g.x = \operatorname{Ad}(g).x$. If $x \in \mathfrak{g}$ and $V \subset \mathfrak{g}$, let V^x denote the subset of elements of V which commute with x. Recall that x is called semisimple if $\operatorname{ad}(x)$ is semisimple and is called nilpotent if $x \in [\mathfrak{g}, \mathfrak{g}]$ and $\operatorname{ad}(x)$ is nilpotent. Following [33, 1.11] a nilpotent element $0 \neq x \in \mathfrak{p}$ is called $\mathfrak{p}$-*distinguished,* or simply *distinguished,* if $\mathfrak{p}^x$ does not contain any non-zero semisimple element.

Let X be an affine algebraic K-variety with structural sheaf $\mathcal{O}_X$. The algebraic quotient $\operatorname{Spec}\mathcal{O}(X)^K$ is denoted by $X/\!\!/K$ and $\varpi_X : X \twoheadrightarrow X/\!\!/K$ is the associated surjective morphism. Recall that the algebraic variety $X/\!\!/K$ identifies with the set of closed orbits in X. When $X = \mathfrak{p}$, and if there is no possible ambiguity, we will write $\varpi_{\mathfrak{p}} = \varpi$. Define the nilpotent cone of $\mathfrak{p}$ by $\mathbf{N}(\mathfrak{p}) = \varpi^{-1}(\varpi(0))$; recall that $\mathbf{N}(\mathfrak{p})$ is the set of nilpotent elements in $\mathfrak{p}$ and is a finite union of K-orbits [15, Propositions 10 & 11, Theorem 2].

Recall [15, 37] the following properties of the action $(K : \mathfrak{p})$. It is stable (there exists a dense open subset of closed orbits) and visible (each fibre of ϖ is a finite union of K-orbits). Let ℓ be the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$; that is, $\ell = \dim \mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace. If $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the associated Weyl group, the Chevalley restriction theorem gives an isomorphism $S(\mathfrak{p}^*)^K \xrightarrow{\sim} S(\mathfrak{a}^*)^W$, and these algebras are polynomial rings in ℓ indeterminates. Define the commuting variety of $\mathfrak{p}$ by

$$\mathcal{C}(\mathfrak{p}) = \{(x, y) \in \mathfrak{p} \times \mathfrak{p} : [x, y] = 0\}.$$

One has $\dim \mathcal{C}(\mathfrak{p}) = \ell + \dim \mathfrak{p}$, see [27, Proof of (3.7)].

Let $b \in \mathfrak{p}$ be semisimple. Then [15, I.6] the decomposition $\mathfrak{g}^b = \mathfrak{k}^b \oplus \mathfrak{p}^b$ defines a symmetric pair (of the same rank); furthermore, $\mathbf{N}(\mathfrak{p}^b) = \mathbf{N}(\mathfrak{p}) \cap \mathfrak{p}^b$ and $a \in \mathfrak{p}^b$ is semisimple in $\mathfrak{g}^b$ if and only if a is semisimple in $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is semisimple and set $\mathfrak{g}' = [\mathfrak{g}^b, \mathfrak{g}^b]$, $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{g}'$, $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$. Then $(\mathfrak{g}', \mathfrak{k}')$ is a symmetric pair with $\mathfrak{g}'$ semisimple. Such a pair is called a *sub-symmetric pair* of $(\mathfrak{g}, \mathfrak{k})$ [33].

Proposition 2.1. *There exists an isomorphism* $\mathfrak{p}/\!\!/K \cong \big(\mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p} \times \mathbf{N}(\mathfrak{p}))\big)/\!\!/K$, *induced by the map* $a \mapsto (a, 0)$ *from* $\mathfrak{p}$ *to* $\mathcal{C}(\mathfrak{p})$. *Under this isomorphism,* $\big(\mathcal{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathbf{N}(\mathfrak{p}))\big)/\!\!/K$ *identifies with* $\{\varpi(0)\}$.

Proof. Let $(x, y) \in \mathcal{C}(\mathfrak{p})$ and $x = x_s + x_n$, $y = y_s + y_n$ be the Jordan decompositions of x and y. Then it is standard that (x_s, y_s) lies in the closure of $K.(x, y)$, see, for example, [29, Theorem 5.2]. Therefore, if $K.(x, y) \in \big(\mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p} \times \mathbf{N}(\mathfrak{p}))\big)/\!\!/K$ is a closed orbit with $y \in \mathbf{N}(\mathfrak{p})$, then y is also semisimple and so $y = 0$. Hence, the map $\imath : \mathfrak{p}/\!\!/K \to \big(\mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p} \times \mathbf{N}(\mathfrak{p}))\big)/\!\!/K$, defined by $K.x \mapsto K.(x, 0)$, is a bijection. It is easily seen that its inverse is induced by the restriction $\mathcal{C}(\mathfrak{p}) \twoheadrightarrow \mathfrak{p}$ of the projection $\eta_1 : \mathfrak{p} \times \mathfrak{p} \twoheadrightarrow \mathfrak{p}$ onto the first component. Thus $\imath$ is an isomorphism. The identification of $\big(\mathcal{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathbf{N}(\mathfrak{p}))\big)/\!\!/K$ with $\{\varpi(0)\}$ follows easily. $\square$

Lemma 2.2. *Let F and T be two closed K-stable subsets of $\mathfrak{p}$ such that F is a finite union of K-orbits. Let $X \subset \mathcal{C}(\mathfrak{p}) \cap (F \times T)$ be closed, irreducible and K-stable. Then:*

(i) $\dim X \leq \dim \mathfrak{p}$.

(ii) *Let* $\eta_1 : \mathfrak{p} \times \mathfrak{p} \twoheadrightarrow \mathfrak{p}$ *be the projection onto the first component. Assume that* $T \subset \mathbf{N}(\mathfrak{p})$ *and that* $\overline{\eta_1(X)} = \overline{K.u}$, *where* $u \in \eta_1(X)$ *is nilpotent but not distinguished. Then,* $\dim X < \dim \mathfrak{p}$.

Proof. (i) Set $Y = \overline{\eta_1(X)}$ and consider $\eta = \eta_{1|X} : X \to Y$. Let U be a dense open subset of Y such that

$$\forall\, v \in U, \quad \dim X - \dim Y = \dim \eta^{-1}(v).$$

Since η is K-equivariant, Y is a K-stable closed irreducible subset of F. Therefore, $Y = \overline{K.u}$ for some $u \in U$. Let $v \in U \cap K.u$. Identify the variety $\mathcal{C}(\mathfrak{p}) \cap \eta_1^{-1}(v)$ with $\mathfrak{p}^v$ through the second projection; then, $\eta^{-1}(v) = X \cap \eta_1^{-1}(v)$ is a closed subset of $\mathfrak{p}^v$. Recall [15, Proposition 5] that $\dim \mathfrak{p}^v - \dim \mathfrak{k}^v = \dim \mathfrak{p} - \dim \mathfrak{k}$. Hence,

$$(\star) \qquad \dim X = \dim K.v + \dim \eta^{-1}(v) \leq \dim \mathfrak{k} - \dim \mathfrak{k}^v + \dim \mathfrak{p}^v = \dim \mathfrak{p}.$$

(ii) By hypothesis, the element v in the proof of (i) is not distinguished and so there exists a non-zero semisimple element $s \in \mathfrak{p}^v$. Since $T \subset \mathbf{N}(\mathfrak{p})$, one has

$(v,s) \in (\mathcal{C}(\mathfrak{p}) \cap \eta_1^{-1}(v)) \setminus \eta^{-1}(v)$. Thus $\dim \eta^{-1}(v) < \dim \mathfrak{p}^v$ and hence, as in $(\star)$, it follows that $\dim X < \dim \mathfrak{p}$. □

Let $0 \neq x \in \mathbf{N}(\mathfrak{p})$. Then [15] there exist $y \in \mathfrak{p}$, $z \in \mathfrak{k}$ such that $[x,y] = z$, $[z,x] = 2x$, $[z,y] = -2y$. The triple (z,x,y) is called *a normal S-triple* containing x. Set $\mathfrak{s} = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$; thus $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{C})$. The $\mathfrak{s}$-module $\mathfrak{g}$ then decomposes as $\mathfrak{g} = \bigoplus_{j=1}^{s} E(\lambda_j)$, where $E(\lambda_j)$ is a simple $\mathfrak{s}$-module of highest weight $\lambda_j \in \mathbb{N}$. We can set $\mathfrak{p}^y = \bigoplus_{i=1}^{m} \mathbb{C}v_i$ with $\mathrm{ad}(z).v_i = -\lambda_i v_i$, and we have:

$$\mathfrak{p} = [x,\mathfrak{k}] \oplus \mathfrak{p}^y \tag{2.1}$$

It is well known ([32, Lemma 1.21], [34, III.5.1, III.7.4]) that $x+\mathfrak{p}^y$ is a transversal slice, in $\mathfrak{p}$, to the orbit $K.x$. Let $\psi : K \times \mathfrak{p}^y \to \mathfrak{p}$ be the K-equivariant morphism given by $\psi((g,v)) = g.(x+v)$. Thus ψ is smooth on $K \times \mathfrak{p}^y$ (and therefore is an open morphism).

The following result is well known in the analytic case, see [36, Chapter 5, Lemma 22], and we leave it to the reader to check that that argument can be easily modified to work in the algebraic setting.

Proposition 2.3. *There exists an affine open neighborhood U of 0 in $\mathfrak{p}^y$ such that:*

(1) *ψ is smooth on $Y = K \times U$, and $\Omega = \psi(Y) = K.(x+U)$ is a K-stable open subset of $\mathfrak{p}$;*
(2) *$\Omega \cap \overline{K.x} = K.x$ and $K.x \cap \{x+U\} = \{x\}$.* □

Assume that $\mathfrak{g}$ is semisimple. Let (z,x,y) be a normal S-triple containing $x \in \mathbf{N}(\mathfrak{p})$. With the previous notation, we set:

$$\lambda_{\mathfrak{p}}(x) = \sum_{j=1}^{m} (\lambda_j + 2) - \dim \mathfrak{p} \tag{2.2}$$

Remark 2.4. The integer $\lambda_{\mathfrak{p}}(x)$ is denoted by $\tilde{\delta}_{\mathfrak{p}}(x)$ in [33, 6.1]. Let us illustrate this number in the special case when $\ell = 1$ and $x \in \mathbf{N}(\mathfrak{p})$ is a regular nilpotent element; that is, $\dim K.x = \dim \mathfrak{p} - 1$. Then x is distinguished [33, (1.9)]. However as $\dim K.x = \dim \mathfrak{p} - 1$, [15, Proposition 5] implies that $m = 1$ and, since $\lambda_1 = 2$, $\lambda_{\mathfrak{p}}(x) = 4 - \dim \mathfrak{p}$. Thus $\lambda_{\mathfrak{p}}(x) > 0$ if, and only if, $\dim \mathfrak{p} < 4$.

We will be interested in symmetric pairs $(\mathfrak{g},\mathfrak{k})$ such that, for all sub-symmetric pairs $(\mathfrak{g}',\mathfrak{k}')$ of $(\mathfrak{g},\mathfrak{k})$, one has $\lambda_{\mathfrak{p}'}(x) > 0$ for each $\mathfrak{p}'$-distinguished nilpotent element $x \in \mathfrak{p}'$. The following theorem, proved in [33, §6], provides examples of such symmetric pairs.

Theorem 2.5. (J. Sekiguchi) *Assume that $\mathfrak{g}$ is semisimple and let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair. Let Σ be the restricted root system associated to a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$. Consider the following condition:*

$$\dim \mathfrak{g}^{\alpha} + \dim \mathfrak{g}^{2\alpha} \leq 2 \qquad \text{for all } \alpha \in \Sigma, \tag{†}$$

(1) *The pair $(\mathfrak{g},\mathfrak{k})$ satisfies (†) if and only if each of its irreducible factors is isomorphic to one of the following pairs:*

- **(O)** $(\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathfrak{g}_1)$, *$\mathfrak{g}_1$ simple Lie algebra (the diagonal case)*
- **(I)** $(\mathfrak{sl}(m,\mathbb{C}), \mathfrak{so}(m,\mathbb{C}))$
- **(II)** $(\mathfrak{sl}(2m,\mathbb{C}), \mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{C})$
- **(III)** $(\mathfrak{sp}(m,\mathbb{C}), \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{C})$
- **(IV)** $(\mathfrak{so}(2m+k,\mathbb{C}), \mathfrak{so}(m+k,\mathbb{C}) \oplus \mathfrak{so}(m,\mathbb{C}))$, *$k = 0,1,2$*

(V) $(\mathfrak{e}_6, \mathfrak{sp}(4,\mathbb{C}))$
(VI) $(\mathfrak{e}_6, \mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$
(VII) $(\mathfrak{e}_7, \mathfrak{sl}(8,\mathbb{C}))$
(VIII) $(\mathfrak{e}_8, \mathfrak{so}(16,\mathbb{C}))$
(IX) $(\mathfrak{f}_4, \mathfrak{sp}(3,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$
(X) $(\mathfrak{g}_2, \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$

(2) *Assume that* $(\mathfrak{g},\mathfrak{k})$ *satisfies* (†) *and let* $(\mathfrak{g}',\mathfrak{k}')$ *be a sub-symmetric pair of* $(\mathfrak{g},\mathfrak{k})$. *Then,* $(\mathfrak{g}',\mathfrak{k}')$ *satisfies* (†) *and* $\lambda_{\mathfrak{p}'}(x) > 0$ *for each* $\mathfrak{p}'$*-distinguished nilpotent element* $x \in \mathfrak{p}'$. □

Definition 2.6. The symmetric pair $(\mathfrak{g},\mathfrak{k})$ is said to be *nice* if the semisimple, sub-symmetric pair $([\mathfrak{g},\mathfrak{g}], \mathfrak{k}\cap[\mathfrak{g},\mathfrak{g}])$ satisfies condition (†) of Theorem 2.5.

Remark 2.7. Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair with $\mathfrak{g}$ semisimple. Then, $(\mathfrak{g},\mathfrak{k})$ is of maximal rank (that is, $\ell = \operatorname{rk}\mathfrak{g}$) if and only if, in the notation of Theorem 2.5, each irreducible factor of $(\mathfrak{g},\mathfrak{k})$ is of type

(I), (III), (IV) $(k = 0, 1)$**, (V), (VII), (VIII), (IX)** or **(X)**

3. Equivariant D-modules with nilpotent support

Let X be a smooth algebraic variety with structural sheaf $\mathcal{O}_X$ and cotangent bundle T^*X. Denote by Θ_X the $\mathcal{O}_X$-module of vector fields and by $\mathcal{D}_X$ the sheaf of differential operators on X. The notation related to algebraic D-modules will be as in [3] or [12]. In particular, if M is a $\mathcal{D}_X$-module we denote its characteristic variety by $\operatorname{Ch} M \subset T^*X$ and its support by $\operatorname{Supp} M \subset X$.

Assume that a reductive algebraic group K acts regularly on X and denote by $\tau_X : \mathfrak{k} \to \Theta_X$ the differential of the K-action on X. One can develop a theory of K-equivariant $\mathcal{D}_X$-modules; for these notions, and related definitions, we will refer to [3] and [35, §6]. We simply recall here the definition of a K-equivariant coherent $\mathcal{D}_X$-module in the case where X is affine: If M is a finitely generated $\mathcal{D}(X)$-module, then M is said to be a *K-equivariant $\mathcal{D}_X$-module* if it satisfies the following conditions:

(i) M is endowed with a compatible action of K; that is, K acts rationally on M and $g.(dv) = (g.d)(g.v)$ for all $g \in K$, $d \in \mathcal{D}(X)$ and $v \in M$;
(ii) $\frac{d}{dt}_{|t=0}(\exp(ta).v) = \tau(a)v$ for all $a \in \mathfrak{k}$, $v \in M$.

Let $\mathfrak{g}$ be a reductive Lie algebra and $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair. Recall that the K-invariant form κ is non-degenerate on $\mathfrak{p}$. Let $n = \dim\mathfrak{p} > 0$ and fix a κ-orthonormal basis of $\mathfrak{p}$. Let $\{x_i \in \mathfrak{p}^*, \partial_{x_i} \in \Theta(\mathfrak{p})\}_{1\leqslant i\leqslant n}$ be the associated coordinate system. Let $e \in S^2(\mathfrak{p}^*)^K$ be defined by $e(x) = \kappa(x,x)$ and write $\partial(e) \in S^2(\mathfrak{p})^K$ for the corresponding differential operator with constant coefficients. Thus, in coordinates, $e = \sum_{i=1}^n x_i^2$ and $\partial(e) = \sum_{i=1}^n \partial_{x_i}^2$. Let $\mathtt{E} = \sum_{i=1}^n x_i\partial_{x_i}$ be the Euler vector field and set

$$f = -\tfrac{1}{4}\partial(e), \qquad h = \mathtt{E} + \tfrac{n}{2}.$$

Then $\mathfrak{u} = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ is a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. By construction, $\mathfrak{u} \subset \mathcal{D}(\mathfrak{p})^K$.

Let (z,x,y) be a normal S-triple containing the nilpotent element $0 \neq x \in \mathbf{N}(\mathfrak{p})$ and adopt the notation of §2. If $\mathfrak{p}^y = \oplus_{i=1}^m \mathbb{C}v_i$ with $\operatorname{ad}(z).v_i = -\lambda_i v_i$, let $\{y_i = v_i^*, \partial_{y_i}\}_{1\leqslant i\leqslant m}$ denote the associated coordinate system. Let U be the affine open neighborhood of 0 in $\mathfrak{p}^y$ found in Proposition 2.3 and $\psi^\# : \mathcal{O}_{\mathfrak{p}} \to \mathcal{O}_Y$ be the comorphism of the dominant K-equivariant morphism $\psi : Y = K \times U \to \mathfrak{p}$. Since ψ is smooth, we have a surjective map of $\mathcal{O}_Y$-modules

$$\Theta_Y \to \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathfrak{p}}} \Theta_{\mathfrak{p}} \to 0 \tag{3.1}$$

Let $\theta \in \Theta(\mathfrak{p})$ and identify θ with $1 \otimes \theta \in \mathcal{O}(Y) \otimes_{\mathcal{O}(\mathfrak{p})} \Theta(\mathfrak{p})$. Then, by (3.1), there exists a lift $\psi^{\#}(\theta) \in \Theta(Y)$ of θ; such a lift satisfies $\psi^{\#}(\theta).\psi^{\#}(f) = \psi^{\#}(\theta.f)$ for all $f \in \mathcal{O}(\mathfrak{p})$. It is easily checked that K acts on each term of (3.1), and that (3.1) is an exact sequence of K-equivariant $\mathcal{O}_Y$-modules. It follows that we have a surjective morphism $\Theta_Y^K \twoheadrightarrow (\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathfrak{p}}} \Theta_{\mathfrak{p}})^K$. In particular, if $\theta \in \Theta(\mathfrak{p})^K$ there exists a lift $\psi^{\#}(\theta) \in \Theta(Y)^K$.

Write R_a for the left invariant vector field on K defined by $a \in \mathfrak{k}$; thus, for all $f \in \mathcal{O}(K)$ and $g \in K$ one has $(R_a.f)(g) = \frac{d}{dt}_{|t=0} f(ge^{ta})$. The enveloping algebra $U(\mathfrak{k})$ then identifies with $\mathbb{C}\langle R_a\,;\, a \in \mathfrak{k}\rangle$. Thus, since K and U are affine, we obtain that: $\mathcal{D}(K) = \mathcal{O}(K) \otimes_{\mathbb{C}} U(\mathfrak{k})$, $\mathcal{D}(U) = \mathbb{C}\langle \mathcal{O}(U)\,;\, \partial_{y_i}, 1 \leqslant i \leqslant m\rangle$, and $\mathcal{D}(Y) = \mathcal{D}(K) \boxtimes \mathcal{D}(U)$. The group K acts on $\mathcal{D}(Y)$ via left translation on $\mathcal{D}(K)$, and so

$$\mathcal{D}(Y)^K = \mathcal{D}(K)^K \boxtimes \mathcal{D}(U) = U(\mathfrak{k}) \boxtimes \mathcal{D}(U) = (\mathbb{C} \boxtimes \mathcal{D}(U)) \oplus (U_+(\mathfrak{k}) \boxtimes \mathcal{D}(U))$$

where $U_+(\mathfrak{k}) = \mathfrak{k}U(\mathfrak{k})$. Therefore, if $\theta \in \Theta(\mathfrak{p})^K$, we can write

$$\psi^{\#}(\theta) = 1 \boxtimes \Delta(\theta) + \textstyle\sum_j R_{a_j} \boxtimes \varphi_j \tag{3.2}$$

for some $a_j \in \mathfrak{k}$ and $\varphi_j \in \mathcal{O}(U)$. We call $\Delta(\theta) \in \Theta(U)$ a *radial component* of θ.

Lemma 3.1. *The vector field* $\sum_{j=1}^m (\frac{1}{2}\lambda_j + 1) y_j \partial_{y_j} + \frac{1}{2} R_z$ *is an invariant lift of the Euler vector field* $\mathtt{E}$*, and* $\Delta(\mathtt{E}) = \sum_{j=1}^m (\frac{1}{2}\lambda_j + 1) y_j \partial_{y_j}$ *is a radial component of* $\mathtt{E}$.

Proof. See [7, Lemma 30] or [36, Lemma 24, p. 93]. □

If M is a coherent $\mathcal{D}_{\mathfrak{p}}$-module, denote by

$$\mathcal{M} = \mathcal{D}_{Y \to \mathfrak{p}} \otimes^L_{\psi^{-1}\mathcal{D}_{\mathfrak{p}}} \psi^{-1} M$$

its inverse image in the D-module sense; thus $\mathcal{M} = \psi^! M[\dim Y - \dim \mathfrak{p}]$ (see, for example, [3, VI.4.2]). Since ψ is smooth the $\mathcal{D}_Y$-module $\mathcal{M}$ is, as an $\mathcal{O}_Y$-module, the usual inverse image $\psi^* M = \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathfrak{p}}} M$ [12, Proposition II.4.2(i)].

Lemma 3.2. *Assume that M is a K-equivariant coherent $\mathcal{D}_{\mathfrak{p}}$-module such that* $\operatorname{Supp} M = \overline{K.x}$*, for some* $0 \neq x \in \mathbf{N}(\mathfrak{p})$*. Then:*

1. *The group K acts on $\mathcal{M}$ by $g.(b \otimes_{\mathcal{O}_{\mathfrak{p}}} v) = g.b \otimes_{\mathcal{O}_{\mathfrak{p}}} g.v$, $\mathcal{M}$ is a K-equivariant $\mathcal{D}_Y$-module, and the canonical map $M \to \mathcal{M}$, $v \mapsto 1 \otimes_{\mathcal{O}_{\mathfrak{p}}} v$, is K-equivariant.*
2. *Suppose that $M = \mathcal{D}_{\mathfrak{p}}.v$. Then, $\mathcal{M} = \mathcal{D}_Y.(1 \otimes_{\mathcal{O}_{\mathfrak{p}}} v)$ and $\mathcal{M} \neq 0$.*
3. *There exists $k \in \mathbb{N}$ such that $\mathcal{M} \cong (\mathcal{O}_K \boxtimes H^m_{[0]}(\mathcal{O}_U))^{\oplus k}$.*
4. *$\mathcal{M}^K \cong \mathbb{C} \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$ as a $(U(\mathfrak{k}) \boxtimes \mathcal{D}_U)$-module.*
5. *$H^m_{[0]}(\mathcal{O}_U) = \bigoplus_{\alpha \in \mathbb{N}^m} \mathbb{C}\partial^\alpha$, where $\partial^\alpha = \prod_{i=1}^m \partial_{y_i}^{\alpha_i}$.*

Proof. The proofs of parts (1) and (2) are standard and are left to the reader. We first compute $\operatorname{Supp} \mathcal{M}$. Since $\mathcal{M} = \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathfrak{p}}} M$ as $\mathcal{O}_Y$-modules, $\operatorname{Supp} \mathcal{M} \subset \psi^{-1}(\operatorname{Supp} M)$. Let $a = (g, u) \in \operatorname{Supp} \mathcal{M}$; then, Proposition 2.3(2) implies that $\psi(a) = g.(x + u) \in \operatorname{Supp} M \cap \psi(Y) = \overline{K.x} \cap \Omega = K.x$. Thus $x + u \in K.x$, forcing $u = 0$ by Proposition 2.3(2), again. Therefore $\emptyset \neq \operatorname{Supp} \mathcal{M} \subset \Phi = K \times \{0\}$. Since $\mathcal{M}$ is K-equivariant it follows that $\operatorname{Supp} \mathcal{M} = \Phi$.

Now consider the K-equivariant closed embedding $\jmath : \Phi \hookrightarrow Y$. Since $\operatorname{Supp} \mathcal{M} = \Phi$, Kashiwara's equivalence [3, VI.7.11] yields $\mathcal{M} = \jmath_+ \jmath^! \mathcal{M}$. Since the $\mathcal{D}_\Phi$-module $\jmath^! \mathcal{M}$

is K-equivariant and $\Phi \cong K$, $\jmath^!\mathcal{M}$ is a finite direct sum of copies of $\mathcal{O}_\Phi$ [3, Proof of VII.12.11]. Setting $\jmath^!\mathcal{M} = (\mathcal{O}_\Phi)^{\oplus k}$ we obtain

$$\mathcal{M} = (\jmath_+\mathcal{O}_\Phi)^{\oplus k} = H^m_{[\Phi]}(\mathcal{O}_Y)^{\oplus k} = (\mathcal{O}_K \boxtimes H^m_{[0]}(\mathcal{O}_U))^{\oplus k}$$

by [3, VI §7] or [12, I §4.2]. Observe that K acts on $\mathcal{M}$ via left translation on $\mathcal{O}_K$. Thus $\mathcal{M}^K = \mathbb{C} \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$. To finish the proof, recall that $H^m_{[0]}(\mathcal{O}_U) \cong \mathcal{D}(U)/\big(\sum_{i=1}^m \mathcal{D}(U)y_i\big)$; see, for example, [12, Corollary I.5.2]. □

Lemma 3.3. *Let M be as in Lemma 3.2. Let $\theta \in \Theta(\mathfrak{p})^K$ and $v \in M^K$. Then $1 \otimes v \in \mathcal{M}^K$. If $\psi^\#$ is defined by* (3.2), *then*

$$\psi^\#(\theta).(1 \otimes v) = 1 \otimes \theta.v = (1 \boxtimes \Delta(\theta)).(1 \otimes v).$$

Proof. The claim follows from the definition of $\psi^\#(\theta)$ and Lemma 3.2(4). □

Lemma 3.4. *Let M be as in Lemma 3.2. Assume that $M = \mathcal{D}(\mathfrak{p}).s$ with $s \in M^K$ such that $\mathtt{E}.s = \mu s$, for some $\mu \in \mathbb{C}$. Then, there exists $\alpha \in \mathbb{N}^m$ such that $\mu = -\frac{1}{2}(\lambda_\mathfrak{p}(x) + \dim \mathfrak{p}) - \frac{1}{2}\sum_{i=1}^m (\lambda_i + 2)\alpha_i$.*

Proof. Let (z, x, y) be a normal S-triple containing x and adopt the notation of Lemma 3.2. Set $H = H^m_{[0]}(\mathcal{O}_U)$. If $v \in M^K$ we write $v^* = 1 \otimes_{\mathcal{O}_\mathfrak{p}} v = 1 \boxtimes \rho(v)$ with $\rho(v) = \oplus_{i=1}^k \rho_i(v) \in H^{\oplus k}$. Recall from Lemma 3.2 that $\mathcal{M} = \mathcal{D}_Y.s^* = (\mathcal{O}_K \boxtimes H)^{\oplus k}$; this implies in particular that $\rho_i(s) \neq 0$ for all i.

The element $s^* = 1 \otimes s$ is a non-zero element of $\mathcal{M}^K$. Therefore, by Lemma 3.3, $1 \otimes \mathtt{E}.s = \mu s^* = (1 \boxtimes \Delta(\mathtt{E})).s^*$ and so $\Delta(\mathtt{E}).\rho(s) = \mu\rho(s)$. Thus each $\rho_i(s) \in H$ is an eigenvector of $\Delta(\mathtt{E})$ with eigenvalue μ. Now, by Lemma 3.1 and Lemma 3.2,

$$\Delta(\mathtt{E}).\partial^\beta = -\big(\textstyle\sum_i (\frac{1}{2}\lambda_i + 1)(\beta_i + 1)\big)\partial^\beta$$

for all $\partial^\beta = \prod \partial_{y_i}^{\beta_i} \in H$. Therefore the eigenvalues of $\Delta(\mathtt{E})$ on H are of the form

$$-\textstyle\sum_{i=1}^m (\frac{1}{2}\lambda_i + 1)(\beta_i + 1) = -\frac{1}{2}(\lambda_\mathfrak{p}(x) + n) - \frac{1}{2}\sum_{i=1}^m (\lambda_i + 2)\beta_i.$$

The existence of $\alpha = (\alpha_i)_i$ follows. □

Remarks. (1) In the notation of Lemma 3.4, assume that $\mu = -\frac{1}{2}(\lambda_\mathfrak{p}(x) + \dim \mathfrak{p})$. Then it is not difficult to prove that:

(i) $\mathcal{M} \cong \mathcal{O}_K \boxtimes H^m_{[0]}(\mathcal{O}_U)$;

(ii) if $v \in M^K$ is an eigenvector of $\mathtt{E}$ for μ, $\operatorname{Supp} \mathcal{D}(\mathfrak{p}).(v - cs) \subsetneq \operatorname{Supp} M$ for some $c \in \mathbb{C}$.

(2) Suppose that we are in the diagonal case $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathfrak{g}_1)$, where $\mathfrak{g}_1$ is the Lie algebra of a semisimple group G_1. Then, $\mathfrak{p} \cong \mathfrak{g}_1$ and a module M satisfying the hypotheses of Lemma 3.2 can be considered as a G_1-equivariant coherent $\mathcal{D}_{\mathfrak{g}_1}$-module for which $\operatorname{Supp} M = \overline{G_1.x}$, and $\mathbf{O} = G_1.x \subset \mathfrak{g}_1$ is a nilpotent orbit. As in [13, §7], set:

$$\lambda_\mathbf{O} = \tfrac{1}{2}\dim \mathbf{O} - \dim \mathfrak{g}_1 = -\tfrac{1}{2}(\dim \mathfrak{g}_1 + \dim \mathfrak{g}_1^x)$$

It is easily seen that $\lambda_\mathfrak{p}(x) = \dim \mathfrak{g}_1^x$ and Lemma 3.4 implies that, if s is a G_1-invariant generator of M and $\mathtt{E}.s = \mu s$, then $\mu \leqslant \lambda_\mathbf{O}$. Moreover, if $\mu = \lambda_\mathbf{O}$ the previous remark yields that the eigenspace for $\lambda_\mathbf{O}$ has dimension one. See [38, Lemma 6.2] for a variant of this result.

In the sequel we will identify the cotangent bundle $T^*\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}^*$ with $\mathfrak{p} \times \mathfrak{p}$ through the isomorphism $\mathfrak{p} \cong \mathfrak{p}^*$ induced by the form κ. We endow $\mathcal{D}(\mathfrak{p})$ with the filtration $\{\mathcal{D}^k(\mathfrak{p})\}_k$ by order of differential operators, and use the induced filtration on its subalgebras and factors. The associated graded algebra $\mathrm{gr}(\mathcal{D}(\mathfrak{p}))$ of $\mathcal{D}(\mathfrak{p})$ identifies with $\mathcal{O}(T^*\mathfrak{p})$. We denote by $\sigma(a) \in \mathcal{O}(T^*\mathfrak{p})$ the principal symbol of the vector field $\tau(a)$, $a \in \mathfrak{k}$. Thus, $\sigma(a)(b,c) = \kappa(a,[c,b])$ for all $(b,c) \in \mathfrak{p} \times \mathfrak{p}$. Let $\mathbf{a}$ be the ideal of $\mathcal{O}(T^*\mathfrak{p})$ generated by $\{\sigma(a),\ a \in \mathfrak{k}\}$, write $\mathbf{q}$ for the radical ideal defining $\mathcal{C}(\mathfrak{p})$ and set $\mathbf{b} = \mathrm{gr}(\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))$. Write $\mathcal{N} = \mathcal{D}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$. One obviously has $\mathbf{a} \subset \mathbf{b}$, with $\sqrt{\mathbf{a}} = \mathbf{q}$, and so $\mathrm{Ch}\,\mathcal{N} \subset \mathcal{C}(\mathfrak{p})$. Similarly, $\mathrm{gr}(\mathcal{D}(\mathfrak{p})^K)$ identifies with $\mathcal{O}(T^*\mathfrak{p})^K = \mathcal{O}((T^*\mathfrak{p})/\!\!/K)$ and for any finitely generated $\mathcal{D}(\mathfrak{p})^K$-module Q one sets $\mathrm{Ch}\,Q = \mathcal{V}(\mathrm{ann}\,\mathrm{gr}(Q))$, which is the variety of zeroes, in $(T^*\mathfrak{p})/\!\!/K$, of the annihilator of the module $\mathrm{gr}(Q)$.

Lemma 3.5. *Let $M = \mathcal{D}(\mathfrak{p})/L$ be such that $L \supset \mathcal{D}(\mathfrak{p})F + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ where F is a power of $S_+(\mathfrak{p}^*)^K$. Then, $\mathrm{Ch}\,M \subset \mathcal{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathfrak{p})$ and any subquotient of M is a K-equivariant holonomic $\mathcal{D}_{\mathfrak{p}}$-module.*

Proof. Since $F \subset L$ and $\tau(\mathfrak{k}) \subset L$, we see that $\mathrm{Ch}\,M \subset (\mathbf{N}(\mathfrak{p})\times\mathfrak{p})\cap\mathcal{C}(\mathfrak{p})$ and it follows from Lemma 2.2 that M is holonomic. Since $L \supset \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$, [35, Lemma 6.2.6(4)] yields the equivariance of M. If Q is a subquotient of M, then it follows immediately that Q is holonomic while the K-equivariance of Q follows from [35, Theorem 6.2.4] (see also [19, Lemma 6.1]). □

Remark 3.6. Let M be as in Lemma 3.5 and set $\lambda(M) = \min\{\lambda_{\mathfrak{p}}(x) : x \in \mathrm{Supp}\,M\}$. Then it can be easily deduced from Lemma 3.4 that any eigenvalue of $\mathtt{E}$ on M^K is less or equal to $-\frac{1}{2}(\lambda(M)+\dim\mathfrak{p})$. In the diagonal case, this result can be compared with [2, Corollary 3.9].

We now consider modules of the following form:

$$M = \mathcal{D}(\mathfrak{p})/L \ \text{ such that } L \supset \mathcal{D}(\mathfrak{p})F + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})F' \tag{3.3}$$

where $F' \subset S(\mathfrak{p})^K$ is an ideal of finite codimension and F is a power of $S_+(\mathfrak{p}^*)^K$.

Note that, since $\mathcal{D}(\mathfrak{p})^K$ contains the Lie algebra $\mathfrak{u} = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$, both M and M^K do have a natural $\mathfrak{u}$-module structure.

Lemma 3.7. *Let $M = \mathcal{D}(\mathfrak{p})/L$ be defined by* (3.3). *Then:*

1. $\mathrm{Ch}\,M \subset \mathcal{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathbf{N}(\mathfrak{p}))$*;*
2. *the action of $\mathfrak{u}$ on M is locally finite.*

Proof. (1) Observe that, since $F' \subset S(\mathfrak{p})^K$ has finite codimension, $\mathrm{gr}(F')$ contains a power of the augmentation ideal $S_+(\mathfrak{p})^K$; thus $\mathrm{Ch}\,M \subset \mathfrak{p} \times \mathbf{N}(\mathfrak{p})$. The claim then follows from Lemma 3.5.

(2) If $v \in \mathcal{D}(\mathfrak{p})$, write $\bar{v} \in M$ for its class modulo L. By hypothesis, there exist $l \in \mathbb{N}$ and a non-zero polynomial $a \in \mathbb{C}[T]$ such that $e^l.\bar{1} = a(f).\bar{1} = 0$. Recall that the elements of $S(\mathfrak{p}^*)$ and $S(\mathfrak{p})$ act ad-nilpotently on $\mathcal{D}(\mathfrak{p})$. Therefore, for all $d \in \mathcal{D}(\mathfrak{p})$, there exists $\ell \in \mathbb{N}$ and $0 \neq b \in \mathbb{C}[T]$ such that $e^\ell.\bar{d} = b(f).\bar{d} = 0$. It is then a classical result that $U(\mathfrak{u}).\bar{d}$ is a finite dimensional $\mathfrak{u}$-module (see [14, Corollary 2.4.11, Remark 2.5.2] for a more general result). □

We can now prove the main result of this section.

Theorem 3.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair (such that $n = \dim\mathfrak{p} > 0$). Assume that $\lambda_{\mathfrak{p}}(x) > 0$ for all distinguished nilpotent elements $x \in \mathfrak{p}$. If M is defined by* (3.3) *then $M = 0$.*

Proof. Suppose that $M \neq 0$. By taking a simple quotient we may assume that M is simple; note that $0 \neq \bar{1} \in M^K$. Then $\operatorname{Supp} M \subset \mathbf{N}(\mathfrak{p})$ is irreducible, closed and K-stable. Hence $\operatorname{Supp} M = \overline{K.x}$ is the closure of a single nilpotent orbit. As already observed, $\operatorname{gr}(L)$ contains a power of $S_+(\mathfrak{p})^K$; it follows that $\operatorname{Ch} M \subset \mathcal{C}(\mathfrak{p}) \cap (\overline{K.x} \times \mathbf{N}(\mathfrak{p}))$.

Suppose that x is not distinguished (this includes the possibility that $x = 0$). Since $\operatorname{Supp} M$ contains x, there exists an irreducible component X of $\operatorname{Ch} M$ such that $(x, \xi) \in X$ for some $\xi \in \mathfrak{p}$. Since X is K-stable, it follows, in the notation of Lemma 2.2, that $\overline{\eta_1(X)} = \overline{K.x}$. Thus $\dim X < n = \dim \mathfrak{p}$ by Lemma 2.2(ii). But, by Lemma 3.5, M is a non-zero holonomic $\mathcal{D}_{\mathfrak{p}}$-module, and so each irreducible component of $\operatorname{Ch} M$ has dimension n (see [6], for example). Hence a contradiction.

Thus, x is distinguished. By Lemma 3.7, the action of $\mathfrak{u}$ is locally finite on the non-zero module M^K. Therefore, we can pick $0 \neq s \in M^K$ which is a highest weight vector for $\mathfrak{u}$. Let $\nu \in \mathbb{N}$ be the weight of s; thus $h.s = \nu s$, or, equivalently, $\mathtt{E}.s = (\nu - n/2)s$. By Lemma 3.5, M is a K-equivariant, coherent $\mathcal{D}_{\mathfrak{p}}$-module and so we may apply Lemma 3.4 to conclude that

$$\nu = -\tfrac{1}{2}\lambda_{\mathfrak{p}}(x) - \tfrac{1}{2}\textstyle\sum_{i=1}^{m}(\lambda_i + 2)\alpha_i$$

for some $\alpha \in \mathbb{N}^m$. The hypothesis on $\lambda_{\mathfrak{p}}(x)$ gives $\nu < 0$ and a contradiction. $\square$

Remarks. (1) The proof of Theorem 3.8 can be applied in a few further cases. For example, assume that $\ell = 1$ and that either $n = 2$ or $n = 2r + 1$ is odd; this forces $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}(3, \mathbb{C}), \mathfrak{so}(2, \mathbb{C}))$ respectively $(\mathfrak{so}(2r+2, \mathbb{C}), \mathfrak{so}(2r+1, \mathbb{C}))$. Then the proof of the theorem can be used to show that *any* $\mathcal{D}(\mathfrak{p})$-module M that satisfies (3.3) is actually zero. Indeed, in the notation of the proof, $m = 1$ and $\lambda_1 = 2$ and the final displayed equation then forces $n = 2\nu + 4(\alpha + 1)$ for some $\alpha \in \mathbb{N}$, giving the required contradiction.

(2) The theorem is, however, false without some restriction. For example, consider the real symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0) = (\mathfrak{so}(1, q+1), \mathfrak{so}(1, q))$ where $q \geqslant 3$ is odd. Then the complexified pair $(\mathfrak{so}(q+2, \mathbb{C}), \mathfrak{so}(q+1, \mathbb{C}))$ is not nice and there exists a *non-zero* $\mathcal{D}(\mathfrak{p})$-module $M = \mathcal{D}(\mathfrak{p})/L$ satisfying the hypotheses (3.3). Specifically, if u is the hyperfunction defined in [33, (6.2)], then $M = \mathcal{D}(\mathfrak{p}).u \neq 0$ has the required properties (see also [1, §4]). We will examine this phenomenon in greater detail in §6.

Return, now, to the case of a general, reductive, symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Set

$$A(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})^K / (\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))^K. \tag{3.4}$$

Notice that $\operatorname{gr}(A(\mathfrak{p})) = (\operatorname{gr}(\mathcal{D}(\mathfrak{p}))/\mathbf{b})^K$ is a factor of $(\operatorname{gr}(\mathcal{D}(\mathfrak{p}))/\mathbf{a})^K = (\mathcal{O}(T^*\mathfrak{p})/\mathbf{a})^K$ and so we can identify $\operatorname{Ch} A(\mathfrak{p}) = \operatorname{Spec} \operatorname{gr}(A(\mathfrak{p}))$ with the closed subvariety $\operatorname{Ch} \mathcal{N}/\!\!/K$ of $\mathcal{C}(\mathfrak{p})/\!\!/K = \mathcal{V}(\mathbf{a}^K)$. Set $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$, and $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$. Let K_1 be the semisimple (connected) subgroup of K such that $\operatorname{Lie}(K_1) = \mathfrak{k}_1$ and write $\mathfrak{z}$ for the centre of $\mathfrak{g}$. Then, $\mathfrak{p} = \mathfrak{p}_1 \oplus (\mathfrak{p} \cap \mathfrak{z})$, $\mathfrak{k} = \mathfrak{k}_1 \oplus (\mathfrak{z} \cap \mathfrak{k})$ and $(\mathfrak{g}_1, \mathfrak{k}_1)$ is a symmetric pair. We retain this notation in the next corollary.

Corollary 3.9. *Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair such that $\mathfrak{p}_1 \neq 0$. Assume that $\lambda_{\mathfrak{p}_1}(x) > 0$ for all $\mathfrak{p}_1$-distinguished nilpotent elements $x \in \mathfrak{p}_1$.*

1. *Let $M = \mathcal{D}(\mathfrak{p})/(\mathcal{D}(\mathfrak{p})F_1 + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})F_1')$, where F_1 is a power of $S_+(\mathfrak{p}_1^*)^{K_1}$ and $F_1' \subset S(\mathfrak{p}_1)^{K_1}$ is an ideal of finite codimension. Then, $M = 0$.*
2. *Let $\mathcal{F}$ be a $(\mathcal{D}(\mathfrak{p}), A(\mathfrak{p}))$-bimodule such that $\mathcal{F}$ has finite length as a left $\mathcal{D}(\mathfrak{p})$-module. Then $\mathcal{F} = 0$.*

Proof. (1) Set $\mathfrak{p}_0 = \mathfrak{z} \cap \mathfrak{p}$. Observe that $\mathcal{D}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p}_0) \otimes_{\mathbb{C}} \mathcal{D}(\mathfrak{p}_1)$ and $\tau(\mathfrak{k}) = \tau(\mathfrak{k}_1)$. Thus, $M \cong \mathcal{D}(\mathfrak{p}_0) \otimes_{\mathbb{C}} M_1$, where $M_1 = \mathcal{D}(\mathfrak{p}_1)/(\mathcal{D}(\mathfrak{p}_1)F_1 + \mathcal{D}(\mathfrak{p}_1)\tau(\mathfrak{k}_1) + \mathcal{D}(\mathfrak{p}_1)F_1')$. The hypotheses ensure that we can apply Theorem 3.8 to the $\mathcal{D}(\mathfrak{p}_1)$-module M_1. Hence $M_1 = 0$ and, therefore, $M = 0$.

(2) Recall that the Euler vector field $\mathtt{E} \in \mathcal{D}(\mathfrak{p})^K$ defines a grading $\mathcal{D}(\mathfrak{p}) = \bigoplus_{j\in\mathbb{Z}}\mathcal{D}(\mathfrak{p})_j$ where $\mathcal{D}(\mathfrak{p})_j = \{D \in \mathcal{D}(\mathfrak{p}) : [\mathtt{E}, D] = jD\}$. Since the action of K on $\mathfrak{p}$ is linear, the rings $\mathcal{D}(\mathfrak{p})^K$, $S(\mathfrak{p}^*)^K$ and $S(\mathfrak{p})^K$ are all graded subalgebras of $\mathcal{D}(\mathfrak{p})$. Moreover, the induced grading on $S(\mathfrak{p}^*)^K$, or $S(\mathfrak{p})^K$, is the natural one given by the degree of polynomials. Notice also that any ideal of $\mathcal{D}(\mathfrak{p})^K$ is graded (see [35, Lemma 2.3]).

Set $\mathbb{k} = \mathrm{End}_{\mathcal{D}(\mathfrak{p})}(\mathcal{F})$ and let I be the annihilator of the right $\mathcal{D}(\mathfrak{p})^K$-module $\mathcal{F}$. Note that, since $\mathcal{F}$ has finite length as a left $\mathcal{D}(\mathfrak{p})$-module, Dixmier's Lemma implies that $\mathbb{k}$ is a finite dimensional $\mathbb{C}$-algebra. As $\mathcal{F}$ is a right $A(\mathfrak{p})$-module, I contains $(\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))^K$ and $\mathcal{D}(\mathfrak{p})^K/I$ is a subalgebra of $\mathbb{k}$ via right multiplication on $\mathcal{F}$. Thus I has finite codimension in $\mathcal{D}(\mathfrak{p})^K$, which implies that $F = I \cap S(\mathfrak{p}^*)^K$ and $F' = I \cap S(\mathfrak{p})^K$ are ideals of $S(\mathfrak{p}^*)^K$, respectively $S(\mathfrak{p})^K$, of finite codimension. From the last paragraph we know that I, F and F' are graded, hence F contains a power of $S_+(\mathfrak{p}^*)^K$ and F' contains a power of $S_+(\mathfrak{p})^K$. Now consider the $\mathcal{D}(\mathfrak{p})$-module $M = \mathcal{D}(\mathfrak{p})/(\mathcal{D}(\mathfrak{p})F + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})F')$. Since K is reductive we have $M^K = A(\mathfrak{p})/(A(\mathfrak{p})F + A(\mathfrak{p})F')$; therefore, by construction, $\mathcal{D}(\mathfrak{p})^K/I$ is a factor of the $\mathcal{D}(\mathfrak{p})^K$-module M^K. However, part (1) implies that $M = 0$; hence $M^K = 0$, $I = \mathcal{D}(\mathfrak{p})^K$ and $\mathcal{F} = 0$. $\square$

4. Proof of Theorem A

The idea of the proof of Theorem A is similar to that of [18, Theorem 5.5]; the main step is to show that the module $\mathcal{L} = \mathcal{K}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ is supported on the nilpotent cone $\mathbf{N}(\mathfrak{p})$, after which the result follows easily from Corollary 3.9. In order to fix the notation, and for the convenience of the reader, we first recall a few facts from [18, §4].

We begin with the relevant notation. For simplicity we will assume that smooth varieties are irreducible. Assume that G is an arbitrary reductive algebraic group and that X is an affine G-variety. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\tau_X : \mathfrak{g} \to \mathcal{D}(X)$ be, as usual, the Lie algebra homomorphism induced by the G-action. Set

$$\mathcal{K}_G(X) = \{d \in \mathcal{D}(X) : \forall f \in \mathcal{O}(X)^G,\, d(f) = 0\}.$$

Notice that $\mathcal{D}(X)\tau_X(\mathfrak{g}) \subset \mathcal{K}_G(X)$ and that $\mathcal{K}_K(\mathfrak{p})$ is the ideal $\mathcal{K}(\mathfrak{p})$ of the introduction. Given a reductive subgroup $M \subset G$ and an affine M-variety Y, define $G \times^M Y = (G \times Y)/\!\!/M$, under the M-action $m.(g, y) = (gm^{-1}, m.y)$. If $\phi \in \mathcal{O}(X)$, write X_ϕ for the principal open subset $\{x \in X : \phi(x) \neq 0\}$. If $\mathcal{F}$ is a $\mathcal{D}(X)$-module, we denote by $\mathcal{F}_\phi$ the localization of $\mathcal{F}$ at the Ore set $\{\phi^k : k \in \mathbb{N}\}$. Note that $\mathcal{D}(X_\phi) = \mathcal{D}(X)_\phi$.

Part (1) of the next lemma is a standard application of Luna's slice theorem. We give the proof, since it illustrates the technique that will be used several times in the paper. For basic facts about the Slice Theorem and excellent morphisms the reader is referred to [28, §6] or [31].

Lemma 4.1. *Let X be a finite dimensional rational G-module.*

(1) *Assume that* $X^G = \{0\}$ *and that* $\mathcal{K}_M(N) = \mathcal{D}(N)\tau_N(\mathfrak{m})$ *for all slice representations* (N, M) *at non-zero closed orbits in* X. *Write* $\mathbf{N}(X) = \varpi_X^{-1}(\varpi_X(0))$ *for the null-cone in* X *and let* $\mathcal{L}(X) = \mathcal{K}_G(X)/\mathcal{D}(X)\tau_X(\mathfrak{g})$. *Then* $\operatorname{Supp}\mathcal{L}(X) \subset \mathbf{N}(X)$.

(2) *Let* $X = E \oplus F$ *be a* G*-stable decomposition of* X *such that* $E \subset X^G$. *Then,* $\mathcal{K}_G(X) = \mathcal{D}(X)\mathcal{K}_G(F)$.

Proof. (1) Observe that $\mathcal{L}(X)$ is a rational G-module; thus $\operatorname{Supp}\mathcal{L}(X)$ is a closed G-stable subset of X. Suppose that there exists $x \in \operatorname{Supp}\mathcal{L}(X)$ with $x \notin \mathbf{N}(X)$. Then, by definition of the null-cone, the unique closed orbit contained in $\overline{G.x} \subset \operatorname{Supp}\mathcal{L}(X)$ is of the form $G.b$ for some $b \neq 0$. Set $M = G^b$ and let (N, M) be the slice representation at the point b. By [18, Proposition 4.4] there exists $\phi \in \mathcal{O}(X)^G$ such that $\phi(b) \neq 0$ and $\mathcal{L}(X)_\phi = 0$. Thus $b \notin \operatorname{Supp}\mathcal{L}(X)$, giving the required contradiction.

(2) This is routine. □

We are now ready to prove Theorem A from the introduction. Thus, in the notation given there, we want to prove the following result.

Theorem 4.2. *Let* $\mathfrak{g}$ *be a reductive Lie algebra and* $(\mathfrak{g}, \mathfrak{k})$ *be a nice symmetric pair. Then* $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.

Proof. By Lemma 4.1 we may assume that $\mathfrak{g}$ is semisimple. We argue by induction on $\dim\mathfrak{g}$, with the case $\dim\mathfrak{g} = 0$ being obvious. (The theorem is also immediate if $\mathfrak{p} = 0$; that is, if $\vartheta = I$.)

Recall that $\mathcal{L} = \mathcal{K}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$. If $0 \neq b \in \mathfrak{p}$ is semisimple, then $\mathfrak{p} = \mathfrak{p}^b \oplus [\mathfrak{k}, b]$ and $(\mathfrak{p}^b, K^b)$ is the slice representation at the point b. Thus, by Lemma 4.1(1) and induction, $\operatorname{Supp}\mathcal{L} \subset \mathbf{N}(\mathfrak{p})$. Since $\mathcal{L}$ is a submodule of $\mathcal{N} = \mathcal{D}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$, we have $\operatorname{Ch}\mathcal{L} \subset \mathcal{C}(\mathfrak{p})$; hence $\operatorname{Ch}\mathcal{L} \subset \mathcal{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathfrak{p})$ and $\mathcal{L}$ is holonomic by Lemma 2.2. As $\mathcal{L}$ is clearly a $(\mathcal{D}(\mathfrak{p}), A(\mathfrak{p}))$-bimodule, Corollary 3.9 yields $\mathcal{L} = 0$. □

As an immediate corollary of Theorem 4.2 one obtains the following generalization of a fundamental result of Harish-Chandra (see [9, Theorem 4] or (1.2)) from the diagonal case to nice symmetric pairs:

Corollary 4.3. *Assume that the symmetric pair* $(\mathfrak{g}, \mathfrak{k})$ *is nice and is the complexification of a real symmetric pair* $(\mathfrak{g}_0, \mathfrak{k}_0)$. *Write* K_0 *for the connected Lie group satisfying* $\operatorname{Lie}(K_0) = \mathfrak{k}_0$. *Let* $U \subset \mathfrak{p}_0$ *be a* K_0*-stable open subset and* T *be a* K_0*-invariant distribution on* U. *Then* $\mathcal{I}(\mathfrak{p}) \cdot T = 0$. □

As observed in Remark 2.7, many of the nice irreducible symmetric pairs are of maximal rank. We give in Proposition 4.6 another sufficient condition to ensure that $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$. This result will be used in the proof of Theorem 6.4 and it also gives a simpler proof of Theorem 4.2 when $(\mathfrak{g}, \mathfrak{k})$ has maximal rank.

Let $x \in \mathfrak{p}$. If $\mathbf{m}_x \subset \mathcal{O}(\mathfrak{p})$ is the maximal ideal corresponding to x, we denote by M_x the localization of an $\mathcal{O}(\mathfrak{p})$-module M with respect to $\mathcal{O}(\mathfrak{p}) \setminus \mathbf{m}_x$. This applies to any $\mathcal{D}(\mathfrak{p})$-module and, when $M = \mathcal{D}(\mathfrak{p})$, $\mathcal{D}(\mathfrak{p})_x$ is the algebra of differential operators on the local ring $\mathcal{O}(\mathfrak{p})_x$.

The Gelfand-Kirillov dimension of a module M, over some $\mathbb{C}$-algebra, is denoted by $\operatorname{GKdim} M$; see [22] for details. Recall, see [23] for example, that if M is a finitely generated module over $\mathcal{D}(\mathfrak{p})$ or $\mathcal{D}(\mathfrak{p})^K$, then $\operatorname{GKdim} M$ is the dimension of the characteristic variety $\operatorname{Ch} M \subset \mathfrak{p} \times \mathfrak{p}$, or $\operatorname{Ch} M \subset (\mathfrak{p} \times \mathfrak{p})/\!\!/K$. The following lemma will be used several times. It is implicit in [4] and proved in [21, Theorem I.(3.2)].

Lemma 4.4. *Let Ξ be a multiplicatively closed set of commuting, regular, locally ad-nilpotent elements of an algebra D. Then, Ξ is an Ore set in D and* $\operatorname{GKdim} M\Xi^{-1} \leqslant \operatorname{GKdim} M$*, for any right D-module M. If M has no Ξ-torsion, then* $\operatorname{GKdim} M\Xi^{-1} = \operatorname{GKdim} M$. □

We remark that, whenever we apply this lemma, D will be (close to being) a ring of differential operators $\mathcal{D}(V)$ with $\Xi \subset \mathcal{O}(V)$, in which case the elements of Ξ will automatically be locally ad-nilpotent and commuting.

We define the set of *generic* elements by $\mathfrak{p}' = \{x \in \mathfrak{p} : x \text{ regular and semisimple}\}$. Recall [11, Proposition III.4.9] that there exists $\zeta \in \mathcal{O}(\mathfrak{p})^K$, called the *discriminant* of $(\mathfrak{g}, \mathfrak{k})$, such that $\mathfrak{p}' = \mathfrak{p}_\zeta = \{x \in \mathfrak{p} : \zeta(x) \neq 0\}$.

Lemma 4.5. *The module $\mathcal{N} = \mathcal{D}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ satisfies* $\operatorname{GKdim} \mathcal{N} = \dim \mathcal{C}(\mathfrak{p}) = n + \ell$.

Proof. Recall that $\operatorname{Ch} \mathcal{N} \subset \mathcal{C}(\mathfrak{p})$ and hence that $\operatorname{GKdim} \mathcal{N} \leqslant \dim \mathcal{C}(\mathfrak{p})$. Let $v \in \mathfrak{p}'$. It is easily seen that there exists a basis $\{\partial_1, \dots, \partial_n\}$ of the $\mathcal{O}(\mathfrak{p})_v$-module $\Theta(\mathfrak{p})_v$ such that $\mathcal{D}(\mathfrak{p})_v \tau(\mathfrak{k}) = \mathcal{K}(\mathfrak{p})_v = \sum_{i=\ell+1}^n \mathcal{D}(\mathfrak{p})_v \partial_i$ (similar results are proved in [19, Lemma 6.7]). Hence, $\mathcal{N}_v = \mathcal{D}(\mathfrak{p})_v \big/ \sum_{i=\ell+1}^n \mathcal{D}(\mathfrak{p})_v \partial_i$, which implies that $\operatorname{GKdim} \mathcal{N}_v = 2n - (n - \ell) = n + \ell = \dim \mathcal{C}(\mathfrak{p})$. However, Lemma 4.4 implies that $\operatorname{GKdim} \mathcal{N}_v \leqslant \operatorname{GKdim} \mathcal{N}$. Thus $\operatorname{GKdim} \mathcal{N} = \dim \mathcal{C}(\mathfrak{p})$. □

Proposition 4.6. *Assume that $\mathcal{C}(\mathfrak{p})$ is irreducible and $\mathbf{q} = \mathbf{a}$, in the notation of Section 3. Then, $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.*

Proof. The hypothesis ensures that $\mathcal{O}(T^*\mathfrak{p})/\mathbf{a}$ is a domain of dimension $n + \ell = \dim \mathcal{C}(\mathfrak{p})$. Since $\mathcal{O}(T^*\mathfrak{p})/\mathbf{b}$ is a factor of $\mathcal{O}(T^*\mathfrak{p})/\mathbf{a}$, Lemma 4.5 shows that $\mathbf{b} = \mathbf{a}$. This implies in particular that $\operatorname{GKdim} \mathcal{N}_\zeta = \operatorname{GKdim}(\mathcal{O}(T^*\mathfrak{p})/\mathbf{b})_\zeta = \dim \mathcal{C}(\mathfrak{p})$. Set $N = \mathcal{D}(\mathfrak{p})/\mathcal{K}(\mathfrak{p})$. Then, $\mathcal{N}_\zeta = N_\zeta$ and therefore $\operatorname{GKdim} N \geqslant \operatorname{GKdim} N_\zeta = \dim \mathcal{C}(\mathfrak{p})$. Now, suppose that $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{p}) \subsetneq \mathcal{K}(\mathfrak{p})$. Then $\operatorname{gr}(\mathcal{L}) \neq 0$ while $\operatorname{gr}(N) = \operatorname{gr}(\mathcal{N})/\operatorname{gr}(\mathcal{L})$. Since $\operatorname{gr}(\mathcal{N}) = \mathcal{O}(T^*\mathfrak{p})/\mathbf{b}$ is a domain, this forces $\operatorname{GKdim} N = \operatorname{GKdim} \operatorname{gr}(N) < \dim \mathcal{C}(\mathfrak{p})$, hence a contradiction. □

Corollary 4.7. ([31, Theorem 9.9]) *Assume that $(\mathfrak{g}, \mathfrak{k})$ has maximal rank. Then, $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.*

Proof. Recall that $(\mathfrak{g}, \mathfrak{k})$ is of maximal rank if and only if the action $(K : \mathfrak{p})$ is locally free; that is, if and only if $\max\{\dim K.x : x \in \mathfrak{p}\} = \dim \mathfrak{k}$. By [27, Theorem 3.2] the hypotheses of Proposition 4.6 are therefore satisfied. □

Remark. It is not difficult to reduce the proof of Theorem 4.2 to the case where the symmetric pair is irreducible. The significance of this is that Corollary 4.7 therefore provides another proof of Theorem 4.2 whenever $(\mathfrak{g}, \mathfrak{k})$ is nice and has no irreducible factors of type **(O), (II), (IV)** ($k = 2$) or **(VI)** (see Remark 2.7). The details are left to the interested reader.

5. Applications

The main aim of this section is to prove Theorem D from the introduction and the basic inductive technique is provided by the Luna Slice Theorem, see [31, Theorem 1.14].

We begin with some technical lemmas which will enable us to apply the Slice Theorem in proving Theorem D. The notation is as in the beginning of §4. Furthermore, if X is an irreducible affine G-variety (G reductive) we set:

$$\mathcal{I}_G(X) = \mathcal{K}_G(X)^G \quad \text{and} \quad R_G(X) = \mathcal{D}(X)^G/\mathcal{I}_G(X)$$

Observe that $R_G(X)$ is a $\mathbb{C}$-algebra and $\mathcal{O}(X)^G \subset R_G(X)$. The next lemma is then easy to prove.

Lemma 5.1. *Let X be a smooth affine G-variety.*

(1) *Let $\phi \in \mathcal{O}(X)^G$. Then, $\mathcal{K}_G(X)_\phi = \mathcal{K}_G(X_\phi)$. If $\mathcal{K}_G(X) = \mathcal{D}(X)\tau_X(\mathfrak{g})$, then $\mathcal{K}_G(X_\phi) = \mathcal{D}(X_\phi)\tau_X(\mathfrak{g})$.*

(2) *Let $U \subset X$ be a ϖ_X-saturated open affine subset and let $b \in U$ such that $G.b$ is closed. There exists $\phi \in \mathcal{O}(X)^G$ such that $b \in X_\phi = U_\phi$. One has $\mathcal{K}_G(U) = \{d \in \mathcal{D}(U) : d(\mathcal{O}(X)^G) = 0\}$.* □

Lemma 5.2. *Let $\varphi : Z \to U$ be an excellent surjective G-morphism of smooth affine G-varieties.*

(1) *$\mathcal{O}(Z)^G \otimes_{\mathcal{O}(U)^G} R_G(U)$ identifies with $R_G(Z)$.*

(2) *$R_G(U)$ is simple if and only if $R_G(Z)$ is simple.*

(3) *Let $\mathcal{F}$ be a finitely generated $R_G(U)$-module and set $\mathcal{G} = R_G(Z) \otimes_{R_G(U)} \mathcal{F}$. Then $\operatorname{GKdim} \mathcal{G} = \operatorname{GKdim} \mathcal{F}$. Conversely if $\mathcal{G}$ is a finitely generated $R_G(Z)$-module, there exists a finitely generated $R_G(U)$-module $\mathcal{F}$ such that $\mathcal{G} = R_G(Z) \otimes_{R_G(U)} \mathcal{F}$.*

Proof. (1) is immediate from the proof of [31, Corollary 4.4].

For simplicity we drop the subscript G in the rest of this proof. Set $A = \mathcal{O}(U)$ and $B = \mathcal{O}(Z)$. Notice that by the faithful flatness of B^G over A^G, $R(U) \hookrightarrow R(Z)$ via $Q \mapsto 1 \otimes_{A^G} Q$. By the definition of $\mathcal{I}(Z)$, $R(Z)$ embeds in $\mathcal{D}(Z/\!\!/G) = \mathcal{D}(B^G)$. It follows that we may endow $R(Z)$ with the filtration by ord, the order of differential operators in $\mathcal{D}(Z/\!\!/G)$.

(2) Suppose that J is a proper, non-zero ideal of $R(Z)$ and pick $0 \neq Q \in J$. If $Q \notin B^G$, there exists $b \in B^G$ such that $[b, Q] \neq 0$ and $\operatorname{ord}[b, Q] < \operatorname{ord} Q$. Thus, by induction, we deduce that $J \cap B^G \neq 0$. Since $A^G \to B^G$ is étale, it follows, for example from [24, Corollary 3.16], that $J \cap A^G \neq 0$. Therefore $J \cap R(U)$ is a proper non-zero ideal of $R(U)$.

Conversely, assume that $R(Z)$ is simple and let $I \subset R(U)$ be a non-zero ideal. As in the previous paragraph, $I \cap A^G \neq 0$. Thus $0 \neq R(U)(I \cap A^G)R(U) \subset I$ and we may as well assume that $I = R(U)(I \cap A^G)R(U)$. Now, we claim that $IR(Z)$ is an ideal of $R(Z)$. Indeed, we have $R(U)IR(Z) \subset IR(Z)$ and

$$\begin{aligned} B^G IR(Z) &= B^G R(U)(I \cap A^G)R(Z) = R(U)B^G(I \cap A^G)R(Z) \\ &= R(U)(I \cap A^G)B^G R(Z) = IR(Z). \end{aligned}$$

Since $R(Z) = R(U)B^G$, this proves the claim.

As $R(Z)$ is simple, this forces $IR(Z) = R(Z) = R(U) \otimes_{A^G} B^G$. But, $IR(Z) = I(R(U) \otimes_{A^G} B^G) = I \otimes_{A^G} B^G$ and, therefore, $I = R(U)$ by faithful flatness.

(3) Recall [31, Proof of Corollary 4.4] that $\mathcal{D}^m(Z)^G = B^G \otimes_{A^G} \mathcal{D}^m(U)^G$, since φ is excellent. Thus, when we endow $R(U)$ and $R(Z)$ with the filtrations induced by $\{\mathcal{D}^m(U)^G\}_m$, respectively $\{\mathcal{D}^m(Z)^G\}_m$, we obtain $\operatorname{gr}(R(Z)) = B^G \otimes_{A^G} \operatorname{gr}(R(U))$.

Let $\mathcal{F}$ be a finitely generated $R(U)$-module. Choose a good filtration $\{\Phi_k \mathcal{F}\}_k$ on $\mathcal{F}$; that is, a filtration such that the associated graded module $\operatorname{gr}_\Phi(\mathcal{F})$ is finitely generated over the affine commutative algebra $\operatorname{gr}(R(U)) \cong \operatorname{gr}(\mathcal{D}(U)^G)/\operatorname{gr}(\mathcal{I}(U))$. Then, it is not difficult to see that $\Phi_k \mathcal{G} = B^G \otimes_{A^G} \Phi_k \mathcal{F}$ defines a good filtration on the $R(Z)$-module $\mathcal{G}$ such that $\operatorname{gr}_\Phi(\mathcal{G}) = B^G \otimes_{A^G} \operatorname{gr}_\Phi(\mathcal{F})$, as a $\operatorname{gr}(R(Z))$-module. Since $A^G \to B^G$ is étale, so is $\operatorname{gr}(R(U)) \to \operatorname{gr}(R(Z)) = B^G \otimes_{A^G} \operatorname{gr}(R(U))$

(base change for étale morphisms). Recall [23] that $\mathrm{GKdim}\,\mathcal{F} = \mathrm{Kdim}\,\mathrm{gr}_\Phi(\mathcal{F})$ and $\mathrm{GKdim}\,\mathcal{G} = \mathrm{Kdim}\,\mathrm{gr}_\Phi(\mathcal{G})$. Now, $\mathrm{gr}_\Phi(\mathcal{G}) = B^G \otimes_{A^G} \mathrm{gr}_\Phi(\mathcal{F})$. Since $A^G \to B^G$ is étale, $\mathrm{Kdim}_{B^G} B^G \otimes_{A^G} \mathrm{gr}_\Phi(\mathcal{F}) = \mathrm{Kdim}_{A^G} \mathrm{gr}_\Phi(\mathcal{F})$ (see, for example, [24, Corollary 3.16]). Thus, $\mathrm{GKdim}\,\mathcal{F} = \mathrm{GKdim}\,\mathcal{G}$.

Conversely, if $\mathcal{G}$ is a finitely generated $R(Z)$-module, write $\mathcal{G} = \sum_{i=1}^s R(Z)v_i$ and set $\mathcal{F} = \sum_{i=1}^s R(U)v_i$. It is then clear that $\mathcal{G} = R(Z) \otimes_{R(U)} \mathcal{F}$. $\square$

The following lemma can be deduced, by standard methods, from [18, Proof of Lemma 4.3] and [31, Corollary 4.5(i)].

Lemma 5.3. *Let M be a reductive subgroup of G, fix a smooth affine M-variety Y and set $Z = G \times^M Y$. Then, $R_M(Y) \cong R_G(Z)$.* $\square$

The three next results will give a proof of Theorem D from the introduction. Recall that $R(\mathfrak{p}) = R_K(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})^K/\mathcal{K}(\mathfrak{p})^K$.

Lemma 5.4. *Let $(\mathfrak{g}, \mathfrak{k})$ be any symmetric pair. Then,* $\mathrm{GKdim}\,R(\mathfrak{p}) = 2\ell$.

Proof. Observe that, since $\mathcal{O}(\mathfrak{p})^K \subset R(\mathfrak{p})$, we can localize $R(\mathfrak{p})$ with respect to any ore set $\Xi \subset \mathcal{O}(\mathfrak{p})^K$ and that, by Lemma 4.4, $\mathrm{GKdim}\,R(\mathfrak{p})\Xi^{-1} = \mathrm{GKdim}\,R(\mathfrak{p})$. Notice also that, when $\mathfrak{p}$ is contained in the centre of $\mathfrak{g}$, K acts trivially on $\mathfrak{p} = \mathfrak{a}$ and $R(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})$. The claim then follows, by induction on $\dim\mathfrak{g}$, from the Slice Theorem [31, Theorem 1.14] and Lemmas 5.2 and 5.3. $\square$

Theorem 5.5. *Let $(\mathfrak{g}, \mathfrak{k})$ be a nice symmetric pair. Then, $R(\mathfrak{p})$ is a simple ring.*

Proof. Set $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$ and let $\mathfrak{z}$ denote the centre of $\mathfrak{g}$. Then Lemma 4.1(2) implies that $R(\mathfrak{p}) = \mathcal{D}(\mathfrak{p} \cap \mathfrak{z}) \otimes_{\mathbb{C}} R(\mathfrak{p}_1)$. It follows that we may restrict to the case when $\mathfrak{g}$ is semisimple. We argue by induction on $\dim\mathfrak{g}$, with the case $\dim\mathfrak{g} = 0$ being obvious.

Let $0 \neq b \in \mathfrak{p}$ be semisimple; thus $(\mathfrak{p}^b, K^b)$ is the slice representation of the K-action on $\mathfrak{p}$ at the point b. Then, by the Slice Theorem, Lemma 5.1 and induction, there exists $\phi \in \mathcal{O}(\mathfrak{p})^K$ such that $\phi(b) \neq 0$ and $R_K(\mathfrak{p})_\phi$ is simple. Let $\bar{J} = J/\mathcal{I}(\mathfrak{p})$ be a non-zero ideal of $R(\mathfrak{p})$, for some $J \subset \mathcal{D}(\mathfrak{p})^K$. Then, since $\bar{J}_\phi \neq 0$ (ϕ is a non-zero divisor in $R(\mathfrak{p})$), we obtain that $\bar{J}_\phi = R_K(\mathfrak{p})_\phi$. In other words, there exists $s \in \mathbb{N}$ such that $\phi^s \in J$.

Now, consider the $\mathcal{D}(\mathfrak{p})$-module $\mathcal{F} = \mathcal{D}(\mathfrak{p})/(\mathcal{D}(\mathfrak{p})J + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))$. Then $\mathcal{F}$ is a compatible K-module and, by the previous paragraph, $\mathrm{Supp}\,\mathcal{F}$ is a closed K-stable subset of $\mathfrak{p}$ which does not contain any non-zero semisimple element. This forces $\mathrm{Supp}\,\mathcal{F} \subset \mathbf{N}(\mathfrak{p})$ and therefore $\mathrm{Ch}\,\mathcal{F} \subset \mathfrak{C}(\mathfrak{p}) \cap (\mathbf{N}(\mathfrak{p}) \times \mathfrak{p})$. Thus, by Lemma 2.2, $\mathcal{F}$ is holonomic, and therefore of finite length as a left $\mathcal{D}(\mathfrak{p})$-module. By Theorem 4.2, $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$, and so $R(\mathfrak{p}) = A(\mathfrak{p})$, in the notation of (3.4). Moreover, as $J \subset \mathcal{D}(\mathfrak{p})^K$, $\mathcal{F}$ is a $(\mathcal{D}(\mathfrak{p}), A(\mathfrak{p}))$-bimodule. Thus, by Corollary 3.9, $\mathcal{F} = 0$ and hence $R(\mathfrak{p})/\bar{J} = \mathcal{F}^K = 0$. $\square$

Theorem 5.6. *Let $(\mathfrak{g}, \mathfrak{k})$ be a nice symmetric pair and let Q be a non-zero finitely generated $R(\mathfrak{p})$-module. Then* $\mathrm{GKdim}\,Q \geqslant \ell$.

Proof. As before, write $\mathfrak{p}_1 = \mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{p}_1 = 0$, then $\mathfrak{p}$ is contained in the centre $\mathfrak{z}$ of $\mathfrak{g}$. Thus, $R(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})$, $\ell = \dim\mathfrak{p}$, and the result is nothing but the Bernstein inequality. We may therefore assume that $\mathfrak{p}_1 \neq 0$. We prove the theorem by induction on $\dim\mathfrak{g}$ and we adopt the notation developed prior to Corollary 3.9.

Observe that $\mathfrak{p}/\!\!/K = \mathfrak{p}_1/\!\!/K_1 \times \mathfrak{z}$ and that the points $\varpi_{\mathfrak{p}}(b)$, where $b \in \mathfrak{p} \setminus \mathfrak{z}$ is semisimple can be identified with the points $(\varpi_{\mathfrak{p}_1}(a), c) \in \mathfrak{p}_1/\!\!/K_1 \times \mathfrak{z}$ such that

$\varpi_{\mathfrak{p}_1}(a) \neq \varpi_{\mathfrak{p}_1}(0)$. Suppose that Q is a non-zero finitely generated $R(\mathfrak{p})$-module with $\operatorname{GKdim} Q < \ell$. Then, using the Slice Theorem, Lemmas 4.4, 5.2 and 5.3, one can prove that the support of the $\mathcal{O}(\mathfrak{p}/\!\!/K)$-module Q is contained in $\{\varpi_{\mathfrak{p}_1}(0)\} \times \mathfrak{z}$. Since $\{\varpi_{\mathfrak{p}_1}(0)\} \times \mathfrak{z}$ is the variety of zeroes of $S_+(\mathfrak{p}_1^*)^{K_1}$, we obtain that, for all $v \in Q$, there exists a power of $S_+(\mathfrak{p}_1^*)^{K_1}$, say F_1, such that $F_1.v = 0$.

Now, recall that the restriction of the form κ to $\mathfrak{p}$ induces a K-equivariant isomorphism $\kappa : \mathfrak{p} \xrightarrow{\sim} \mathfrak{p}^*$ such that $\kappa(\mathfrak{p}_1) = \mathfrak{p}_1^*$. It is well known that one can extend κ to a K-equivariant automorphism $\varkappa : \mathcal{D}(\mathfrak{p}) \xrightarrow{\sim} \mathcal{D}(\mathfrak{p})$ (the Fourier transform, see [19, §7]) which satisfies: $\varkappa(\tau(a)) = \tau(a)$ for $a \in \mathfrak{k}$, $\varkappa(S(\mathfrak{p})^K) = S(\mathfrak{p}^*)^K$ and $\varkappa(S(\mathfrak{p}_1)^{K_1}) = S(\mathfrak{p}_1^*)^{K_1}$. Since $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) = \mathcal{K}(\mathfrak{p})$ by Theorem 4.2, we have $\varkappa(\mathcal{I}(\mathfrak{p})) = \mathcal{I}(\mathfrak{p})$ and $\varkappa$ induces an automorphism of $R(\mathfrak{p})$. Return to the module Q and define the $R(\mathfrak{p})$-module $Q^{\varkappa}$ to be the abelian group Q with action $a * v = \varkappa(a).v$, for $a \in R(\mathfrak{p})$, $v \in Q$. One clearly has $\operatorname{GKdim} Q^{\varkappa} = \operatorname{GKdim} Q < \ell$. Hence each element of $Q^{\varkappa}$ is killed by a power of $S_+(\mathfrak{p}_1^*)^{K_1}$. In other words, each $v \in Q$ is killed by a power, say F_1', of $S_+(\mathfrak{p}_1)^{K_1}$.

Thus, by the last two paragraphs, $R(\mathfrak{p}).v$ is a factor of $M_1 = R(\mathfrak{p})/(R(\mathfrak{p})F_1 + R(\mathfrak{p})F_1')$. Set

$$M = \mathcal{D}(\mathfrak{p})/(\mathcal{D}(\mathfrak{p})F_1 + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})F_1').$$

By Corollary 3.9, $M = 0$. On the other hand, $M^K = M_1$, whence $v = 0$ and $Q = 0$, as desired. □

Corollary 5.7. *Let $(\mathfrak{g}, \mathfrak{k})$ be a nice symmetric pair, $\mathbf{n} \subset S(\mathfrak{p})^K$ be a maximal ideal and $0 \neq \phi \in \mathcal{O}(\mathfrak{p})^K$. Then, $\mathcal{D}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\phi + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})\mathbf{n}$.*

Proof. Set $M = \mathcal{D}(\mathfrak{p})/(\mathcal{D}(\mathfrak{p})\phi + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})\mathbf{n})$. It suffices to prove that $M^K = R(\mathfrak{p})/(R(\mathfrak{p})\phi + R(\mathfrak{p})\mathbf{n}) = 0$, or, by Theorem 5.6, that $\operatorname{GKdim} M^K < \ell$. Recall from §3 that the filtration $\{\mathcal{D}^k(\mathfrak{p})\}_k$ on $\mathcal{D}(\mathfrak{p})$ induces a filtration on $A(\mathfrak{p}) = R(\mathfrak{p})$ such that $\operatorname{Spec} \operatorname{gr}(R(\mathfrak{p})) = \operatorname{Ch} \mathcal{N}/\!\!/K \subset \mathcal{C}(\mathfrak{p})/\!\!/K$. Now, since $\operatorname{gr}(\mathbf{n}) = S_+(\mathfrak{p})^K$, we have

$$\operatorname{Ch} M \subset \mathcal{X} = \mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap (\{\phi = 0\} \times \mathfrak{p})$$

It follows from Proposition 2.1 that $\mathcal{X}/\!\!/K$ identifies with the subvariety $\{\phi = 0\}$ of $\mathfrak{p}/\!\!/K$. Since $\mathfrak{p}/\!\!/K$ is an irreducible variety of dimension ℓ, this implies that $\operatorname{GKdim} M^K = \dim \operatorname{Ch} M/\!\!/K \leqslant \dim \mathcal{X}/\!\!/K < \ell$. □

We now prove Corollary C from the introduction. Let $\mathfrak{g}_0$ be a real reductive Lie algebra and $(\mathfrak{g}_0, \vartheta_0)$ be a symmetric pair with associated decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Let ϑ be the extension of ϑ_0 to $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$; then $(\mathfrak{g}, \vartheta)$ is a symmetric pair and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}_0$ and $\mathfrak{p} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{p}_0$. If U is a subset of $\mathfrak{p}_0$ we denote by U' the set of regular semisimple elements in U; thus $U' = U \cap \mathfrak{p}' = \{u \in U : \zeta(u) \neq 0\}$. Recall that a distribution T on the open subset $U \subset \mathfrak{p}_0$ is locally invariant if $\tau(\mathfrak{k}).T = 0$, and is an eigendistribution if there exists a maximal ideal $\mathbf{n} \subset S(\mathfrak{p})^K$ such that $\mathbf{n}.T = 0$.

Corollary 5.8. *Assume that $(\mathfrak{g}, \mathfrak{k})$ is a nice symmetric pair. Let T be a locally invariant eigendistribution on the open subset $U \subset \mathfrak{p}_0$ such that $\operatorname{Supp} T \subset U \setminus U'$. Then $T = 0$.*

Proof. (We mimic the proof given in [38, p. 798] in the diagonal case.) Let $\mathbf{n} \subset S(\mathfrak{p})^K$ be a maximal ideal such that $\mathbf{n}.T = 0$. Recall [7, Lemma 21] that, for all $u \in U$, there exists $s \in \mathbb{N}$ and an open subset $u \in V \subset U$ such that $\zeta^s.T_{|V} = 0$. Let L be the annihilator in $\mathcal{D}(\mathfrak{p})$ of $T_{|V}$. Since $T_{|V}$ is locally invariant, we see that L contains

$\tau(\mathfrak{k})$, $\mathbf{n}$ and ζ^s. Thus $L = \mathcal{D}(\mathfrak{p})$ by Corollary 5.7 and therefore $T_{|V} = 0$. Hence $T = 0$. $\square$

Remark. Let $0 \neq \phi \in \mathcal{O}(\mathfrak{p})^K$ and set $U_\phi = \{u \in U : \phi(u) \neq 0\}$. Replacing ζ by ϕ in the previous proof shows that there exists no non-zero locally invariant eigendistribution supported on $U \setminus U_\phi$. However, the case $\phi = \zeta$ readily implies the more general result, since any locally invariant eigendistribution is a real analytic function on U', see [33, Theorem 5.3].

Now suppose that $X \subsetneq \mathfrak{p}$ is an algebraic K-subvariety defined by an ideal $0 \neq I \subset \mathcal{O}(\mathfrak{p})$. Since the generic K-orbit in $\mathfrak{p}$ is closed, one easily sees that $I^K \neq 0$. We can therefore also conclude that there is no non-zero locally invariant eigendistribution supported on $U \cap X$.

6. The rank one case

In the case when $(\mathfrak{g}, \mathfrak{k})$ has rank one it is easy to calculate $\mathcal{K}(\mathfrak{p})$ and $R(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})^K/\mathcal{K}(\mathfrak{p})^K$. We do so in this section, since it shows that the main results of this paper can fail for symmetric pairs that are not nice. Curiously, these computations also show how one may be able to modify those theorems so that they hold for all symmetric pairs.

Thus, *throughout this section, we will assume that* $\mathfrak{g}$ *is semisimple and that* $(\mathfrak{g}, \mathfrak{k})$ *is a symmetric pair of rank one*, unless we explicitly say otherwise (the one exception will be in Theorem 6.3 and the comments immediately preceding it).

Recall from §3 that the K-invariant elements $\sum_{i=1}^n \partial_{x_i}^2$, $\sum_{i=1}^n x_i^2$ and $\mathtt{E}+\frac{n}{2}$ generate a Lie algebra $\mathfrak{u} \cong \mathfrak{sl}(2, \mathbb{C})$ inside $\mathcal{D}(\mathfrak{p})$. In order to work with highest weight modules, we slightly change our notation and set

$$e = \tfrac{1}{2} \sum_{i=1}^n \partial_{x_i}^2, \quad f = -\tfrac{1}{2} \sum_{i=1}^n x_i^2, \quad h = -\mathtt{E} - \tfrac{n}{2}.$$

(These elements still satisfy $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.) Recall the definition of the radial component map: The Chevalley isomorphism $S(\mathfrak{p}^*)^K \xrightarrow{\sim} S(\mathfrak{a}^*)^W$ induces an isomorphism $\mathcal{D}(\mathfrak{p}/\!\!/K) \xrightarrow{\sim} \mathcal{D}(\mathfrak{a}/W)$; by composing this isomorphism with the natural restriction map $\mathcal{D}(\mathfrak{p})^K \to \mathcal{D}(\mathfrak{p}/\!\!/K)$, we obtain the radial component map $rad : \mathcal{D}(\mathfrak{p})^K \longrightarrow \mathcal{D}(\mathfrak{a}/W)$. Note that $\operatorname{Ker} rad = \mathcal{I}(\mathfrak{p})$ and $R(\mathfrak{p})$ is isomorphic to $R = \operatorname{Im} rad$. Set

$$\bar{e} = rad(e), \quad \bar{f} = rad(f), \quad \bar{h} = rad(h).$$

Choose the coordinate system $\{x_i, \partial_{x_i}\}_i$ of $\mathcal{D}(\mathfrak{p})$ such that $t = x_1$ is the coordinate function on $\mathfrak{a}$, and $x_2, \ldots, x_n$ are coordinate functions on a κ-orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Then the Weyl group $W = \{\pm 1\}$ acts on $S(\mathfrak{a}^*) = \mathbb{C}[t]$ by $t \mapsto -t$ and $S(\mathfrak{a}^*)^W = \mathbb{C}[z]$, where $z = t^2$. Furthermore, $rad(\sum_{i=1}^n x_i^2) = z$, $rad(\sum_{i=1}^n \partial_{x_i}^2) = 4z\partial_z^2 + 2n\partial_z$, and $rad(\mathtt{E}) = 2z\partial_z$. Hence,

$$\bar{e} = 2z\partial_z^2 + n\partial_z, \quad \bar{f} = -\tfrac{1}{2}z, \quad \bar{h} = -2z\partial_z - \tfrac{n}{2}. \tag{6.1}$$

Lemma 6.1. (1) *Set* $U = \mathbb{C}\langle \bar{e}, \bar{f}, \bar{h}\rangle$. *Then,* $R = U \cong U(\mathfrak{sl}(2, \mathbb{C}))/\big(\omega - \frac{1}{4}n(n-4)\big)$, *where* $\omega = h^2 + 2h + 4fe$ *is the Casimir element of the enveloping algebra* $U(\mathfrak{sl}(2, \mathbb{C}))$.

(2) *Assume that* $n = 2r \geqslant 4$. *Then,* $V = \mathbb{C}[z]z^{1-r}/\mathbb{C}[z]$ *is a finite dimensional simple* R*-module. Indeed,* $V \cong R/(R\bar{f}^{r-1} + R(\bar{h} - r + 2) + R\bar{e})$.

Proof. Since U is a factor of $U(\mathfrak{sl}(2, \mathbb{C}))$, the isomorphism in part (1) follows by computing the character of the U-module $\mathbb{C}[z]$. By construction, $U \subset R$. The equality $U = R$ then follows easily from the fact that, by (6.1), $U[\bar{f}^{-1}] = R[\bar{f}^{-1}]$

and, dually, $U[\bar{e}^{-1}] = R[\bar{e}^{-1}]$. This proves (1), while part (2) follows by the obvious computation. $\square$

Recall that, when $n \geqslant 4$, the rank one symmetric pair is not nice (see Remark 2.4). The following result shows that Theorem 3.8 fails for such a pair and, consequently, that the proof of Theorem 4.2 will also not work in this situation. See, however, the first remark following Theorem 3.8. This result is related to the computation of the "minimal extension" $\mathcal{L}(\mathbf{N}(\mathfrak{p}), \mathfrak{p})$, see [25, pp. 220-221].

Proposition 6.2. *Assume that* $n = 2r \geqslant 4$ *and set*

$$L = \mathcal{D}(\mathfrak{p})f^{r-1} + \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})e.$$

Then, $M = \mathcal{D}(\mathfrak{p})/L$ *is a non-zero module that satisfies* (3.3).

Proof. Here, $\mathbf{N}(\mathfrak{p})$ is the variety of zeroes, in $\mathfrak{p}$, of the polynomial f and $S(\mathfrak{p})^K = \mathbb{C}[e]$. Thus M does have the form of (3.3). Consider the R-module V of Lemma 6.1(2) as an $A(\mathfrak{p})$-module. By the final assertion of that lemma, V is a factor of $V' = A(\mathfrak{p})/(A(\mathfrak{p})f^{r-1} + A(\mathfrak{p})e)$. Recall that $\mathcal{N} = \mathcal{D}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$. Thus $M = \mathcal{N} \otimes_{A(\mathfrak{p})} V' = \mathcal{D}(\mathfrak{p})/L$ and, since $M^K = V' \neq 0$, we have $M \neq 0$. $\square$

In order to describe the left ideal $\mathcal{K}(\mathfrak{p})$ in the rank one case, we need to introduce a Lie subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathfrak{p})$. Since the definition can be made in the general case, we return to the situation where $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair of arbitrary rank. Recall that any $a \in \operatorname{End}_{\mathbb{C}}(\mathfrak{p})$ defines a vector field $\tilde{\tau}(a) \in \Theta(\mathfrak{p})$ by the formula $(\tilde{\tau}(a).\varphi)(v) = \frac{d}{dt}_{|t=0}\varphi(v - ta.v)$, for all $\varphi \in \mathcal{O}(\mathfrak{p})$, $v \in \mathfrak{p}$. Since $\tau(a) = \tilde{\tau}(\operatorname{ad}(a))$ for all $a \in \mathfrak{k}$ we may, without confusion, write $\tau(a) = \tilde{\tau}(a)$ for $a \in \operatorname{End}_{\mathbb{C}}(\mathfrak{p})$. Following [16, §4], we define a Lie subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathfrak{p})$ by

$$\tilde{\mathfrak{k}} = \left\{a \in \operatorname{End}_{\mathbb{C}}(\mathfrak{p}) : \tau(a).\varphi = 0 \text{ for all } \varphi \in \mathcal{O}(\mathfrak{p})^K\right\}. \tag{6.2}$$

We want to study the inclusions $\operatorname{ad}(\mathfrak{k}) \subset \tilde{\mathfrak{k}}$ and $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) \subset \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}}) \subset \mathcal{K}(\mathfrak{p})$. Obviously, we only need to work in the case where $\mathfrak{g}$ is semisimple, which we now assume. In the rank one case, $\mathcal{O}(\mathfrak{p})^K = \mathbb{C}[f]$, where f is the quadratic form $\sum_i x_i^2$, and the Lie algebra $\tilde{\mathfrak{k}}$ is isomorphic to $\mathfrak{so}(n)$. (We drop the base field $\mathbb{C}$ in the sequel.) We have the the following general result, which depends upon a case by case analysis.

Theorem 6.3. ([20]) *Assume that* $\mathfrak{g}$ *is a semisimple Lie algebra and let* $(\mathfrak{g}, \mathfrak{k})$ *be a symmetric pair (of any rank). Then,* $\operatorname{ad}(\mathfrak{k}) = \tilde{\mathfrak{k}}$ *if and only if each irreducible, rank one factor of* $(\mathfrak{g}, \mathfrak{k})$ *is of type* $(\mathfrak{so}(n+1), \mathfrak{so}(n))$. $\square$

This implies that, if $(\mathfrak{g}, \mathfrak{k})$ is irreducible of rank bigger than one, then $\tilde{\mathfrak{k}} = \operatorname{ad}(\mathfrak{k})$. Conversely, if $\ell = 1$, then $\operatorname{ad}(\mathfrak{k}) \neq \tilde{\mathfrak{k}}$ except when $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}(n+1), \mathfrak{so}(n))$.

We now look to the inclusion $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) \subset \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$ when $(\mathfrak{g}, \mathfrak{k})$ is of rank one. In the notation of [10, Chap. X, Table V], there are four possible cases:

(BDI) $(\mathfrak{so}(n+1), \mathfrak{so}(n))$, $n \geqslant 2$
(AIII) $(\mathfrak{sl}(m+1), \mathfrak{gl}(m))$, $m \geqslant 2$
(CII) $(\mathfrak{sp}(m+1), \mathfrak{sp}(m) \oplus \mathfrak{sp}(1))$, $m \geqslant 1$
(FII) $(\mathfrak{f}_4, \mathfrak{so}(9))$

For the first three cases, the relationship between $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ and $\mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$ has been determined by H. Ochiai [26]: if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(3), \mathfrak{gl}(2))$, then $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) \subsetneq \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$, but otherwise $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) = \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$. In contrast, there is no exceptional case in the description of $\mathcal{K}(\mathfrak{p})$:

Theorem 6.4. *Let* $(\mathfrak{g}, \mathfrak{k})$ *be any symmetric pair of rank one with* $\mathfrak{g}$ *semisimple. Then,* $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$ *and, consequently,* $\mathcal{I}(\mathfrak{p}) = (\mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}}))^K$.

Proof. Suppose, first, that $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}(n+1), \mathfrak{so}(n))$. Then $\tilde{\mathfrak{k}} = \mathrm{ad}(\mathfrak{k}) \cong \mathfrak{so}(n)$ and $\mathfrak{p} = \mathbb{C}^n$ with the standard $\mathrm{SO}(n)$-action. In the coordinate system $\{x_i, \partial_{x_i}\}_i$ the Lie algebra $\tau(\mathfrak{k})$ has basis $\{\tau(e_{ij}) : 1 \leqslant i < j \leqslant n\}$ where $\tau(e_{ij}) = x_i\partial_{x_j} - x_j\partial_{x_i}$. Let $y_j = \sigma(\partial_{x_j}) \in \mathcal{O}(T^*\mathfrak{p})$ be the principal symbol of ∂_{x_j}. Then, $\mathcal{O}(T^*\mathfrak{p}) = \mathbb{C}[x_i, y_i; 1 \leqslant i \leqslant n]$ and the ideal generated by the $\sigma(a)$, $a \in \mathfrak{k}$, is $\mathbf{a} = (x_iy_j - x_jy_i; 1 \leqslant i < j \leqslant n)$. It is easy to see that $\mathcal{C}(\mathfrak{p}) = \mathcal{V}(\mathbf{a})$. Moreover, it is well known that $\mathbf{a}$ is a prime ideal (for instance, by classical invariant theory [39], $\mathbb{C}[x_i, y_j]/\mathbf{a} \cong \mathbb{C}[u_1, u_2; v_i, 1 \leqslant i \leqslant n]^{\mathbb{C}^*}$, where $\mathbb{C}^*$ acts by $\lambda.u_i = \lambda u_i$, $\lambda.v_i = \lambda^{-1}v_i$). It then follows from Proposition 4.6 that $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$.

Now consider the general case. Notice that the $\tilde{\mathfrak{k}}$-module $\mathfrak{p}$ identifies with the standard $\mathfrak{so}(n)$-module $\mathbb{C}^n$. Consider the symmetric pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}) = (\mathfrak{so}(n+1), \mathfrak{so}(n))$ where $\tilde{\mathfrak{p}} = \mathbb{C}^n$ with standard action of $\tilde{K} = \mathrm{SO}(n)$. We may identify the $\tilde{\mathfrak{k}}$-modules $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ and we have $\mathcal{O}(\mathfrak{p})^K = \mathcal{O}(\mathfrak{p})^{\tilde{K}} = \mathcal{O}(\tilde{\mathfrak{p}})^{\tilde{K}} = \mathbb{C}[f]$. Therefore, since $\mathcal{D}(\mathfrak{p}) = \mathcal{D}(\tilde{\mathfrak{p}})$,

$$\begin{aligned}\mathcal{K}_K(\mathfrak{p}) &= \{d \in \mathcal{D}(\tilde{\mathfrak{p}}) : d(\mathcal{O}(\tilde{\mathfrak{p}})^{\tilde{K}}) = 0\} \\ &= \mathcal{K}_{\tilde{K}}(\tilde{\mathfrak{p}}) = \mathcal{D}(\tilde{\mathfrak{p}})\tau(\tilde{\mathfrak{k}}) \quad \text{(by the first case)} \\ &= \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}}).\end{aligned}$$

Hence the result. □

Corollary 6.5. *Suppose that* $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(3), \mathfrak{gl}(2))$. *Then,* $\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k})$ *is strictly contained in* $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$. *In contrast,* $\mathcal{I}(\mathfrak{p}) = (\mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}}))^K = (\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))^K$.

Proof. The first assertion follows from Theorem 6.4 and Ochiai's result [26]. The final claim can be proved using [26, Theorem 2]. □

7. Questions

Theorem 4.2 and Theorem 6.4 provide the only known cases where the ideal $\mathcal{K}(\mathfrak{p})$ has been described. We have seen that in these cases $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$. Recall from §3 and Lemma 4.5 that $\mathrm{Ch}\,\mathcal{N} \subset \mathcal{C}(\mathfrak{p})$ with $\dim \mathrm{Ch}\,\mathcal{N} = \dim \mathcal{C}(\mathfrak{p})$. Set $\tilde{\mathcal{N}} = \mathcal{D}(\mathfrak{p})/\mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$. We then have $\mathrm{Ch}\,\tilde{\mathcal{N}} \subset \mathrm{Ch}\,\mathcal{N} \subset \mathcal{C}(\mathfrak{p})$ and $\dim \mathrm{Ch}\,\tilde{\mathcal{N}} = \dim \mathcal{C}(\mathfrak{p})$. In view of these observations, and Corollary 6.5, we make the following conjectures.

Conjectures. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair (with $\mathfrak{g}$ reductive) and let $\tilde{\mathfrak{k}}$ be the Lie algebra defined in (6.2). Then:

(C.1) $\mathcal{K}(\mathfrak{p}) = \mathcal{D}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$

(C.2) $\mathcal{I}(\mathfrak{p}) = (\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}))^K$

(C.3) $\mathrm{Ch}\,\tilde{\mathcal{N}} = \mathcal{R} = \overline{K.(\mathfrak{a} \times \mathfrak{a})}$; thus $\mathrm{Ch}\,\tilde{\mathcal{N}}$ is an irreducible component of $\mathcal{C}(\mathfrak{p})$.

Let $\lambda \in \mathfrak{a}^*$ and denote by $\mathbf{n}_\lambda \subset S(\mathfrak{p})^K$ the maximal ideal $\big(q - q(\lambda); q \in S(\mathfrak{p})^K\big)$. Set $\mathcal{N}_\lambda = \mathcal{D}(\mathfrak{p})\big/\big(\mathcal{D}(\mathfrak{p})\tau(\mathfrak{k}) + \mathcal{D}(\mathfrak{p})\mathbf{n}_\lambda\big)$.

Conjectures. Assume that the pair $(\mathfrak{g}, \mathfrak{k})$ is nice. Then:

(C.4) $\mathcal{N}_\lambda$ is a semisimple $\mathcal{D}(\mathfrak{p})$-module

(C.5) $\mathcal{N}_\lambda$ has no ζ-torsion.

By [19], conjectures (C.4) and (C.5) are true in the diagonal case. One significance of (C.5) is that, if true, it ought to provide a method for proving [33, Conjecture 7.2].

The question of describing $\mathcal{K}(\mathfrak{p})$ is a particular case of the more general problem of determining the left ideals $\mathcal{K}_K(V)$ (cf. §4) where $(K : V)$ is any finite dimensional representation of a reductive group K, see [30, 31]. In this general setting one can still define, as in (6.2), a Lie subalgebra

$$\tilde{\mathfrak{k}} = \left\{a \in \operatorname{End}_{\mathbb{C}}(V) : a.\varphi = 0 \text{ for all } \varphi \in \mathcal{O}(V)^K\right\}$$

and one again has $\mathcal{D}(V)\tau_V(\mathfrak{k}) \subset \mathcal{D}(V)\tau_V(\tilde{\mathfrak{k}}) \subset \mathcal{K}_K(V)$. The representations $(K : \mathfrak{p})$ are particular cases of the so-called θ-groups [37] or, more generally, of orthogonal polar representations, see [5] and [28, 8.6] (in these cases one still has a Cartan subspace). One may generalize the previous conjectures to this wider class of representations and the following questions arise:

Questions. Suppose that $(K : V)$ is a θ-group, or an orthogonal polar representation.

(Q.1) Is $\mathcal{K}_K(V) = \mathcal{D}(V)\tau_V(\tilde{\mathfrak{k}})$?
(Q.2) Is $\mathcal{I}_K(V) = (\mathcal{D}(V)\tau_V(\mathfrak{k}))^K$?
(Q.3) Do we have $\operatorname{Ch}\mathcal{D}(V)/\mathcal{D}(V)\tau_V(\tilde{\mathfrak{k}}) = \overline{K.(\mathfrak{c}\times\mathfrak{c})}$, where $\mathfrak{c} \subset V$ is a Cartan subspace?

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