Modules over the 4-dimensional Sklyanin Algebra

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RÉSUMÉ. *Cet article étudie les 'point modules' et 'line modules' sur l'algèbre définie par E.K. Sklyanin dans [17]. Ces modules sont précisément les modules de Cohen-Macaulay de multiplicité 1 et dimension de Gelfand-Kirillov 1 et 2 respectivement. Il a été démontré en [21] que les 'point modules' sont en bijection avec les points d'une courbe elliptique E dans $\mathbb{P}^3$ augmentée de 4 autres points. On prouve ici que les 'line modules' sont en bijection avec les droites sécantes de E. On montre que d'autres propriétés algébriques de ces modules sont conséquences et/ou analogues de propriétés géométriques de E et des 4 points. Par exemple, si deux droites non concourantes sont sur une quadrique lisse contenant E, alors les deux modules correspondant ont le même annulateur. On démontre également que l'algèbre de Sklyanin peut être définie à l'aide des formes bilinéaires s'annulant sur une certaine sous-variété de $\mathbb{P}^3 \times \mathbb{P}^3$.*

ABSTRACT. *This paper studies point modules and line modules over the algebra defined by E.K. Sklyanin in [17]. It was proved in [21] that the point modules are in bijection with the points of an elliptic curve E in $\mathbb{P}^3$ together with 4 other points. Here it is proved that the line modules are in bijection with the lines in $\mathbb{P}^3$ which are secant lines to E. The point and line modules are precisely the Cohen-Macaulay modules of multiplicity 1, and Gelfand-Kirillov dimension 1 and 2 respectively. Further algebraic properties of these modules are shown to be consequences and analogues of the geometric properties of the elliptic curve and the 4 points. For example, if two lines lie on a smooth quadric containing E, and they do not intersect, then the two corresponding line modules have the same annihilator. It is also shown that the Sklyanin algebra may be defined in terms of the bilinear forms vanishing on a certain subvariety of $\mathbb{P}^3 \times \mathbb{P}^3$.*

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Introduction.

The 4-dimensional Sklyanin algebra is the graded algebra $A = \mathbb{C}[x_0, x_1, x_2, x_3]$ defined by the six relations:

$$x_0x_i - x_ix_0 = \alpha_i(x_jx_k + x_kx_j)$$
$$x_0x_i + x_ix_0 = x_jx_k - x_kx_j$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ lies on the surface $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. (One also excludes a certain finite set of points on this surface – see §1 for details). This two parameter family of algebras was defined and first studied by E.K. Sklyanin in 1982 [17]. Although the above succinct description of A suffices for the purposes of this introduction, an alternative "better" definition (which explains the restriction on the α_i) is given at the start of Section 1. There A will be defined in terms of an elliptic curve $\mathbb{C}/\Lambda$, and a point $\tau \in \mathbb{C}/\Lambda$ which is not of order 4.

Sklyanin's initial study of A showed (among other things) that A has various finite dimensional representations in spaces of theta functions, and that A has two homogeneous central elements of degree two [17], [18]. Further algebraic properties of A were obtained in [21]. It was proved there that A is a noetherian domain, and has the same Hilbert series as the polynomial ring in 4 variables, namely $(1-t)^{-4}$. Furthermore, A is a Koszul algebra of global homological dimension 4, and is regular in the sense of Artin and Schelter [1]. The methods in [21] follow closely those in [2] and [3]. Further homological properties of A were established in [11]; in particular A is Auslander-regular.

The purpose of this paper is to begin a study of the representation theory of A. All the results mentioned in the previous paragraph will play a key role in our analysis. Following the ideas in [2] and [3], we study three classes of A-modules: point modules, line modules and plane modules. These are defined to be cyclic modules with Hilbert series $(1 - t)^{-n}$ where $n = 1, 2, 3$ respectively. Thus the Hilbert series of a line module is the same as that of the homogeneous coordinate ring of the projective line $\mathbb{P}^1$. The point modules were classified in [21]. They are in bijection with the points of a subvariety $E \cup \mathcal{S}$ of $\mathbb{P}^3$, where E is a smooth elliptic curve, and $\mathcal{S}$ consists of 4 more points. If $p \in E \cup \mathcal{S}$, we write $M(p)$ for the corresponding point module. It is rather easy to see that plane modules are in bijection with the hyperplanes in $\mathbb{P}^3$. Thus our main interest is in line modules.

We will prove that the line modules are in bijection with the set of lines in $\mathbb{P}^3$ which are secant lines of E (we note that E has no trisecants). If $p, q \in E$, then $M(p, q)$ denotes the corresponding line module. Both $M(p)$ and $M(q)$ are quotients of $M(p, q)$, and the kernel of each of these surjections is again a line module. The line modules can also be characterized by their homological properties. They are precisely the Cohen-Macaulay modules of projective dimension 2, and multiplicity 1 (see §1 for definitions).

As in [21], the proof of these results is closely related to the geometry of E. The algebra A determines not just $E \cup \mathcal{S}$ (as the space parametrizing the point modules), but also an automorphism σ of $E \cup \mathcal{S}$. It is shown in [21] that $\sigma|_{\mathcal{S}} = \mathrm{Id}$ and $\sigma|_E$ is translation $p \mapsto p + \tau$ by a certain point $\tau \in E$.

For this paragraph, suppose that τ is of infinite order. Then the center of A is a polynomial ring $\mathbb{C}[\Omega_1, \Omega_2]$ in the two central elements found by Sklyanin. The annihilator of $M(p,q)$ is generated by a non-zero element $\Omega \in \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$. Moreover, up to scalar multiples, Ω depends only on $p+q$, so we write $\mathrm{Ann}\, M(p,q) = \langle \Omega(p+q) \rangle$. Furthermore, if $r, s \in E$, then $\mathbb{C}\Omega(r) = \mathbb{C}\Omega(s)$ if and only if either $r = s$ or $r+s = -2\tau$. As an indication of the parallels between the algebraic properties of A and the geometric properties of $E \subset \mathbb{P}^3$ we will prove that, if $z \in E$ is fixed, then all the secant lines through p and $z-p$ lie on a common quadric containing E, and all the line modules $M(p, z-p)$ have a common annihilator.

The results are presented as follows. Section 1 begins with a definition of the Sklyanin algebra, and shows that the defining relations have a succinct geometric description. This way of viewing the relations, and the degree two central elements of A, will be extremely useful for us. Section 1 also contains background material on homological properties (e.g. the Auslander condition) and Hilbert series of graded algebras. Section 2 gives a homological classification of point, line and plane modules: they are precisely the Cohen-Macaulay modules of multiplicity 1. Section 3 examines the geometry of the secant lines of E, and the quadrics on which they lie. Section 4 proves that the line modules are in bijection with the secant lines of E. Section 5 examines point modules, their finite dimensional simple quotients, and their relationship to line modules. For example, if $p, q \in E$ then there is a short exact sequence $0 \to M(p+\tau, q-\tau) \to M(p,q) \to M(q) \to 0$. Section 6 describes the annihilators of the line modules.

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§1. Preliminaries.

1.1. The defining equations of the Sklyanin Algebra.

We now define the Sklyanin algebra in a way which will be more useful than that given in the introduction. From our point of view (1.2b) below is the best way to define the Sklyanin algebra.

Fix, once and for all $\eta \in \mathbb{C}$ with $Im(\eta) > 0$, and write $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\eta$. Let $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ be Jacobi's four theta functions associated to Λ, as defined in Weber's book [24, p.71]. In particular,

$$\theta_{ab}(z+1) = (-1)^a \theta_{ab}(z), \quad \theta_{ab}(z+\eta) = \exp(-\pi i \eta - 2\pi i z - \pi i b)\theta_{ab}(z)$$

and the zeroes of θ_{ab} are at the points $(\frac{1+b}{2}) + (\frac{1+a}{2})\eta + \Lambda$. Furthermore θ_{11} is an odd function, and the other θ_{ab} are even functions.

Fix, once and for all $\tau \in \mathbb{C}$, such that τ is not of order 4 in $\mathbb{C}/\Lambda$. Whenever $\{ab,\, ij,\, kl\} = \{00,\, 01,\, 10\}$, define

$$\alpha_{ab} = (-1)^{a+b} \left[\frac{\theta_{11}(\tau)\theta_{ab}(\tau)}{\theta_{ij}(\tau)\theta_{k\ell}(\tau)} \right]^2 ,$$

and set $\alpha_1 = \alpha_{00}, \alpha_2 = \alpha_{01}, \alpha_3 = \alpha_{10}$. It is not difficult to show that these satisfy $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$, and $\{\alpha_1, \alpha_2, \alpha_3\} \cap \{-1, 0, 1\} = \emptyset$.

Let V be a 4-dimensional vector space with basis x_0, x_1, x_2, x_3. Define A to be the quotient of the tensor algebra $T(V)$, with defining relations as in the introduction. Thus $A = T(V)/I$ where I is the graded ideal generated by its six dimensional subspace of degree two elements, namely $I_2 \subset V \otimes V$. There is another ideal in $T(V)$ which is important for us. In [17] Sklyanin found two central elements in A_2; see also [21, §3.9]. It will be convenient for us to take these central elements to be any two of the following:

$$\begin{aligned}
\Omega_0 &= (1+\alpha_3)x_1^2 + (1+\alpha_1\alpha_3)x_2^2 + (1-\alpha_1)x_3^2 \\
\Omega_1 &= (1+\alpha_3)x_0^2 + (\alpha_1\alpha_3 - \alpha_3)x_2^2 - (\alpha_1+\alpha_3)x_3^2 \\
\Omega_2 &= (1+\alpha_1\alpha_3)x_0^2 + (\alpha_3 - \alpha_1\alpha_3)x_1^2 - (\alpha_1+\alpha_1\alpha_3)x_3^2 \\
\Omega_3 &= (1-\alpha_1)x_0^2 + (\alpha_1+\alpha_3)x_1^2 + (\alpha_1+\alpha_1\alpha_3)x_2^2.
\end{aligned}$$

(The hypothesis that $\tau \notin E_4$ ensures that none of the coefficients of the x_i^2 is zero.) We will write Z_2 for the two dimensional space spanned by these elements, and define $B := A/\langle Z_2 \rangle$ to be the algebra obtained by quotienting out these elements. Define J to be the kernel of the map $T(V) \to B$. Thus J is generated by its 8-dimensional subspace $J_2 \supset I_2$.

For each $ab \in \{00,\, 01,\, 10,\, 11\}$ define

$$g_{ab}(z) = \gamma_{ab}\theta_{ab}(\tau)\theta_{ab}(2z) \quad \text{where} \quad \gamma_{ab} = \begin{cases} i = \sqrt{-1} & ab = 00, 11 \\ 1 & ab = 01, 10 \end{cases}$$

Define $j_\tau : \mathbb{C}/\Lambda \to \mathbb{P}(V^*) = \mathbb{P}^3$ by

$$j_\tau(z) = (g_{11}(z), g_{00}(z), g_{01}(z), g_{10}(z))$$

with respect to the homogeneous coordinates x_0, x_1, x_2, x_3. Write $E = j_\tau(\mathbb{C}/\Lambda)$. It follows from [21, §2.4] that E is defined by any two of the following quadratic forms:

$$\begin{aligned}
g_0 &= (1+\alpha_2\alpha_3)x_1^2 + (1+\alpha_3)x_2^2 + (1-\alpha_2)x_3^2 \\
g_1 &= (1+\alpha_2\alpha_3)x_0^2 + (\alpha_2\alpha_3-\alpha_3)x_2^2 + (\alpha_2\alpha_3+\alpha_2)x_3^2 \\
g_2 &= (1+\alpha_3)x_0^2 + (\alpha_3-\alpha_2\alpha_3)x_1^2 + (\alpha_2+\alpha_3)x_3^2 \\
g_3 &= (\alpha_2-1)x_0^2 + (\alpha_2+\alpha_2\alpha_3)x_1^2 + (\alpha_2+\alpha_3)x_2^2.
\end{aligned}$$

(The hypothesis that $\tau \notin E_4$ ensures that none of the coefficients of the x_i^2 is zero.)

We will show that the defining relations of A have a succinct description in terms of the geometry of E. Consider V as linear forms on $\mathbb{P}(V^*)$, and $V \otimes V$ as bi-homogeneous forms on $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$. We define the following subvarieties:

$$e_0 = (1,0,0,0), \quad e_1 = (0,1,0,0), \quad e_2 = (0,0,1,0), \quad e_3 = (0,0,0,1)$$

$$\begin{aligned}
\mathcal{S} &= \{e_i \mid 0 \le i \le 3\} \\
\Delta_{\mathcal{S}} &= \{(e_i, e_i) \mid 0 \le i \le 3\} \\
\Delta_\tau &= \{(p, p+\tau) \mid p \in E\} \\
\Gamma &= \Delta_{\mathcal{S}} \cup \Delta_\tau.
\end{aligned}$$

Thus Γ is the graph of the automorphism σ of $E \cup \mathcal{S}$ which was introduced in [21, §2.8]. Recall that $\sigma(p) = p + \tau$ for $p \in E$, and $\sigma(e_i) = e_i$ for $i = 0,1,2,3$. The following is one of the main results in [21, §§2,3].

Theorem 1.1. The subvarieties of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ defined by I_2 and J_2 are $\mathcal{V}(I_2) = \Gamma$, and $\mathcal{V}(J_2) = \Delta_\tau$.

Theorem 1.2.

(a) I_2 is the subspace of J_2 consisting of those forms which vanish on $\Delta_{\mathcal{S}}$. Thus I_2 is the subspace of $V \otimes V$ consisting of those forms which vanish on $\Gamma = \Delta_\tau \cup \Delta_{\mathcal{S}}$.

(b) If $f \in J_2$ vanishes at two points of $\Delta_{\mathcal{S}}$ then $f \in I_2$.

Proof. (a) Since I_2 vanishes on Γ, we need to show that if $f \in J_2$ vanishes on $\Delta_{\mathcal{S}}$, then $f \in I_2$. Let $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ be preimages of the central elements Ω_1 and Ω_2 in $V \otimes V$. Thus $J_2 = I_2 \oplus \mathbb{C}\tilde{\Omega}_1 \oplus \mathbb{C}\tilde{\Omega}_2$. Since $\tilde{\Omega}_1, \tilde{\Omega}_2 \in \bigoplus_{0 \le i \le 3} \mathbb{C}(x_i \otimes x_i)$ never vanish at all the points e_i, if $f \in J_2$ and $f(\Delta_{\mathcal{S}}) = 0$, then $f \in I_2$.

(b) The result follows from the fact that a non-zero element in $\mathbb{C}\tilde{\Omega}_1 \oplus \mathbb{C}\tilde{\Omega}_2$ cannot be a linear combination of just two of the $x_i \otimes x_i$. ■

Remarks. 1. In Section 3 it will be proved that the points $e_i \in \mathcal{S}$ are the only points in $\mathbb{P}^3$ which lie on infinitely many secant lines of E. Thus $\mathcal{S}$ is determined by E, and hence $\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$ is determined by E and τ. Therefore (1.2a) gives a succinct geometric description of the defining relations of A.

2. We shall use (1.2) as a way of recognizing non-zero central elements in A_2. If $f \in V \otimes V$ vanishes on Δ_τ but not on $\Delta_{\mathcal{S}}$, then the image of f in A is a non-zero central element. Furthermore if two elements of J_2 vanish at a common point $(e_i, e_i) \in \Delta_{\mathcal{S}}$, then the elements are scalar multiples of one another.

1.2. Homological properties of the Sklyanin Algebra.

Let k be a field. For the sake of simplicity the algebras considered in this section will be noetherian graded k-algebras of the form $A = k \oplus A_1 \oplus A_2 \oplus \ldots$. The dimension of a k-vector space E is denoted by $dimE$.

We denote by $\mathrm{Mod}(A)$, respectively $\mathrm{Mod}^g(A)$, the category of left or right A-modules, respectively $\mathbb{Z}$-graded A-modules. The subcategories of finitely generated A-modules will be denoted by $\mathrm{Mod}_f(A)$ and $\mathrm{Mod}_f^g(A)$, respectively. Let $M = \bigoplus_m M_m$ be in $\mathrm{Mod}^g(A)$ and $p \in \mathbb{Z}$. The *shifted module* $M[p]$ is defined by setting $M[p]_m := M_{p+m}$. If M and N are in $\mathrm{Mod}^g(A)$, then $HOM_A(M,N)$ denotes the $\mathbb{Z}$-graded group such that $HOM_A(M,N)_p = \{\phi : M \to N \mid \phi \text{ is } A\text{-linear and } \phi(M_m) \subset M_{m+p} \text{ for all } m\}$. It is well know that if $M \in \mathrm{Mod}_f^g(A)$, then $HOM_A(M,N)$ coincides with the usual $\mathrm{Hom}_A(M,N)$. In that case the derived functors of $HOM_A(M,-)$ and $\mathrm{Hom}_A(M,-)$ are the same, namely $\mathrm{Ext}_A^q(M,-)$.

When not otherwise specified a map $M \xrightarrow{\phi} N$ between modules of $\mathrm{Mod}^g(A)$ will be an element of $HOM_A(M,N)_0$, i.e. $\phi(M_m) \subset N_m$ for all m.

The projective dimension of $M \in \mathrm{Mod}(A)$ is denoted by $pd(M)$. The algebra A is said to have *finite global homological dimension* if $d = \sup\{pd(M) \mid M \in \mathrm{Mod}(A)\}$ is finite. In this case we write $gldim(A) = d$. We say that A has *finite injective dimension* if the left and the right A-module A both have finite injective dimension. They are then equal (because A is noetherian) and we set $\mu = injdim(A)$.

Let ${}_AM$ be in $\mathrm{Mod}_f^g(A)$ with $p = pd(M) < \infty$. There exists a minimal graded free resolution of length p of M. That is: $P_\bullet \to M \to 0$, $P_j = 0$ if $j > p$, $P_j = \bigoplus_i A[-i]^{a_{ij}}$, and $\partial_j : P_{j+1} \to P_j$ is given by a matrix where the non-zero entries are elements of A of positive degree. We call $P_\bullet \xrightarrow{\partial} M \to 0$ the *minimal resolution* of M.

The *dual* of M is an element M^D of the derived category $D_r^b(A)$ of bounded complexes of finitely generated graded right modules. It is defined by $R\mathrm{Hom}_A(M,A)$ and is represented by $0 \leftarrow P_\bullet^\vee \leftarrow \ldots$, where $P_\bullet^\vee = \mathrm{Hom}_A(P_\bullet, A)$. The cohomology of this complex gives the groups $\mathrm{Ext}_A^q(M,A) \in \mathrm{Mod}_f^g(A)$. The Hilbert series of M is defined to be $h_M(t) := \sum_m (\dim M_m) t^m$ and is then equal to $\sum_j (-1)^j h_{P_j}(t) = \sum_j (-1)^j [\sum_i a_{i,j} t^i h_A(t)]$.

Let $M = \bigoplus_m M_m$ be in $\mathrm{Mod}_f^g(A)$. Define a function $f_M : \mathbb{Z} \to \mathbb{N}$ by $f_M(n) := \sum_{m \le n} \dim M_m$. From now on we shall assume that, for all $M \in \mathrm{Mod}_f^g(A)$, the growth of f_M, as defined in [9, Chapter 1], is a polynomial of degree s. The *Gelfand-Kirillov dimension* of M, $GKdim(M)$, is defined to be $s = \overline{\lim}_n \frac{\log f(n)}{\log n}$. A module of Gelfand-Kirillov dimension s is said to be *s-homogeneous* if every non-zero submodule is also of Gelfand-Kirillov dimension s. A module is *s-critical* if it is of Gelfand-Kirillov dimension s and all its proper quotients have strictly smaller Gelfand-Kirillov dimension.

We shall be interested in the case where $h_M(t) = g_M(t)(1-t)^{-d}$ for some fixed $d \in \mathbb{N}$ and $g_M(t) \in \mathbb{Z}[t, t^{-1}]$. As in [3, 2.21] one can see that f_M has polynomial growth of degree s which is the order of the pole at $t = 1$ of $h_M(t)$. Hence $h_M(t) = g_M(t)(1-t)^{-s}$, where $g_M(t) \in \mathbb{Z}[t, t^{-1}]$, and the *multiplicity* of M is defined to be $e(M) := g_M(1) = [(1-t)^s h_M(t)]_{t=1} \in \mathbb{N}$.

Let M be in $\mathrm{Mod}_f(A)$. The *grade* of M is the element $j(M) \in \mathbb{N} \cup \{+\infty\}$ defined by $j(M) := \inf\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}$. When $injdim(A) < \infty$ and $M \neq 0$ we have $j(M) < \infty$, (see [10, Theorem 3.1] for instance). We say that M is *pure* if $j(N) = j(M)$ for all non-zero submodules N of M. When $n = j(M)$ we say that M is *n-pure*. We abbreviate $\mathrm{Ext}^q_A(M, A)$ by $E^q_A(M)$ or $E^q(M)$.

Definition. The algebra A is *Auslander-Gorenstein*, respectively *Auslander-regular*, of dimension μ if

(a) $inj.dim(A) = \mu < \infty$, respectively $gldim(A) = \mu < \infty$, and

(b) for all $M \in \mathrm{Mod}_f(A)$, and for all $q \ge 0$, $j(N) \ge q$ for every A-submodule N of $E^q(M)$.

Remarks. 1. Condition (b) is called the *Auslander* condition. It implies that $E^p(E^q(M)) = 0$ for all $p < q$, which may be taken as the definition of a Gorenstein ring in the commutative case. This condition is discussed in detail in [6].

2. By [8, Theorem 0.1] one can replace $\mathrm{Mod}_f(A)$ by $\mathrm{Mod}_f^g(A)$ in the definition.

3. Assume A is Auslander-Gorenstein. Then the *grade number* $j(M)$ is exact on short exact sequences: that is, if $0 \to M' \to M \to M'' \to 0$ is exact then $j(M) = \inf\{j(M'), j(M'')\}$, (see [6,§1.8]).

Proposition 1.3. [11] Let A be Auslander-regular. Then A is a domain and is regular in the sense of Artin-Schelter (cf. [1] and [2, 2.12]).

The following summarizes results in [5, Chapter 2], [6, §1], [7], [10, §4], and [12].

Theorem 1.4. Let A be an Auslander-Gorenstein algebra of dimension μ, and let M be a non-zero finitely generated A-module.

(a) There exists a convergent spectral sequence in $\mathrm{Mod}_f(A)$:

$$E_2^{p,-q} := E^p_A(E^q_A(M)) \Longrightarrow \mathbb{H}^n$$

where $\mathbb{H}^n = 0$ if $n \neq 0$ and $\mathbb{H}^0 = M$. The resulting filtration on M has the form:

$$0 = F^{\mu+1}M \subset F^{\mu}M \subset \ldots \subset F^1M \subset F^0M = M.$$

(b) There is an exact sequence

$$0 \to \frac{F^pM}{F^{p+1}M} \to E^p(E^p(M)) \to Q(p) \to 0$$

where $Q(p)$ is a subquotient of $\bigoplus_{i\geq 1} E^{p+i+1}(E^{p+i}(M))$ and satisfies $j(Q(p)) \geq p+2$.
(c) F^pM is the largest submodule X of M such that $j(X) \geq p$. In particular $j(M) = \max\{p \mid F^pM = M\}$.
(d) $E^{j(M)}(M)$ is pure and $E^p(E^p(M))$ is either 0 or p-pure.

Remark. When $M \in \mathrm{Mod}_f^g(A)$ the spectral sequence in (1.4a) takes place in $\mathrm{Mod}_f^g(A)$. Hence the filtration $F^\bullet M$ consists of finitely generated graded submodules of M.

Definition. Let μ be an integer. We say that A satisfies the *Cohen-Macaulay property* (CM-property for short) if $GKdim M + j(M) = \mu$ for all $0 \neq M \in \mathrm{Mod}_f(A)$.

Remarks. 1. If A satisfies the Cohen-Macaulay property, then $GKdim A = \mu$.
2. If $(A, \mathfrak{M})$ is a commutative noetherian local ring, then A is Cohen-Macaulay if and only if $Kdim(M) + j(M) = Kdim(A)$ for all $0 \neq M \in \mathrm{Mod}_f(A)$ (here $Kdim$ is the Krull dimension). This explains the terminology in the definition.

Corollary 1.5. Suppose that A is Auslander-Gorenstein of dimension μ. Let $0 \neq M \in \mathrm{Mod}_f(A)$ and suppose that $GKdim(A) = \mu$. If A satisfies the Cohen-Macaulay property, then:
(a) M is n-pure if and only if M is $(\mu - n)$-homogeneous
(b) The submodule F^pM is the largest submodule of M of GK-dimension $\leq \mu - p$.
(c) If N is a submodule of $E_A^q(M)$, then $GKdim(N) \leq \mu - q$.
Proof. The assertions follow easily from the definitions and Theorem 1.4. ■

There is an example of an Auslander-Gorenstein algebra which is of particular importance for us. For simplicity assume that k is algebraically closed.

Let E be a smooth elliptic curve, $\mathcal{L}$ an invertible sheaf of degree ≥ 3 on E, σ a k-automorphism of E. As in [2], [4], or [21] one can construct a graded algebra $B := B(E, \sigma, \mathcal{L}) = k \oplus B_1 \oplus B_2 \oplus \ldots$, which for $\sigma = \mathrm{Id}_E$ is the homogeneous coordinate ring of the projective embedding $E \hookrightarrow \mathbb{P}(H^0(E, \mathcal{L})^*)$. By [2] and [4] it is known that B is a noetherian graded algebra generated by B_1, $GKdim(B) = 2$, and B is a domain.

The next two results are proved in [11].

Proposition 1.6. [11] Let $B = B(E, \sigma, \mathcal{L})$ be as above. Then B is Auslander-Gorenstein of dimension 2 and satisfies the Cohen-Macaulay property.

Remarks. 1. This result (which is well known in the commutative case viz. $\sigma = \mathrm{Id}_E$) is based on results of A. Yekutieli [25].

2. It is easily seen that if $M \in \mathrm{Mod}_f^g(B)$ then the Hilbert series $h_M(t)$ is of the form $q_M(t)(1-t)^{-2}$ for some $q_M(t) \in \mathbb{Z}[t, t^{-1}]$.

Definition. An element $\Omega \in A$, is said to be *normal* if $\Omega A = A\Omega$. Recall also that a sequence $\{\Omega_1, \ldots, \Omega_\ell\} \subset A$ is a *regular normalizing* (respectively *centralizing*) sequence if Ω_{i+1} is a regular (i.e. non zero divisor) normal (respectively central) element in the ring $A/\langle\Omega_1, \ldots, \Omega_i\rangle$, for all $0 \leq i \leq \ell - 1$.

Theorem 1.7. [11] Suppose that $\{\Omega_1, \ldots, \Omega_\ell\}$ is a regular normalizing sequence of homogeneous elements of positive degree in A. If $B = A/\langle\Omega_1, \ldots, \Omega_\ell\rangle$ is Auslander-Gorenstein of dimension ν and satisfies the Cohen-Macaulay property, then A is Auslander-Gorenstein of dimension $\mu = \nu + \ell$ and satisfies the Cohen-Macaulay property.

Corollary 1.8. Suppose that A contains a regular normalizing sequence $\{\Omega_1, \ldots, \Omega_\ell\}$ of homogeneous elements of positive degree, such that $B := A/\langle\Omega_1, \ldots, \Omega_\ell\rangle \cong B(E, \sigma, \mathcal{L})$ for some triple $(E, \sigma, \mathcal{L})$ as above. If $gldim(A) < \infty$, then A is Auslander-regular of dimension $\ell + 2$ and satisfies the Cohen-Macaulay property: $GKdim(M) + j(M) = \ell + 2$ for all $0 \neq M \in \mathrm{Mod}_f(A)$.

By [21, 5.4], Corollary 1.8 applies to the Sklyanin algebra. Since we are mainly interested in modules over this algebra we isolate:

Corollary 1.9. The 4-dimensional Sklyanin algebra is an Auslander-regular ring of dimension 4 which satisfies the Cohen-Macaulay property.

Remark. There are some other cases where (1.8) can be applied. In [2,§2] some "regular" algebras are introduced which have the property $A/\langle\Omega\rangle \cong B(E, \sigma, \mathcal{L})$ for a central element $\Omega \in A_3$ and a triple $(E, \sigma, \mathcal{L})$ with $deg\mathcal{L} = 3$. In [23], J.T. Stafford modifies the construction of the Sklyanin algebra to construct some families of "regular" algebras which satisfy $A/\langle\Omega_1, \Omega_2\rangle \cong B(E, \sigma, \mathcal{L})$, where $\{\Omega_1, \Omega_2\}$ is a regular normalizing (not centralizing in general) sequence of quadratic elements and $deg\mathcal{L} = 4$.

For the rest of Section 1 we make the following assumptions on A:

(a) the Hilbert series of A is $h_A(t) = (1-t)^{-\mu}$,

(b) A is Auslander-regular of dimension μ,

(c) A satisfies the Cohen-Macaulay property.

The 4-dimensional Sklyanin algebra satisfies these conditions.

As noticed in (1.3), A is then regular in the sense of Artin-Schelter and by [3, §2.18] every $M \in \mathrm{Mod}_f^g(A)$ satisfies $h_M(t) = q_M(t)(1-t)^{-s}$, for some $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ such that $q_M(1) \neq 0$. Furthermore in this situation $s = GKdim(M) = \mu - j(M)$ whenever $M \neq 0$.

If $0 \neq M \in \mathrm{Mod}_f^g(A)$ with $n = j(M)$, then we define the *dual* of M to be $M^\vee := E^n(M)$. When A is Auslander-Gorenstein $M^\vee$ is an n-pure graded right module, so (1.4b) gives a natural map $0 \to M/F^{n+1}M \to M^{\vee\vee}$. This map generalizes the well known map from M modulo its torsion to the bidual of M.

Definition. A non-zero module $M \in \mathrm{Mod}_f^g(A)$ is called *Cohen-Macaulay* (CM for short) if $pd(M) = j(M)$, i.e. $E_A^i(M) = 0$ if $i \neq n = j(M)$.

Proposition 1.10. The duality $M \to M^\vee$ gives a bijection between left and right Cohen-Macaulay-modules of projective dimension n. In particular $M \cong M^{\vee\vee}$, $h_{M^\vee}(t) = (-1)^{\mu-n}t^{-\mu}h_M(t^{-1})$, and $e(M^\vee) = e(M)$.

Proof. We first show that $M^\vee$ is CM. We have $E^i(E^n(M)) = 0$ if $i < n$ by the Auslander condition and $E^n(E^n(M)) \neq 0$ by (1.4). Thus $j(M^\vee) = n$. We must prove $E^j(M^\vee) = E^j(E^n(M)) = 0$ if $j > n$. Recall the spectral sequence of (1.4):

$$E_2^{p,-q} = E^p(E^q(M)) \Longrightarrow \mathbb{H}^{p-q} = \begin{cases} 0 & p \neq q \\ M & p = q \end{cases}$$

It follows that $E_\infty^{p,-q} = 0$ if $p > q$. If $r \geq 2$, the differential d_r in the spectral sequence gives:

$$E_r^{j-r,-n+(r-1)} \xrightarrow{d_r} E_r^{j,-n} \xrightarrow{d_r} E_r^{j+r,-n-(r-1)}.$$

Since $n \mp (r-1) \neq n$, $E^{n\mp(r-1)}(M) = 0$. Hence $E_r^{j+r,-n\pm(r-1)} = 0$ for all $r \geq 2$ and it follows that $E_2^{j,-n} = E_3^{j,-n} = \ldots = E_\infty^{j,-n} = 0$ for all $j > n$. Thus $E^j(E^n(M)) = E_2^{j,-n} = 0$ if $j > n$.

Because $E^p(M) = 0$ when $p \neq n$, the filtration on M has the form: $M = F^0M = \ldots = F^nM \supsetneqq F^{n+1}M = \ldots = F^\mu M = 0$. By (1.4b) $M = F^nM \cong M^{\vee\vee} = E^n(E^n(M))$. Since M is CM we have $h_{M^D}(t) = (-1)^n h_{M^\vee}(t)$. From [3, §2.35,§2.36] we deduce that $h_{M^D}(t) = (-1)^\mu t^{-\mu} h_M(t^{-1})$, and $e(M^D) = (-1)^n e(M)$ (recall $\mu = n + GKdimM$) so the proof is complete. See also the remark after (1.12). ■

Corollary 1.11. Let $M \in \mathrm{Mod}_f^g(A)$ be a Cohen-Macaulay module. Set $n = j(M)$, and $m = GKdimM$. Then M is n-pure, (equivalently m-homogeneous). Furthermore, if $e(M) = 1$, then M is critical for GK-dimension.

Proof. By (1.9) $M = (M^\vee)^\vee = E^n(M^\vee)$, so the first assertion follows from (1.4d) and the CM-property. The second assertion is then obvious by additivity of the multiplicity.■

Lemma 1.12. Let $0 \to M' \to M \to N \to 0$ be an exact sequence in $\mathrm{Mod}^g_f(A)$. Suppose that $j(M) = n$, and $j(N) = n+1$.
(a) If M and N are Cohen-Macaulay then so is M'
(b) If M and M' are Cohen-Macaulay then so is N.

Proof. Use the definition and the long exact sequence in cohomology associated to $0 \to M' \to M \to N \to 0$. ■

Remark. Let $P_\bullet \to M \to 0$ be the minimal resolution of a Cohen-Macaulay module M. Then $P_\bullet^\vee \to M^\vee \to 0$ is the minimal resolution of $M^\vee$, where $P_\bullet^\vee = \mathrm{Hom}_A(P_\bullet, A)$. This follows from the following two observations. Firstly, if $\partial_j : P_{j+1} = \bigoplus_i A[-i]^{a_{i,j+1}} \to P_j = \bigoplus_i A[-i]^{a_{i,j}}$ is the differential in $P_\bullet$, its dual $\partial_j^\vee : P_j^\vee = \bigoplus_i A[i]^{a_{i,j}} \to P_{j+1}^\vee = \bigoplus_i A[i]^{a_{i,j+1}}$ is also given by a matrix where entries have positive degree. Secondly, by definition of a CM-module, the complex $(P_\bullet^\vee, \partial^\vee)$ has cohomology only in the top degree $n = j(M)$ and this cohomology is $M^\vee$.

¿From this remark and the fact that $h_A(t) = (1-t)^{-\mu}$, one obtains the equalities $h_{M^\vee}(t) = (-1)^{\mu-n} t^{-\mu} h_M(t^{-1})$ and $e(M^\vee) = e(M)$ of (1.10).

We shall be interested in Cohen-Macaulay modules of multiplicity 1 over the Sklyanin algebra. It may be useful to recall the situation for a commutative polynomial ring in μ variables (which satisfies the assumptions made after (1.9) which are in force for the remainder of this section). Because there seems to be no suitable reference we include a proof of the following:

Theorem 1.13. Let $A = k[X_1, \dots, X_\mu]$ be a polynomial ring in μ variables, graded by setting $\deg X_i = 1$. Let M be in $\mathrm{Mod}^g_f(A)$. Then the following are equivalent:
(a) M is cyclic with Hilbert series $t^p(1-t)^{-d}$ for some $p \in \mathbb{Z}$.
(b) there exists $p \in \mathbb{Z}$ such that $M[p] \cong A/\langle Y_1, \dots, Y_{\mu-d}\rangle$ where $Y_1, \dots, Y_{\mu-d}$ are linearly independent elements of A_1.
(c) M is a Cohen-Macaulay module, $GKdim(M) = d$ and $e(M) = 1$.

Proof. Replacing M by a suitable shift $M[p]$ if necessary, we can assume that $M = \oplus_{m \geq 0} M_m$ with $M_0 \neq 0$.

(a) $\Longrightarrow$ (b) Write $M \cong A/I$ for some graded ideal I. Since $h_M(t) = (1-t)^{-d}$ we have $\dim I_1 = \mu - d$. If $\{Y_1, \dots, Y_{\mu-d}\}$ is a basis of I_1 we get a natural graded surjective map: $A/\langle Y_1, \dots, Y_{\mu-d}\rangle \xrightarrow{\phi} M$. Since these two modules have the same Hilbert series, ϕ is an isomorphism.

(b) $\Longrightarrow$ (c) This is easy and well known (for instance use (1.12b)).

(c) $\Longrightarrow$ (a) Notice that (a) and (c) remain true under base extension. If $k \subset L$ is a field extension, set $A_L = L \otimes_k A = L[X_1, \dots, X_\mu]$, and $M_L = L \otimes_k M = \bigoplus_{m=0}^\infty (L \otimes_k M)_m \in \mathrm{Mod}^g_f(A_L)$. We make three useful observations. Firstly M is CM if and only if M_L is CM (because $A \hookrightarrow A_L$ is faithfully flat). Secondly $h_M(t) = h_{M_L}(t)$, so $GKdim M_L =$

$GKdimM$ and $e(M_L) = e(M)$. Thirdly $M_L/\langle X_1, \ldots, X_\mu\rangle M_L \cong L \otimes_k M/\langle X_1, \ldots, X_\mu\rangle M$ so M is cyclic if and only if M_L is cyclic.

Thus to prove the implication we may assume that k is an uncountable field.

1. If $m = (m_1, \ldots, m_\mu) \in \mathbb{N}^\mu$ with $m_1 < m_2 < \ldots < m_\mu$ we put $f_m(X) = \det[X_i^{jm_i}]_{1\leq i,j\leq\mu} = \prod_{1\leq i\leq\mu} X_i^{m_i} \prod_{1\leq i<j\leq\mu}(X_j^{m_j} - X_i^{m_i})$. Since k is uncountable we can find $0 \neq \lambda \in k$ which is not a root of unity. If we set $\lambda = (\lambda, \lambda, \ldots, \lambda) \in \mathbb{N}^\mu$ then $f_m(\lambda) \neq 0$ for all m as above.

2. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ be proper ideals of A and assume $A_1 \subset \bigcup_{j=1}^t \mathfrak{M}_j$. Choose λ as above and notice that the set $S = \{\lambda^m X_1 + \lambda^{2m} X_2 + \ldots + \lambda^{\mu m} X_\mu \mid m \in \mathbb{N}\}$ is infinite. Since $S \subset A_1 \subset \bigcup_{j=1}^t \mathfrak{M}_j$ there exists $\ell \in \{1, \ldots, t\}$ such that $S \cap \mathfrak{M}_\ell$ is infinite. Choose $m = (m_1 < m_2 < \ldots < m_\mu) \in \mathbb{N}^\mu$ such that $\sum_{i=1}^\mu \lambda^{im_j} X_i \in \mathfrak{M}_\ell$ for all $j \in \{1, \ldots, \mu\}$. Since $f_m(\lambda) \neq 0$ we deduce that $X_i \in \mathfrak{M}_\ell$ for all i, that is $\mathfrak{M}_\ell = \langle X_1, \ldots, X_\mu\rangle$.

3. We now prove the implication by induction on $d = GKdimM$. If $d = 0$, then $\dim M = e(M) = 1$, hence $M = M_0 = A/\langle X_1, \ldots, X_\mu\rangle$. If $d \geq 1$, denote by $\mathfrak{M}_1, \ldots, \mathfrak{M}_t$ the associated prime ideals of M. Recall that $\bigcup_{j=1}^t \mathfrak{M}_j$ is the set of zero-divisors in M. Suppose $A_1 \subset \bigcup_{j=1}^t \mathfrak{M}_j$. By part 2 we get $\mathfrak{M}_\ell = \langle X_1, \ldots, X_\mu\rangle$ for some ℓ. It follows that$A/\langle X_1, \ldots, X_\mu\rangle \hookrightarrow M$ which implies $pd(M) = \mu$, i.e. $d = 0$, which is a contradiction. Therefore we can find $a \in A_1$ which is a non-zero divisor in M. The graded module $\overline{M} = M/aM$ is CM with Hilbert series $h_{\overline{M}}(t) = (1-t)h_M(t)$, and $\overline{M}/\langle X_1, \ldots, X_\mu\rangle\overline{M} = M/\langle X_1, \ldots, X_\mu\rangle M$. By induction $\overline{M}$ is cyclic, $h_{\overline{M}}(t) = (1-t)^{1-d}$, and from above we conclude that M is cyclic and $h_M(t) = (1-t)^{-d}$. ∎

§2. Homological Characterization of Point, Line and Plane Modules.

Throughout this section we will assume that A is a $\mathbb{C}-$algebra of finite global dimension containing a regular normalizing sequence $\{\Omega_1, \Omega_2\}$, where $\Omega_1, \Omega_2 \in A_2$ are such that $B := A/\langle\Omega_1, \Omega_2\rangle \cong B(E, \sigma, \mathcal{L})$ (graded isomorphism) for some $(E, \sigma, \mathcal{L})$ with $\mathcal{L}$ of degree 4. As we noticed after (1.8) the Skylyanin algebra is an example of such a ring; also see the remark after (1.9).

Since $h_B(t) = (1+t)^2(1-t)^{-2} = (1-t^2)^2 h_A(t)$ we have $h_A(t) = (1-t)^{-4}$. By (1.8) A is Auslander regular of dimension 4 and satisfies the Cohen-Macaulay property. Furthermore, since B is generated by B_1, the algebra A is generated by A_1. Hence if $\mathfrak{M} = \bigoplus_{m\geq 1} A_m$ is the augmentation ideal we have: $\mathfrak{M} = A_1 A = A A_1$.

The following proposition provides a version of [3, Theorem 4.1, Corollary 4.2].

Proposition 2.1 Let $0 \neq M \in Mod_f^g(A)$, set $n = j(M)$, and $m = GKdimM$. Then

(a) $n + m = 4$;

(b) $M^\vee = E^n(M)$ is m-homogeneous and $e(M^\vee) = e(M)$;

(c) $GKdimE^q(M) \leq 4 - q$ and the following are equivalent:

(i) $GKdimE^q(M) = 4 - q$,

(ii) $E^q(E^q(M)) \neq 0$,

(iii) $GKdimN = 4 - q$ for some submodule $0 \neq N \subseteq M$;

(d) There is a canonical map $M \xrightarrow{\phi} M^{\vee\vee}$ and an exact sequence $0 \to M/ker\phi \to M^{\vee\vee} \to Q \to 0$ with $Ker\phi$ equal to the maximal submodule of M of $GKdim < m$, and $GKdimQ \leq m - 2$;

(e) The following are equivalent:

(i) $pd(M) < 4$,

(ii) the socle of M is $Soc(M) = Hom_A(A/\mathfrak{M}, M) = 0$,

(iii) $E^4(M) = 0$,

(iv) the torsion submodule of M is $T(M) := \{x \in M \mid \mathfrak{M}^i x = 0 \text{ for some } i\} = 0$;

furthermore $E^4(M) \cong E^4(T(M))$ is a finite dimensional vector space of the same dimension as $T(M)$ and $E^i(M) \cong E^i(M/T(M))$ for all $i < 4$;

(f) If $m \in \{0, 1, 2\}$, $M^\vee$ is Cohen-Macaulay. If $m = 3$, $E^j(M^\vee) = 0$ for $j = 0, 3, 4$ and $E^2(M^\vee) \subset E^4(E^2(M))$ is a finite dimensional vector space.

Proof. (a) This is the Cohen-Macaulay property.

(b) This follows from (1.4) and [3,§2.8] using the inequality $GKdimE^q(M) \leq 4 - q$ of (c).

(c) The inequality is (1.5c). By §1.2, $GKdimE^q(M) = 4 - q \Leftrightarrow j(E^q(M)) = q \Leftrightarrow E^q(E^q(M))$ is q-pure $\Leftrightarrow F^q M/F^{q+1}M \neq 0 \Leftrightarrow M$ contains a submodule of GK-dimension $4 - q$.

(d) The map ϕ is the composition of $M/F^{n+1}M \to M^{\vee\vee}$ with the natural projection $M \to M/F^{n+1}M$. Hence $Ker\phi = F^{n+1}M$, so (d) is a consequence of (1.4) and (1.5).

(e) This is proved in [3,§2.46].

(f) We shall use the notation in the proof of (1.10). Recall that the spectral sequence $E_2^{p,-q} \Rightarrow \mathbb{H}^{p-q}$ implies $E_\infty^{p,-q} = 0$ if $p - q \neq 0$. Furthermore $E_{r+1}^{p,-q} = Kerd_r/Imd_r$. If $r \geq 2$, then $n+1-r < n$ so $E_2^{p-r,-n-1+r} = 0$ and $E_{r+1}^{p,-n} = Ker(d_r)$. By the Auslander condition $E^p(M^\vee) = E^p(E^n(M)) = 0$ if $p < n = 4-m$.

Suppose that $p > n$, and that $m \in \{0,1,2\}$ or equivalently $n \geq 2$. If $r \geq 2$, then $p+r > 4$, so $E_2^{p-r,-n-1+r} = 0$ because $gldim(A) = 4$. It follows that $E_2^{p,-n} = E_3^{p,-n} = \cdots = E_\infty^{p,-n} = 0$.

Suppose that $m = 3$, or equivalently that $n = 1$. When $p \geq 3$ and $r \geq 2$ the previous argument shows that $E_2^{p,-n} = \cdots = E_\infty^{p,-n} = 0$ because $p + r \geq 5$. Assume $p = 2$. If $r \geq 3$ then $2 + r \geq 5$ so $0 = E_2^{2+r,-1}$, whence $0 = E_\infty^{2,-1} = \cdots = E_3^{2,-1} = Ker(E_2^{2,-1} \xrightarrow{d_2} E_2^{4,-2})$. Thus $E_2^{2,-1} = E^2(M^\vee)$ is a submodule of $E^4(E^2(M)) = E_2^{4,-2}$, hence is finite dimensional by (e). ■

We want to study Cohen-Macaulay modules with multiplicity 1. There are two easy cases. If M is Cohen-Macaulay and $GKdimM = 0$ then $M \cong k[p]$, for some $p \in \mathbb{Z}$. If M is Cohen-Macaulay, and $GKdimM = 4$ then $M \cong A[p]$, for some $p \in \mathbb{Z}$. Therefore we shall only consider the case $1 \leq GKdimM \leq 3$. If A is a commutative polynomial ring in 4 variables, Theorem 1.13 shows that these modules are (up to a shift) in bijection with the linear subvarieties of $\mathbb{P}(A_1^*) \cong \mathbb{P}^3$. On the other hand [3] describes such modules over a regular algebra A of dimension 3 with $A/\langle\Omega\rangle \cong B(E, \sigma, \mathcal{L})$ where Ω is a central cubic element and $\mathcal{L}$ has degree 3. They can be classified (up to a shift) as follows:

–If $GKdimM = 1$, M is a "point module", i.e. M is cyclic, and $h_M(t) = (1-t)^{-1}$. Point modules are parametrized by the points of $E \subset \mathbb{P}^2 \cong \mathbb{P}(A_1^*)$. See [3,§6.17].

–If $GKdimN = 2$, M is a "line module", i.e. M is cyclic, $h_M(t) = (1-t)^{-2}$. Line modules are parametrized by the lines in $\mathbb{P}^2 \cong \mathbb{P}(A_1^*)$. See [3,§6.1].

In view of these remarks we make the following definition:

Definition. Let M be in $Mod_f^g(A)$. We say that
M is a *plane module* if M is cyclic and $h_M(t) = (1-t)^{-3}$,
M is a *line module* if M is cyclic and $h_M(t) = (1-t)^{-2}$,
M is a *point module* if M is cyclic and $h_M(t) = (1-t)^{-1}$.

A shift of a plane, respectively line or point, module is a cyclic module with Hilbert series $t^p(1-t)^{-i}, p \in \mathbb{Z}$, $i = 3, 2, 1$ respectively. The main result in this section is the following.

Theorem 2.2. The module $M \in Mod_f^g(A)$ is a shift of a plane, respectively line, respectively point, module if and only if M is a Cohen-Macaulay module, $e(M) = 1$ and $GKdimM = 3$, respectively 2, respectively 1.

The proof of Theorem 2.2 occupies the rest of §2. We shall examine separately the different types of modules. Because of the following consequence of (1.10) and (1.11), we will work with left $A-$modules unless otherwise specified.

Corollary 2.3. The duality $M \to M^\vee$ gives a bijection between left and right shifted plane, respectively line or point, modules. These modules are critical.

We first investigate the case of plane modules. Proposition 2.5 below is a particular case of [3,§2.43]. For the convenience of the reader we include a proof of it in our setting. We first begin with a lemma taken from [3, §2.41].

Lemma 2.4. Let $f : \bigoplus_i A[-i]^{b_i} \longrightarrow \bigoplus_i A[-i]^{a_i}$ be an injective linear map of degree 0 between finitely generated graded free modules. Assume that the non-zero matrix entries of f all have positive degree. Then $b_j \leq \sum_{i<j}(a_i - b_i)$ for all j.

Proof. The assumptions on f imply that $f(\bigoplus_{i\leq j} A[-i]^{b_i}) \subset \bigoplus_{i<j} A[-i]^{a_i}$. If X is the quotient of these two modules, then $h_X(t) = (1-t)^{-4}(\sum_{i<j} a_i t^i - \sum_{i\leq j} b_i t^i)$. Since $[(1-t)^{GKdimX} h_X(t)]_{t=1} = e(X) \geq 0$, we have $0 \leq [(1-t)^4 h_X(t)]_{t=1} = \sum_{i<j} a_i - \sum_{i\leq j} b_i$. The result follows. ■

Proposition 2.5. If $M \in Mod_f^g(A)$ the following are equivalent:

(a) M is a shift of a plane module;

(b) $M \cong (A/Aa)[p]$, for some $p \in \mathbb{Z}$, where $0 \neq a \in A_1$.

(c) M is Cohen-Macaulay, $GKdimM = 3$ and $e(M) = 1$.

Proof. We may assume that $M = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$, and $M_0 \neq 0$.

$(a) \Rightarrow (b)$ We have $dimM_0 = 1, dimM_1 = 3$, and $dimA_1 = 4$. Hence there exists $0 \neq a \in A_1$ such that $aM_0 = 0$. It follows that $M = AM_0$ is a quotient of A/Aa. But A is a domain, so $h_{A/Aa}(t) = (1-t)^{-3} = h_M(t)$. Thus $M = A/Aa$.

$(b) \Rightarrow (c)$. The short exact sequence $0 \to A[-1] \xrightarrow{\times a} A \longrightarrow M \to 0$ is the minimal resolution of M and $h_M(t) = (1-t)^{-3}$. This shows that $GKdimM = 3, e(M) = 1$, and $pd(M) = 1$.

$(c) \Rightarrow (a)$. Let $0 \to \bigoplus_i A[-i]^{b_i} \xrightarrow{\partial} \bigoplus_i A[-i]^{a_i} \to M \to 0$ be the minimal resolution of M. As noticed in §1, $0 \to \bigoplus_i A[i]^{a_i} \xrightarrow{f=\partial^\vee} \bigoplus_i A[i]^{b_i} \to M^\vee \to 0$ is the minimal resolution of $M^\vee$. By Lemma 2.4 applied to f we have $a_j \leq \sum_{i>j}(b_i - a_i)$. Therefore

$$(*) \qquad \sum_{i\geq 1} i(b_i - a_i) = \sum_{i\geq 1}(b_i - a_i) + \sum_{i\geq 2}(b_i - a_i) + \sum_{i\geq 3}(b_i - a_i) + \cdots \geq \sum_{j\geq 0} a_j \geq 1.$$

It follows from the presentation of M that

$$h_M(t) = \frac{1}{(1-t)^4}\sum_{i\geq 0}(a_i - b_i)t^i = \frac{1}{(1-t)^4}\sum_{j\geq 0}(-1)^j \left(\sum_{i\geq j} \tbinom{i}{j}(a_i - b_i)\right)(1-t)^j.$$

Since $GKdimM = 3$ and $e(M) = 1$, this implies that $\sum_{i\geq 0}(a_i - b_i) = 0$ and $\sum_{i\geq 1} i(b_i - a_i) = 1$. By $(*)$ it follows that $\sum_{j\geq 0} a_j = 1$. Thus $\sum_{j\geq 0} b_j = 1$.

But $a_0 \neq 0$ because $M_0 \neq 0$. Hence $a_0 = 1$ and $a_i = 0$ if $i \geq 1$. Thus $1 = \sum_{i\geq 1} i(b_i - a_i) = \sum_{i\geq 1} ib_i$. But $\sum_{j\geq 0} b_j = 1$ so $b_1 = 1$, and $b_i = 0$ if $i \neq 1$. Hence the minimal resolution takes the form $0 \to A[-1] \to A \to M \to 0$, which proves (a). ■

We now consider the point modules.

Proposition 2.6. Let M be a Cohen-Macaulay module, with $GKdimM = 1$ and $e = e(M)$. Then

(a) $h_M(t) = t^p\left(\sum_{0\leq i\leq d-1} f_i t^i + et^d(1-t)^{-1}\right)$ for some $p \in \mathbb{Z}, d \in \mathbb{N}$, and $f_i \in \mathbb{N}$ such that $1 \leq f_i \leq e-1$ for all $i \in \{0, \cdots, d-1\}$.

(b) If $e = 1$, then $h_M(t) = t^p(1-t)^{-1}, p \in \mathbb{Z}$.

(c) If $e = 2$, then $h_M(t) = t^p(1+t^d)(1-t)^{-1}$, for some $p \in \mathbb{Z}, d \in \mathbb{N}$.

Proof. By shifting the grading we may assume that $M = \bigoplus_{n\geq 0} M_n$, and that $M_0 \neq 0$.

(a) Since $GKdimM = 1$ we have $h_M(t) = g(t)(1-t)^{-1}$, where $g(t) \in \mathbb{Z}[t]$. Write $g(t) = e + (1-t)p(t)$, where $p(t) \in \mathbb{Z}[t]$ is of degree $d-1$. It follows that $h_M(t) = f_0 + f_1 t + \cdots + f_{d-1}t^{d-1} + et^d(1-t)^{-1}$, with $f_i \in \mathbb{N}$, and $f_0 \neq 0$. We can choose d as small as possible with respect to this property. For this choice we obviously must have $f_{d-1} \neq e$.

Note that $f_j \neq 0$ for all $j \in \{0, \cdots, d-1\}$. If this doesn't hold choose j minimal such that $f_{j-1} \neq 0$, and $f_j = 0$. Then $A_1 M_{j-1} \subset M_j = 0$ implies $M_{j-1} \subset Soc(M)$; but M is Cohen-Macaulay so its socle is zero by (2.5e). This contradiction shows that $f_j \neq 0$.

Recall that $M^\vee := E^3(M)$ is a CM-module whose Hilbert series is $-t^{-4}h_M(t^{-1})$ by (1.10). Hence $h_{M^\vee}(t) = (e-f_{d-1})t^{-d-3} + (e-f_{d-2})t^{-d-2} + \cdots + (e-f_0)t^{-4} + et^{-3}(1-t)^{-1}$. In particular $e \geq f_j$ for all j. Since $e > f_{d-1}$ and $Soc(M^\vee) = 0$ we can conclude as above that $e > f_j$ for all j. Hence $1 \leq f_j \leq e-1$ for all j and (a) is proved.

(b) If $e = 1$ the condition $1 \leq f_j \leq 0$ forces $d = 0$, i.e., $h_M(t) = (1-t)^{-1}$.

(c) If $e = 2$ it follows from $1 \leq f_j < 2$ that $h_M(t) = 1 + t + \cdots + t^{d-1} + 2t^d(1-t)^{-1} = (1+t^d)(1-t)^{-1}$. ■

Proposition 2.7. Let M be in $Mod_f^g(A)$. Then the following are equivalent:

(a) M is a shift of a point module;

(b) M is Cohen-Macaulay, and $GKdim(M) = e(M) = 1$.

Proof. $(a) \Rightarrow (b)$ As usual we may assume that M is a point module. Say $M = Av_0$ where $v_0 \in M_0$ and $h_M(t) = (1-t)^{-1}$. By (2.1e) it suffices to show that $Soc(M) = 0$. Suppose that $0 \neq x \in Soc(M)_p$. Then $A_1 x = 0$ and $M_p = \mathbb{C}x = A_1^p v_0$, so $A_1^{p+1}.v_0 = 0$ which is a contradiction.

$(b) \Rightarrow (a)$ By (2.6b) $h_M(t) = t^p(1-t)^{-1}$. We must show that M is cyclic. Pick $0 \neq x \in M_p$. Since $T(M) = 0$ we know that $A_n.x = A_1^n.x \neq 0$ for all $n \geq 0$. Thus $A_n.x = M_{n+p}$ and $M = A.x$. ■

To prove Theorem 2.2 it remains to investigate the line modules. We begin with a characterization of line modules which furnishes half of the desired result: a shift of a line module is Cohen-Macaulay of GK-dimension 2 and multiplicity 1.

Proposition 2.8.

(a) If $u, v \in A_1$ are linearly independent and $A_1 u \cap A_1 v \neq 0$ then $L = A/Au + Av$ is a line module

(b) If L is a line module, then there exist linearly independent $u, v \in A_1$ such that $A_1 u \cap A_1 v \neq 0$ and $L \cong A/Au + Av$.

(c) If L is a line module then L is Cohen-Macaulay, $GKdim L = 2$, and $e(L) = 1$.

Proof. (a) Let $0 \neq a \in A_1$ satisfy $au \in A_1 v$. Write $M = A/Av$ and $\overline{u} \in M_1$ for the image of u. There is a surjective map $\psi : A/Aa \to A\overline{u}[1]$. Since u and v are linearly independent $A\overline{u}$ is a non-zero submodule of the plane module M. Since M is critical and $GKdim M = 3$ it follows that $GKdim(A\overline{u}) = 3$. But A/Aa is also a plane module, so critical of $GKdim 3$. Hence ψ is an isomorphism and $h_L(t) = h_M(t) - h_{A\overline{u}}(t) = (1-t)^{-3} - t(1-t)^{-3} = (1-t)^{-2}$. Thus L is a line module.

(b) If L is a line module then $dim L_1 = 2$ so there exist linearly independent $u, v \in A_1$ such that $uL_0 = vL_0 = 0$. Hence there is a surjective map $\phi : A/Au + Av \to L$. However $dim L_2 = 3$, $dim A_2 = 10$, $dim A_1 u = dim A_1 v = 4$, so $A_1 u \cap A_1 v \neq 0$. By (1) $h_{A/Au+Av}(t) = h_L(t) = (1-t)^{-2}$ so ϕ is an isomorphism.

(c) Write $L = A/Au + Av$ as in (b). We must show that L is CM. As in the proof of (1) write $M = A/Av$ and $\overline{u}$ for the image of u. Then $M' = A\overline{u}$ is a shift of a plane module and we have an exact sequence $0 \to M' \to M \to L \to 0$ where M' and M are CM, $j(M) = 1$, and $j(L) = 2$. By (1.12b) L is Cohen-Macaulay. ∎

Corollary 2.9.

(a) A line module L has a minimal resolution of the form

$$0 \to A[-2] \xrightarrow{\partial_1} A[-1] \oplus A[-1] \xrightarrow{\partial_0} A \to L \to 0.$$

(b) The line modules are in bijection with those lines ℓ in $\mathbb{P}^3 := \mathbb{P}(A_1^*)$ such that $\ell = \mathcal{V}(u, v)$ where $u, v \in A_1$ are linearly independent elements satisfying $A_1 u \cap A_1 v \neq 0$. The line module corresponding to such an ℓ is $M(\ell) := A/Au + Av$.

Proof. (a) Write $L \cong A/Au + Av$ as in (2.8) and denote by β a non-zero element in $A_1 u \cap A_1 v$. Define ∂_0 by the matrix $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ and ∂_1 by $[\beta]$. It is clear that this gives the minimal resolution of L.

(b) If L and L' are isomorphic line modules then $Ann_{A_1}(L_0) = Ann_{A_1}(L'_0)$ so the elements $u, v \in A_1$ and $u', v' \in A_1$ guaranteed by (2.8) span the same subspace of A_1 and hence define the same line in $\mathbb{P}^3$. ∎

Remember that throughout this section we are assuming that A_2 contains a regular normalizing sequence $\{\Omega_1, \Omega_2\}$ such that $A/\langle \Omega_1, \Omega_2 \rangle := B \cong B(E, \sigma, \mathcal{L})$.

Our next goal is to prove a converse of (2.8c). Two preliminary results are necessary before this is done in (2.12).

Lemma 2.10. Let $0 \neq M \in Mod_f(A)$ be critical and let $0 \neq \Omega$ be a normal element of A. Then Ω is a non-zero divisor in M if and only if $\Omega M \neq 0$.

Proof. Since A is a domain we can define an automorphism α of A by the formula $\Omega\alpha(a) = a\Omega$, for all $a \in A$. Denote by ${}^{\alpha}M$ the abelian group M with A acting by $a \cdot x = \alpha(a)x$, for $a \in A$, and $x \in M$. Then ${}^{\alpha}M$ is a critical A-module and multiplication by Ω defines an A-module map ${}^{\alpha}M \xrightarrow{\Omega} M$ with image ΩM and kernel $K = \{x \in M \mid \Omega x = 0\}$. The isomorphism ${}^{\alpha}M/K \xrightarrow{\sim} \Omega M$ shows that $GKdim(M/K) = GKdim(\Omega M) < GKdim(M)$ if $K \neq 0$. Since $GKdim(\Omega M) = GKdim(M)$ if $\Omega M \neq 0$, the result follows.■

Proposition 2.11.

(a) Suppose $M \in Mod_f^g(A)$ is 2–homogeneous with $e(M) = 1$. Then M is critical. Furthermore if $\Omega_1 M = 0$, then Ω_2 is a non-zero divisor in M.

(b) Let M be a Cohen-Macaulay module, with $GKdim(M) = 2$ and $e(M) = 1$. Then $h_M(t) = t^p(1 + t^d)(1 - t)^{-1}(1 - t^2)^{-1}$ where $p \in \mathbb{Z}$ and $d \in \mathbb{N}$ is odd.

Proof. (a) The first assertion is obvious. Suppose $\Omega_1 M = 0$ and that $0 \neq y \in M$ is such that $\Omega_2 y = 0$. We may assume that y is homogeneous of degree q. Then $N = Ay$ is a B-module and $N = By \cong \frac{B}{L}[-q]$ for some graded left ideal L of B. Since B is a domain and $GKdim\ B = GKdim\ N = 2$ we must have $L = 0$. Thus $h_N(t) = t^q \frac{(1+t)^2}{(1-t)^2}$ which implies $e(N) = 4$ which contradicts $e(N) \leq e(M) = 1$. Thus $\Omega_2 M \neq 0$ and by (2.10) Ω_2 is a non-zero divisor in M.

(b) Without loss of generality we may assume that $M = \bigoplus_{n \geq 0} M_n$, and $M_0 \neq 0$. By (a) and (2.10) one of the elements Ω_1, Ω_2 is a non-zero divisor in M, say Ω_2. Set $\overline{M} = M/\Omega_2 M$ and notice that $h_{\overline{M}}(t) = (1 - t^2)h_M(t)$ so $GKdim\overline{M} = 1$, and $e(\overline{M}) = 2$. Denote by α the automorphism of A defined by $\Omega_2\alpha(a) = a\Omega_2$ for $a \in A$, and define ${}^{\alpha}M \in Mod_f^g(A)$ as in the proof of (2.10). It is clear that ${}^{\alpha}M$ is Cohen-Macaulay and that there is an exact sequence $0 \longrightarrow {}^{\alpha}M \xrightarrow{\times\Omega_2} M[2] \longrightarrow \overline{M}[2] \longrightarrow 0$. Thus, by (1.12b), $\overline{M}[2]$ is Cohen-Macaulay. From (2.6c) we deduce $h_{\overline{M}}(t) = (1 + t^d)(1 - t)^{-1}$, for some $d \in \mathbb{N}$. Therefore $h_M(t) = (1+t^d)(1-t)^{-1}(1-t^2)^{-1}$. But $h_M(t)$ must have the form $f(t)(1-t)^{-4}$ for some $f(t) \in \mathbb{Z}[t]$, whence $1 + t$ divides $1 + t^d$. Thus d is odd. ■

Proposition 2.12. Let M be a Cohen-Macaulay module, with $d(M) = 2$, and $e(M) = 1$. Then M is a shift of a line module.

Proof. As usual we shift the grading and assume that $M = M_0 \oplus M_1 \oplus \cdots$, and $M_0 \neq 0$. Write the minimal resolution of M in the form

$$0 \longrightarrow \bigoplus_i A[-i]^{b_i} \xrightarrow{\partial_1} \bigoplus_i A[-i]^{a_i} \xrightarrow{\partial_0} \bigoplus_i A[-i]^{c_i} \xrightarrow{\varphi} M \longrightarrow 0.$$

By the Remark after (1.12), the minimal resolution of $M^\vee$ is

$$0 \longrightarrow \bigoplus_i A[i]^{c_i} \xrightarrow{\partial_0^\vee} \bigoplus_i A[i]^{a_i} \xrightarrow{\partial_1^\vee} \bigoplus_i A[i]^{b_i} \xrightarrow{\varepsilon^\vee} M^\vee \longrightarrow 0$$

Notice that c_i, resp. b_i, is the number of elements of degree i, resp. $-i$, in a minimal set of homogeneous generators for M, resp. $M^\vee$. Since $M = \bigoplus_{n\geq 0} M_n$ it is easily seen that $a_0 = b_0 = b_1 = 0$, and $a_i = b_i = c_i$ for all $i < 0$. Thus $h_M(t) == (1-t)^{-4}\sum_{i\geq 0}(b_i - a_i + c_i)t^i$.

By (2.11b), there is an odd integer $d \in \mathbb{N}$, such that

$$\begin{aligned} h_M(t) &= \frac{1+t^d}{(1-t)^2(1+t)} \\ &= 1 + t + 2t^2 + 2t^3 + \cdots \\ &\qquad + (\tfrac{d-1}{2})t^{d-3} + (\tfrac{d-1}{2})t^{d-2} + (\tfrac{d+1}{2})t^{d-1} + (\tfrac{d+3}{2})t^d + \cdots . \end{aligned}$$

We first have to show that $d = 1$, or equivalently that $h_M(t) = (1-t)^{-2}$. Suppose to the contrary that $d \geq 3$. Then $dimM_0 = 1$, and $dimM_2 = 2$. Since $A_2M_0 \neq 0$ it follows that $c_2 \leq 1$. We will show in the next paragraph that $b_2 \leq 1$, but assume for the moment that this is true. By comparing the coefficient of t^2 in the two expressions for $h_M(t)$ above, it follows that $b_2 - a_2 + c_2 = 4$. Therefore $4 \leq a_2 + 4 = b_2 + c_2 \leq 2$ which is absurd. Thus $d = 1$.

Now we prove that $b_2 \leq 1$. First we have

$$\begin{aligned} h_{M^\vee}(t) &= t^{-4}h_M(t^{-1}) \\ &= t^{-(d+1)}h_M(t) \\ &= t^{-(d+1)} + t^{-d} + 2t^{-(d-1)} + \cdots \\ &\qquad + (\tfrac{d-1}{2})t^{-4} + (\tfrac{d-1}{2})t^{-3} + (\tfrac{d+1}{2})t^{-2} + \cdots . \end{aligned}$$

By (1.10) $M^\vee$ is Cohen-Macaulay, $GKdimM^\vee = 2$ and $e(M^\vee) = 1$. By (2.11a) applied to $M^\vee$, either Ω_1 or Ω_2 is a non-zero divisor in $M^\vee$. Let Ω be this element. Let $w_1, \cdots, w_{b_2}$ be elements of $M^\vee_{-2}$ which are part of a minimal set of homogeneous generators for $M^\vee$. Then the sum $\Omega M^\vee_{-4} + \sum_{1\leq i\leq b_2}\mathbb{C}w_i$ must be direct and $dimM^\vee_{-4} = dim\Omega M^\vee_{-4}$. Hence $b_2 + dimM^\vee_{-4} \leq dimM^\vee_{-2}$. Thus $b_2 \leq dimM^\vee_{-2} - dimM^\vee_{-4} = \frac{d+1}{2} - \frac{d-1}{2} = 1$.

It remains to prove that M is cyclic. Since $M \cong (M^\vee)^\vee$, it is enough to show that $M^\vee$ is cyclic. Since $h_{M^\vee}(t) = t^{-2} + 2t^{-1} + \cdots$, it follows that $b_2 = 1$, and $b_i = 0$ if $i \geq 3$. But $b_0 = b_1 = 0$, so $M^\vee$ is cyclic generated by an element of degree 2 (cf (2.9a)). ■

Proposition 2.13. Let M be a point module. Then $M \cong A/Au + Av + Aw$ for some $u, v, w \in A_1$. In fact a minimal projective resolution of M has the form

$$0 \longrightarrow A[-3] \xrightarrow{\partial_2} A[-2]^3 \xrightarrow{\partial_1} A[-1]^3 \xrightarrow{\partial_0} A[0] \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Proof. Take a minimal resolution for M, say

$$0 \longrightarrow \bigoplus_i A[-i]^{d_i} \xrightarrow{\partial_2} \bigoplus_i A[-i]^{c_i} \xrightarrow{\partial_1} \bigoplus_i A[-i]^{b_i} \xrightarrow{\partial_0} \bigoplus_i A[-i]^{a_i} \xrightarrow{\varepsilon} M \longrightarrow 0.$$

The dual complex is a minimal resolution of $M^\vee$ by the remark after (1.12). Recall that $M^\vee$ is a shifted point module, generated in degree -3. Because M and $M^\vee$ are cyclic, $a_0 = d_3 = 1$ and all other a_i and d_i are zero. As usual we have $H_M(t) = H_A(t).\sum_i (a_i - b_i + c_i - d_i)t^i$ and thus $\sum_i(-b_i + c_i)t^i = -3t + 3t^2$. Because $dim(M_1) = dim(A_1) - 3$ and the resolution is minimal it follows that $b_1 = 3$; the same argument applied to $M^\vee$ shows that $c_2 = 3$. The minimality of the resolution ensures that $b_i = 0$ for $i \leq 0$ and $c_i = 0$ for $i \leq 1$. Therefore $\sum_{i\geq 2} b_i t^i = \sum_{i \geq 3} c_i t^i$. In particular $b_2 = 0$ so it remains to show that for all $i \geq 3$ we have $b_i = c_i = 0$. Suppose this is not the case.

Because A is noetherian, and M is finitely generated we can set $k := max\{i \mid b_i \neq 0\}$. Thus $k \geq 3$. Write

$$F' := \bigoplus_{i<k} A[-i]^{c_i}, \quad G' := A[-k]^{c_k}, \quad F := \bigoplus_{i<k} A[-i]^{b_i}, \quad G := A[-k]^{b_k}.$$

Thus $\partial_1 : F' \oplus G' \to F \oplus G$. Since $deg(\partial_1) = 0$ and $G_m = 0$ for $m < k$ it follows that $\partial_1(F') \subset F$. Since the resolution is minimal, $\partial_1(G') \subset \mathfrak{M}F \oplus \mathfrak{M}G$. But G' is generated in degree k and $\mathfrak{M}G$ is zero in degree k so $\partial_1(G') \subset F$. Thus $\partial_1(F' \oplus G') \subset F$.

On the other hand $\bigoplus_i A[-i]^{a_i} \cong A[0]$ so $\partial_0(F)$ and $\partial_0(G)$ are left ideals of A. The minimality of the resolution ensures they are both non-zero. Since A is a noetherian domain it follows that $\partial_0(F) \cap \partial_0(G) \neq 0$. Hence there exists $0 \neq f \in F$ and $0 \neq g \in G$ such that $\partial_0(f) = \partial_0(g)$. It follows that $f - g \in ker(\partial_0) = Im(\partial_1)$ but $f - g \notin F$. This contradicts the conclusion of the previous paragraph, so the result follows. ■

§3. Quadrics and secant lines.

This section gives proofs of the following properties of the quadrics which contain E, and the secant lines of E. There is a pencil of quadric hypersurfaces containing E, and each point of $\mathbb{P}^3 \backslash E$ belongs to a unique quadric in this pencil. Each line on one of these quadrics is a secant line of E, and every secant line lies on some (in fact, a unique) quadric in the pencil. There are exactly 4 singular quadrics in the pencil, say Q_j $(0 \leq j \leq 3)$. Each Q_j may be characterized as the unique quadric which contains E and e_j where e_j is as defined in §1.1. Furthermore, Q_j is defined by a rank 3 quadratic form so has a unique singular point, and that point is e_j. If $p \in Q_j$ then the line through p and e_j is contained in Q_j, so e_j lies on infinitely many secant lines of E, and Q_j is the union of all the secant lines of E which pass through e_j. In contrast any point in $\mathbb{P}^3 \backslash (E \cup \mathcal{S})$ lies on at most two secant lines of E. Thus the points e_j are in very special position relative to E: they are the only points in $\mathbb{P}^3 \backslash E$ lying on infinitely many secant lines of E.

The map $\mathbb{C}/\Lambda \to E$ described in the introduction fixes the group law on E, and the identity element $0 \in E$. The group law is related to the geometry of the secant lines. Since E is a degree 4 curve, being an intersection of two quadrics, any hyperplane in $\mathbb{P}^3$ meets E at 4 points (counted with multiplicity). The sum of these 4 points of E is zero, and conversely, if 4 points of E sum to zero then they are precisely the points of intersection (counted with multiplicity) of some hyperplane with E. Since E is not contained in any hyperplane, E has no trisecants, so a secant line meets E at two points (counted with multiplicity, where $p \in \ell \cap E$ has multiplicity two if ℓ is tangent to E at p). If $p, q \in \mathbb{P}^3$ we will write ℓ_{pq} for the line through p and q. The subgroup E_2 of E, of points of order 2, contains 4 elements. These elements may be labeled as ω_j $(0 \leq j \leq 3)$ in such a way that if $p, q \in E$, then ℓ_{pq} passes through $e_j \iff p + q = \omega_j$. In fact, Q_j is the union of all the secant lines ℓ_{pq} such that $p + q = \omega_j$. Now suppose that Q is a smooth quadric in the pencil. Hence $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there are two families of lines on Q (all of which are secant lines of E). There is a point $z \in E$ (determined by Q up to sign) such that if $p, q \in E$, then ℓ_{pq} lies on $Q \iff p + q = \pm z$. Moreover, all those ℓ_{pq} such that $p + q = z$ do not intersect, and ℓ_{pq} intersects $\ell_{p'q'} \iff p + q = -(p' + q')$.

All the results in this section are straightforward and rather elementary. In fact they are all well-known to the average 19th century geometer; see for example [16, Art. 347]. Hence the point of this section is to state these facts, and to provide proofs for the convenience of some readers.

If $0 \neq x \in A_1$, then E meets the hyperplane $x = 0$ at 4 points, counted with multiplicity, and the corresponding degree 4 divisor on E will be denoted by $(x)_0 = p_1 + p_2 + p_3 + p_4$. There will be potential for confusion, since the sum of points of E, in the group law on E, will also be denoted by $p_1 + p_2 + p_3 + p_4$. The context should make it clear whether $+$ denotes addition in E or in $Div E$.

Lemma 3.1. Let $p_1, p_2, p_3, p_4 \in E$. Then there exists $0 \neq u \in A_1$ such that $(u)_0 = p_1 + p_2 + p_3 + p_4$ if and only if the sum $p_1 + p_2 + p_3 + p_4$ in $(E, +)$ is zero.

Proof. By Abel's theorem if $p_1, \ldots, p_r, q_1, \ldots, q_r \in E$, then $\mathcal{O}(p_1 + \ldots + p_r) \cong \mathcal{O}(q_1 + \ldots + q_r) \Longleftrightarrow p_1 + \ldots + p_r = q_1 + \ldots + q_r$ (on the LHS + is in Div$E$, on the RHS + is in E). The 4 points of E which lie on the plane $x_0 = 0$ are the four $z \in \mathbb{C}/\Lambda$, such that $\theta_{11}(z) = 0$, namely $0, \frac{1}{2}, \frac{1}{2}\eta, \frac{1}{2} + \frac{1}{2}\eta$ (the points of E_2). Hence, if $j : E \to \mathbb{P}^3$ is the inclusion, then $j^*\mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}(0 + (\frac{1}{2}) + (\frac{1}{2}\eta) + (\frac{1}{2} + \frac{1}{2}\eta))$. Therefore, if $p_1, p_2, p_3, p_4 \in E$, then they are the points of intersection of a hyperplane and $E \Longleftrightarrow \mathcal{O}(p_1 + p_2 + p_3 + p_4) \cong j^*\mathcal{O}_{\mathbb{P}^3}(1) \Longleftrightarrow p_1 + p_2 + p_3 + p_4 = 0 + \frac{1}{2} + \frac{1}{2}\eta + (\frac{1}{2} + \frac{1}{2}\eta) = 0$. ■

Remark. It is easy to see that E has no trisecants. Suppose to the contrary that ℓ is a trisecant. Since $E = X \cap Y$ is an intersection of quadrics, the defining equations of X and Y restricted to ℓ are quadratic forms so either have two zeroes or vanish identically. However, by hypothesis they must have at least 3 zeroes, so we are forced to conclude that $\ell \subseteq X \cap Y = E$. This is absurd.

Let g_1 and g_2 be any two (linearly independent) quadratic forms which define E. To be definite we can take g_1 and g_2 to be the functions so labelled in §1.1. For each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{P}^1$, define $Q_\lambda := \mathcal{V}(\lambda_1 g_1 + \lambda_2 g_2)$. This gives a pencil of quadrics, each of which contains E.

Proposition 3.2.

(a) If Q is a quadric containing E, then $Q = Q_\lambda$ for some $\lambda \in \mathbb{P}^1$.
(b) If $p \in \mathbb{P}^3 \backslash E$, then there is a unique $\lambda \in \mathbb{P}^1$ such that $p \in Q_\lambda$.
(c) If $\lambda \neq \mu$, then $Q_\lambda \cap Q_\mu = E$.
(d) If ℓ is a secant line of E, then there is a unique Q_λ such that $\ell \subset Q_\lambda$.
(e) If ℓ is a line lying on a quadric Q containing E then ℓ is a secant line of E.

Proof. (a) By [21, 2.5], the polynomial ring modulo $\langle g_1, g_2 \rangle$ is reduced. In particular, if g is a quadratic form vanishing on E, then $g \in \sqrt{\langle g_1, g_2 \rangle} = \langle g_1, g_2 \rangle$, so g is a linear combination of g_1 and g_2.

(b) Since $p \notin E$, either $g_1(p) \neq 0$ or $g_2(p) \neq 0$. Hence there is a unique $\lambda = (\lambda_1, \lambda_2) \in \mathbb{P}^1$ such that $(\lambda_1 g_1 + \lambda_2 g_2)(p) = 0$. Of course (c) is an immediate consequence of (b).

(d) The uniqueness of such a Q_λ is guaranteed by (b). Set $\{p, q\} = \ell \cap E$ (counted with multiplicity), and note that each g_λ vanishes at both p and q. Now fix $r \in \ell \backslash E$, and note that either $g_1(r) \neq 0$ or $g_2(r) \neq 0$. So there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbb{P}^1$ such that $\lambda_1 g_1(r) + \lambda_2 g_2(r) = 0$. But then $g_\lambda = \lambda_1 g_1 + \lambda_2 g_2$ vanishes at 3 points of ℓ, and hence on all of ℓ since $deg(g_\lambda) = 2$.

(e) Let g be a quadratic form vanishing on E but not on Q. Then $\ell \cap E$ is given by the zeroes of the restriction of g to ℓ. Since there are two such zeroes, ℓ meets E with multiplicity 2. ■

Lemma 3.3. Let $p, q, r, s \in E$ be such that ℓ_{pq} and ℓ_{rs} are distinct. Then $\ell_{pq} \cap \ell_{rs} \neq \emptyset$ if and only if $p + q = -(r + s)$.

Proof. There is a plane containing both ℓ_{pq} and ℓ_{rs} if and only if $\ell_{pq} \cap \ell_{rs} \neq \emptyset$. This happens if and only if p, q, r, s are coplanar. The result now follows from (3.1). ■

Proposition 3.4.

(a) For each $j = 0, 1, 2, 3$ there is a unique quadric, Q_j say, which contains e_j and E.

(b) Each Q_j is singular, of rank 3, and these are the only singular quadrics in the pencil of quadrics which contain E.

(c) The only singular point on Q_j is e_j.

(d) If $p \in Q_j$ then the line ℓ_{pe_j} lies on Q_j. Furthermore, every line on Q_j passes through e_j, and Q_j is the union of the lines it contains.

(e) For each j, Q_j is the union of all those secant lines of E which pass through e_j.

Proof. (a) This is a special case of (3.2b).

(b) Since E is not contained in any hyperplane, E can not be contained in any rank 2 quadric. Thus the defining equation of Q_j is the unique (up to scalar multiple) linear combination of g_1 and g_2 in which the coefficient of x_j^2 is zero. This is the equation denoted by g_j in §1.1.

A quadratic form q on $\mathbb{P}^n$, defines a singular variety $\Leftrightarrow$ $rank(q) < n + 1$. Hence a pencil $\lambda_1 q_1 + \lambda_2 q_2$ generated by two quadratic forms, contains $n + 1$ singular forms (given by the zeroes of the determinant of an $(n + 1) \times (n + 1)$ matrix whose entries are linear in λ_1, λ_2). Hence in our situation there are 4 such, and these must be the Q_j.

(c) Since $rank(q_j) = 3$ it has a unique singular point. Indeed, the singular point of $\mathcal{V}(y_1^2 + y_2^2 + y_3^2)$ in coordinates y_0, y_1, y_2, y_3 is $(1, 0, 0, 0)$.

(d) Let Q be a rank 3 quadric in $\mathbb{P}^3$ defined by $y_1^2 + y_2^2 + y_3^2$ in suitable coordinates y_0, y_1, y_2, y_3. Then e=$(1, 0, 0, 0)$ is the unique singular point of Q. Let $p = (p_0, p_1, p_2, p_3)$ and $q = (q_0, q_1, q_2, q_3)$ be distinct points of Q such that ℓ_{pq}, the line through p and q, lies on Q. Then for each $(s, t) \in \mathbb{P}^1$, $sp+tq \in Q_j$. In particular, this gives $st(p_1q_1+p_2q_2+p_3q_3) = 0$ because $p, q \in Q_j$. Thus $p_1q_1 + p_2q_2 + p_3q_3 = 0$. Hence $\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = 0.$
If $rank \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} = 2$, then its transpose is also of rank 2, and it is impossible for such a product to be zero. Hence the rank is $\leq$1. So we can choose $s, t \in \mathbb{P}^1$ such that $sp_1 + tq_1 = sp_2 + tq_2 = sp_3 + tq_3 = 0$. Therefore $sp + tq = (1, 0, 0, 0)$, so e lies on ℓ_{pq}.

To see that Q is the union of the lines it contains, suppose that $p = (p_0, p_1, p_2, p_3) \in Q$ and $p \neq e$. Then a typical point of the line ℓ_{ep} through e and p is $(sp_0 + t, sp_1, sp_2, sp_3)$ for $(s, t) \in \mathbb{P}^1$. It is clear that $y_1^2 + y_2^2 + y_3^2$ vanishes at this point, so $\ell_{ep} \subseteq Q$. In particular, p lies on a line contained in Q.

(e) If $p, q \in E$ and ℓ_{pq} passes through e_j, then the unique quadric which contains ℓ_{pq} (such a quadric exists by (3.2d)) also contains e_j. But the only quadric containing e_j is

Q_j, so $\ell_{pq} \subseteq Q_j$. ■

Corollary 3.5 If $p \in E$, then the line ℓ_{pe_j} is a secant line of E. In particular, each e_j lies on infinitely many secant lines of E.

Notation. The points of E_2 are labelled as follows:

$$\omega_0 = 0, \quad \omega_1 = \tfrac{1}{2} + \tfrac{1}{2}\eta, \quad \omega_2 = \tfrac{1}{2}\eta, \quad \omega_3 = \tfrac{1}{2}.$$

Proposition 3.6. Set $\zeta_0 = 0$, $\zeta_1 = \frac{1}{4} + \frac{1}{4}\eta$, $\zeta_2 = \frac{1}{4}\eta$, $\zeta_3 = \frac{1}{4}$. Then

(a) the cosets of E_2 in E_4 are $\zeta_j + E_2 \quad (j = 0, 1, 2, 3)$;

(b) if $p \in E$, then ℓ_{pe_j} is tangent to E at $p \Longleftrightarrow p \in \zeta_j + E_2 \Longleftrightarrow 2p = \omega_j$.

Proof. The zeroes of g_{ab} occur at $\frac{1+a}{4}\eta + \frac{1-b}{4} + \{0, \frac{1}{2}, \frac{1}{2}\eta, \frac{1}{2} + \frac{1}{2}\eta\} + \Lambda$. Hence $\zeta_0 + E_2$ is the zero set of g_{11}, $\zeta_1 + E_2$ is the zero set of g_{00}, $\zeta_2 + E_2$ is the zero set of g_{01}, and $\zeta_3 + E_2$ is the zero set of g_{10}. It is clear that (a) is true.

Let $p = (p_0, p_1, p_2, p_3) \in E_2$. The line ℓ_{pe_0} is tangent to E at $p \Leftrightarrow \ell_{pe_0}$ meets E at p alone. Since $e_0 = (1, 0, 0, 0)$, ℓ_{pe_0} consists of the points $\{(sp_0 + t, sp_1, sp_2, sp_3) \mid (s, t) \in \mathbb{P}^1\}$. It follows from the defining equations of E, that such a point of ℓ_{pe_0} will lie on E if and only if $2stp_0 + t^2 = 0$. Hence ℓ_{pe_0} is tangent to E at $p \Leftrightarrow$ the only solution $(s, t) \in \mathbb{P}^1$ to the equation $2stp_0 + t^2 = 0$ is $(s, t) = (1, 0) \Leftrightarrow p_0 = 0 \Leftrightarrow p = j(z)$ where z is a zero of $g_{11} \Leftrightarrow p \in \zeta_0 + E_2 = E_2$.

This proves (b) for $j = 0$. The proof for $j = 1, 2, 3$ is similar. Finally, since $2\zeta_j = \omega_j$, it is clear that $p \in \zeta_j + E_2 \Leftrightarrow 2p = \omega_j$. ■

Proposition 3.7. Let $p, q \in E$. The line ℓ_{pq} passes through e_j if and only if $p+q = \omega_j$; that is, $\ell_{pq} \subset Q_j \Leftrightarrow p + q = \omega_j$.

Proof. $(\Rightarrow)$ Suppose that $e_j \in \ell_{pq}$. If $p = \zeta_j$, then (3.6b) implies that $q = \zeta_j$ also, whence $p + q = 2\zeta_j = \omega_j$. Suppose that $p \neq \zeta_j$. Then the lines ℓ_{pq} and $\ell_{\zeta_j e_j}$ are distinct and intersect at e_j. Hence they are contained in a single plane. Therefore, by (3.1), $p + q + \zeta_j + \zeta_j = 0$, whence $p + q = \omega_j$.

$(\Leftarrow)$ Suppose that $p + q = \omega_j$. If $p = \zeta_j$, then $q = \zeta_j$ too, so ℓ_{pq} is tangent to E at ζ_j, and (3.6b) shows this line passes through e_j. Suppose that $p \neq \zeta_j$. The three points p, q, ζ_j lie on a common plane, H say. By (3.1) H meets E at a fourth point, namely $-(p + q + \zeta_j) = \zeta_j$. Hence H meets E at ζ_j with multiplicity 2, so H contains the tangent line to E at ζ_j. But this tangent line passes through e_j, so $e_j \in H$, whence $\ell_{pe_j} \subset H$. But ℓ_{pe_j} is a secant line, so meets E at another point of $E \cap H$. Since $p \neq \zeta_j$, that other point must be q. Hence $\ell_{pe_j} = \ell_{pq}$, and $e_j \in \ell_{pq}$. ■

Corollary 3.8. A line through two distinct points of $\mathcal{S}$ is not a secant line of E.

Let $Q \subseteq \mathbb{P}^3$ be a smooth quadric. Then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ (via the Segre embedding), so $Pic(Q) \cong \mathbb{Z} \bigoplus \mathbb{Z}$, where the hyperplane section is (1,1) and the intersection pairing is given by $(m,n)\cdot(m',n') = mn' + nm'$. There are two families of lines on Q, corresponding under the isomorphism to those of the form $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$. Lines in the first family correspond to (1,0) in $Pic(Q)$, and those in the second to $(0,1)$. Each point of Q lies on exactly two lines, one from each family. If Q' is any other rank 4 quadric in $\mathbb{P}^3$, then $Q \cap Q'$ is a curve of class (2,2) in $Pic(Q)$. Hence any line on Q meets $Q \cap Q'$ at two points.

Lemma 3.9. If $p \in \mathbb{P}^3 \backslash (E \cup \mathcal{S})$ then p lies on at most two secant lines of E. In fact if p lies on a smooth quadric containing E then p lies on two secant lines, whereas if p lies on a singular quadric then p lies on one secant line.

Proof. Let Q be the quadric containing p and E. If Q is singular then it follows from (3.4) that p lies on a unique secant line. If Q is smooth, then by the above comments, p lies on two lines contained in Q, say ℓ and ℓ'. Since $E = (2,2)$ in $Pic(Q)$, both ℓ and ℓ' are secant lines of E. If ℓ'' is any secant line through p, then the quadric containing ℓ'' also contains p, so by uniqueness must be Q. Thus $\ell'' \subset Q$, so ℓ'' is either ℓ or ℓ'. ■

Theorem 3.10. Fix $r, s \in E$ such that $r + s \notin E_2$. Let Q be the unique quadric containing E and ℓ_{rs}. Then

(a) Q is smooth;
(b) every line on Q is a secant line of E;
(c) if ℓ is a line on Q with $\ell \cap E = \{p, q\}$, then
   (i) $\ell \cap \ell_{rs} = \emptyset \Leftrightarrow p + q = r + s$,
   (ii) $\ell \cap \ell_{rs} \neq \emptyset \Leftrightarrow p + q = -(r + s)$;
(d) if $p, q \in E$, then $\ell_{pq} \subset Q \Leftrightarrow$ either $p + q = r + s$, or $p + q = -(r + s)$.

Proof. (a) Since $r + s \in E_2$, ℓ_{rs} is not contained in any of the four singular quadrics Q_j by (3.7b). Hence Q is smooth.

(b) This is contained in the remarks prior to (3.9).

(c)(ii) This is (3.3).

(c)(i) ($\Leftarrow$). Since $r + s \notin E_2$, $p + q = r + s \Rightarrow p + q \neq -(r + s)$ so by (ii), $\ell_{pq} \cap \ell_{rs} = \emptyset$.

(c)(i) ($\Rightarrow$). Since $\ell \cap \ell_{rs} = \emptyset$, ℓ and ℓ_{rs} give the same element of $Pic(Q)$. Hence there is a line ℓ' on Q such that $\ell' \cdot \ell = \ell' \cdot \ell_{rs} = 1$. Hence ℓ and ℓ' are coplanar; let $a = 0$ be the plane containing ℓ and ℓ'. Similarly, there is a plane $b = 0$ containing ℓ_{rs} and ℓ'. If $\ell' \cap E = \{z, w\}$, it follows that the divisors on E consisting of the zeroes of a and b are $(a)_0 = p + q + z + w$, and $(b)_0 = r + s + z + w$. Hence the divisor of the rational function $\frac{a}{b} \in \mathbb{C}(E)$ is $div(\frac{a}{b}) = p + q - r - s$. By Abel's Theorem $p + q = r + s$ in E.

(d) ($\Rightarrow$) This follows at once from (c).

(d) ($\Leftarrow$). Since $p \in E \subset Q$, and Q is the union of the lines it contains, there is a line on Q passing through p. In fact there are two such lines, ℓ' and ℓ'' say, and they satisfy $\ell' \cdot \ell_{rs} = 1$ and $\ell'' \cdot \ell_{rs} = 0$.

Suppose that $p + q = r + s$. Since $\ell'' \cap \ell_{rs} = \emptyset$, if $\ell'' \cap E = \{p, t\}$ then $p + t = r + s$ by (c)(i). Therefore $t = q$, so $\ell'' = \ell_{pq}$, and $\ell_{pq} \subset Q$.

Suppose that $p+q=-(r+s)$. Since $\ell' \cap \ell_{rs} = \emptyset$, if $\ell' \cap E = \{p, t\}$ then $p+t=-(r+s)$ by (c)(ii). Therefore $t=q$, so $\ell' = \ell_{pq}$, and $\ell_{pq} \subset Q$. ■

Corollary 3.11. Given any quadric $Q_\lambda, \lambda \in \mathbb{P}^1$, in the pencil of quadrics containing E, there exists $z_\lambda \in E$ such that, if $p, q \in E$ then $\ell_{pq} \subset Q_\lambda \Leftrightarrow p+q = \pm z_\lambda$.

Proof. Combine (3.4), (3.7) and (3.10). ■

§4. Geometric Classification of Line Modules.

The main result in this section is that the line modules are in bijection with the secant lines of E.

Throughout this section u and v will denote linearly independent elements of A_1, and a and b will denote non-zero elements of A_1. We shall make frequent use of (1.1) and (1.2).

Proposition 4.1. If $A/Au + Av$ is a line module, then $\mathcal{V}(u, v)$ is a secant line of E.

Proof. By (2.8), $A_1 u \cap A_1 v \neq 0$, so there exist non-zero $a, b \in A_1$ such that $a \otimes v - b \otimes u \in I_2$. Since the proposition is only concerned with the subspace of A_1 spanned by u and v, we may replace u and v by any linear combinations which are themselves linearly independent. In particular, we can assume that there exist distinct $e_j, e_k \in \mathcal{S}$ such that $u(e_j) = v(e_k) = 0$ and $u(e_k) \neq 0, v(e_j) \neq 0$. By (1.1) $a \otimes v - b \otimes u$ vanishes at (e_j, e_j) and (e_k, e_k), so $a(e_j) = b(e_k) = 0$.

Let $p \in E$. Since $u(e_j) = 0$, if $u(p) = 0$ then so too is $u(\omega_j - p) = 0$ by (3.9). Hence there exist $p_1, p_2, q_1, q_2 \in E$ such that the zeroes of u and v on E are given by

$$(*) \begin{cases} (u)_0 = (p_1) + (\omega_j - p_1) + (p_2) + (\omega_j - p_2) \\ (v)_0 = (q_1) + (\omega_k - q_1) + (q_2) + (\omega_k - q_2). \end{cases}$$

Suppose that u and v have no common zero on E. Since $a(p)v(p+\tau) - b(p)u(p+\tau) = 0$ for all $p \in E$, taking p to be each of the zeroes of u in turn, it follows that $(a)_0 = (p_1 - \tau) + (\omega_j - p_1 - \tau) + (p_2 - \tau) + (\omega_j - p_2 - \tau)$. Since these 4 points are coplanar, their sum is zero. Hence $4\tau = 0$. But this contradicts the fact that $\tau \notin E_4$. Hence u and v have at least one common zero.

Suppose that u and v have exactly one common zero on E. With a change of notation if necessary, we may assume that $p_1 = q_1$. Therefore $\{\omega_j - p_1,\ p_2,\ \omega_j - p_2\} \cap \{\omega_k - p_1,\ q_2,\ \omega_k - q_2\} = \emptyset$. It will be useful to rewrite this as

$$(\dagger) \qquad \{\omega_j - p_1 - \tau,\ p_2 - \tau,\ \omega_j - p_2 - \tau\} \cap \{\omega_k - p_1 - \tau,\ q_2 - \tau,\ \omega_k - q_2 - \tau\} = \emptyset.$$

Since $a(p-\tau)v(p) - b(p-\tau)u(p) = 0$ for all $p \in E$, evaluating this at the points p which are zeroes of only one of u and v, it follows that a vanishes at $\omega_j - p_1 - \tau$, $p_2 - \tau$, $\omega_j - p_2 - \tau$, and b vanishes at $\omega_k - p_1 - \tau$, $q_2 - \tau$, $\omega_k - q_2 - \tau$. Both a and b have a fourth zero on E which can be determined from the other three zeroes by the fact that four points are coplanar if and only if their sum is zero. Hence

$$(a)_0 = (\omega_j - p_1 - \tau) + (p_2 - \tau) + (\omega_j - p_2 - \tau) + (p_1 + 3\tau)$$
$$(b)_0 = (\omega_k - p_1 - \tau) + (q_2 - \tau) + (\omega_k - q_2 - \tau) + (p_1 + 3\tau).$$

Since $a(e_j) = 0$, it follows that $a(p) = 0 \Leftrightarrow a(\omega_j - p) = 0$, for $p \in E$. Hence

$$(1) \ \{\omega_j - p_1 - \tau, p_2 - \tau, \omega_j - p_2 - \tau, p_1 + 3\tau\} = \{p_1 + \tau, \omega_j - p_2 + \tau, p_2 + \tau, \omega_j - p_1 - 3\tau\}.$$

Similarly, since $b(e_k) = 0$, it follows that

(2) $\{\omega_k - p_1 - \tau, q_2 - \tau, \omega_k - q_2 - \tau, p_1 + 3\tau\} = \{p_1 + \tau, \omega_k - q_2 + \tau, q_2 + \tau, \omega_k - p_1 - 3\tau\}$.

Since $p_1 + \tau$ belongs to the right hand side of both (1) and (2), it belongs to the intersection of the left hand side of (1) with the left hand side of (2). However, by (†) this intersection is $\{p_1 + 3\tau\}$. Hence $p_1 + \tau = p_1 + 3\tau$, whence $2\tau = 0$. This contradicts the fact that $\tau \notin E_4$.

Hence u and v have at least two common zeroes on E. ■

Lemma 4.2. Let $u, v \in A_1$ define a secant line $\ell = \mathcal{V}(u, v)$ of E. Suppose that the divisor of the rational function $\frac{u}{v}$ on E is $div(\frac{u}{v}) = (x + \tau) + (y + \tau) - (w + \tau) - (z + \tau)$.

(a) Define a symmetric and transitive relation $\sim$ on E, by $s \sim t$ if $s + t = -(x + y)$. If $s \sim t$, then there exists a unique plane $a = 0$, such that $(a)_0 = x + y + s + t$.

(b) If $0 \neq a \in A_1$ is such that $(a)_0 = x + y + s + t$, then there exists a unique $b \in A_1$ such that $a \otimes v - b \otimes u$ vanishes on Δ_τ; furthermore $(b)_0 = w + z + s + t$.

(c) Let $\approx$ denote the equivalence relation generated by $\sim$ i.e. $s \approx t$ if either $s = t$ or $x + y + s + t = 0$; then there is a bijective map

$$\varphi : E/\approx \ \longrightarrow \{(a, b) \in \mathbb{P}(A_1 \times A_1) \mid (a \otimes v - b \otimes u)(\Delta_\tau) = 0\},$$

given by $\varphi(s) = (a, b)$ where $a \in A_1$ satisfies $(a)_0 = x + y + s + (-x - y - s)$ and b is determined as in (b).

Proof. (a) If $s \sim t$, then $x + y + s + t = 0$, so there certainly exists $0 \neq a \in A_1$ (unique up to non-zero scalar multiple) such that $(a)_0 = x + y + s + t$.

(b) By Abel's Theorem $x + y = w + z$, so $w + z + s + t = 0$. Hence there exists $b \in A_1$ such that $(b)_0 = w + z + s + t$. Notice that b is only determined up to a (non-zero) scalar multiple. Both $a \otimes v$ and $b \otimes u$ vanish at the six points $(x, x + \tau)$, $(y, y + \tau)$, $(s, s + \tau)$, $(t, t + \tau)$, $(w, w + \tau)$, and $(z, z + \tau)$.

By [21, 2.8], there are cubic forms f and g on E such that $(\frac{u}{v})^\sigma = \frac{f}{g}$; that is, $(\frac{f}{g})(p) = \left(\frac{u}{v}\right)(p + \tau)$ for all $p \in E$. Since E is a degree 4 curve in $\mathbb{P}^3$, and f and g are forms of degree 3, both f and g have 12 zeroes on E. However, since $div\left(\frac{u}{v}\right)$ is as above, $div\left(\frac{f}{g}\right) = x + y - z - w$, so f and g must have 10 common zeroes on E; let these common zeroes be $p_1, \cdots, p_{10}$. Hence both ag and bf vanish at the 16 points $x, y, s, t, w, z, p_1, \cdots, p_{10}$ of E. Replace b by a suitable scalar multiple of itself, such that $ag - bf$ vanishes at some seventeenth point of E. Since the degree 4 form $ag - bf$ now has 17 zeroes on E, it is identically zero on E. Hence $(a \otimes v - b \otimes u)(\Delta_\tau) = 0$.

If some other $b' \in A_1$ also satisfies $(a \otimes v - b' \otimes u)(\Delta_\tau) = 0$, then $((b - b') \otimes u)(\Delta_\tau) = 0$. If $b \neq b'$, then $\Delta_\tau \subseteq \mathbb{P}^3 \times \mathcal{V}(u)$, which is absurd since E is not contained in a hyperplane.

(c) There certainly is such a map φ. The injectivity of φ is clear: if $\varphi(s) = \varphi(s')$, then $(a)_0 = (a')_0$, whence $\{s, t\} = \{s', t'\}$, so the equivalence classes of s and s' coincide. To see that φ is surjective, suppose that $a, b \in A_1$ are such that $(a \otimes v - b \otimes u)(\Delta_\tau) = 0$. Evaluating

$a \otimes v - b \otimes u$ at the points $(x, x+\tau)$ and $(y, y+\tau)$ gives $a(x)v(x+\tau) = a(y)v(y+\tau) = 0$, whence $a(x) = a(y) = 0$. So $(a)_0 = x + y + s + t$ for some s, t. ■

Remarks. 1. By (1.2), the element $av - bu$ determined by (4.2) is in the center of A, so annihilates the line module $A/Av + Au$.

2. We may restate part of (4.2) as follows. If $\mathcal{V}(u, v) = \ell_{pq}$ and $a, b \in A_1$ satisfy $(a \otimes v - b \otimes u)(\Delta_\tau) = 0$, then $\mathcal{V}(a, b)$ is a secant line ℓ_{rs} for some r, s such that $r + s = p + q + 2\tau$, and every such r and s may occur.

Lemma 4.3. Let $p, q \in E$. Let $u, v \in A_1$ be linearly independent. Then there exists at most one point $(a, b) \in \mathbb{P}(A_1 \times A_1)$ such that $(a \otimes v - b \otimes u)(\Gamma) = 0$.

Proof. By (1.2) $(a \otimes v - b \otimes u)(\Gamma) = 0 \Leftrightarrow av = bu$. Hence we must show that there is at most one $b \in \mathbb{P}(A_1)$ such that $bu \in Av$ (such b then uniquely determines a, and hence determines $(a, b) \in \mathbb{P}(A_1 \times A_1)$).

Consider $\overline{u} \in A/Av$. Since A/Av is a plane module, it is 3-critical. If $bu \in Av$, there is a surjective map $A/Ab \longrightarrow A\overline{u}$. Since A/Ab is also 3-critical, this map must be an isomorphism; hence b is unique up to (non-zero) scalar multiples. ■

Proposition 4.4. Let $p, q \in E$. Suppose that $u, v \in A_1$ are such that $\mathcal{V}(u, v) = \ell_{p+\tau, q+\tau}$ and $div(\frac{u}{v}) = (x+\tau) + (y+\tau) - (w+\tau) - (z+\tau)$. If $s \in E$, and $\varphi(s) = (a, b) \in \mathbb{P}(A_1 \times A_1)$ where φ is as in (4.2), then $av - bu \in A_2$ is a central element. Furthermore, $av - bu = 0$ if and only if $(a)_0 = x + y + (p + 2\tau) + (q + 2\tau)$. In that case $(b)_0 = w + z + (p + 2\tau) + (q + 2\tau)$.

Proof. By (4.2) and (1.2) $av - bu$ is central. By (4.3) there is at most one pair $(a, b) \in \mathbb{P}(A_1 \times A_1)$ such that $av = bu$. Hence it is enough to show that $(a, b) = \varphi(p + 2\tau)$ satisfies $(a \otimes v - b \otimes u)(\Gamma) = 0$. By (1.2) and (4.2) it therefore suffices to prove that $(a \otimes u - b \otimes v)(e_i, e_i) = 0$ for two $e_i \in \mathcal{S}$.

There are at least two i such that $p + 2\tau \neq \omega_i - p - 2\tau$ and $p + \tau \neq \omega_i - p - \tau$. For such an i, e_i lies on the secant lines $\ell_{p+2\tau, \omega_i - p - 2\tau}$ and $\ell_{p+\tau, \omega_i - p - \tau}$. Since $p + 2\tau \neq \omega_i - p - 2\tau$ and $p + \tau \neq \omega_i - p - \tau$ there are scalars λ, ξ, μ, ρ such that

$$e_i = \lambda(p + 2\tau) + \mu(\omega_i - p - 2\tau) = \xi(p + \tau) + \rho(\omega_i - p - \tau).$$

All these scalars are non-zero, since $e_i \notin E$. Now

$$\begin{aligned} a(e_i)v(e_i) - b(e_i)u(e_i) = \\ &\big(\lambda a(p + 2\tau) + \mu a(\omega_i - p - 2\tau)\big).\big(\xi v(p + \tau) + \rho v(\omega_i - p - \tau)\big) \\ &\quad - \big(\lambda b(p + 2\tau) + \mu b(\omega_i - p - 2\tau)\big).\big(\xi u(p + \tau) + \rho u(\omega_i - p - \tau)\big) \\ &\qquad = \mu\rho\big(a(\omega_i - p - 2\tau)v(\omega_i - p - \tau) - b(\omega_i - p - 2\tau)u(\omega_i - p - \tau)\big). \end{aligned}$$

But $(a \otimes v - b \otimes u)(\Delta_\tau) = 0$, so this expression is zero, and the result follows. ■

Theorem 4.5. The isomorphism classes of line modules are in bijection with the secant lines of E.

Proof. By (2.8) and (4.1), if M is a line module, then $M \cong A/Au + Av$ where $u, v \in A_1$ are such that $\ell{=}\mathcal{V}(u, v)$ is a secant line of E. Conversely, if $u, v \in A_1$ are such $\ell{:=}\mathcal{V}(u, v)$ is a secant line of E, then by (4.4) there exist $0{\neq}a, b{\in}A_1$ such that $av = bu \neq 0$, so $A_1 u \cap A_1 v \neq 0$. By (2.8) $A/Au + Av$ is a line module.

To show that distinct secant lines give non-isomorphic line modules, let $\mathcal{V}(u, v)$ and $\mathcal{V}(u', v')$ be distinct secant lines. If $\varphi : A/Au + Av \longrightarrow A/Au' + Av'$ is a (graded) isomorphism of the corresponding line modules, then φ maps the degree 0 part to the degree 0 part; say $\varphi(\overline{1}) = e \in (A/Au' + Av')_0$. But $Ann(\overline{1})_1 = \mathbb{C}u + \mathbb{C}v$ and $Ann(e)_1 = \mathbb{C}u' + \mathbb{C}v'$. However, since the lines are distinct $\mathbb{C}u + \mathbb{C}v \neq \mathbb{C}u' + \mathbb{C}v'$, so the line modules cannot be isomorphic. ■

Proposition 4.6. Let $M(p, q, r, s)$ be a plane module. Then there is a short exact sequence $0 \to M(p + \tau, q + \tau, r - \tau, s - \tau)[-1] \to M(p, q, r, s) \to M(p, q) \to 0$.

Proof. Let $u, v \in A_1$, be such that $(u)_0 = p + q + r + s$, and $\mathcal{V}(u, v) = \ell_{pq}$. Thus $M(p, q) \cong A/Au + Av$, giving an exact sequence $0 \to A.\overline{v} \to A/Au \to A/Au + Av \to 0$. By (4.3) and (4.4) there is a unique $a \in \mathbb{P}(A_1)$ such that $av \in A_1 u$. Since plane modules are 3-critical, the map $A/Aa \to A\overline{v}$ is an isomorphism. By (4.2) and (4.4), $(a)_0 = (p + \tau) + (q + \tau) + (r - \tau) + (s - \tau)$. This proves the result. ■

Remark. Iterating this proposition, $M(p, q, r, s)$ contains a submodule isomorphic to $M(p + n_1\tau, q + n_2\tau, r + n_3\tau, s + n_4\tau)$ if $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ satisfies $n_1 + n_2 + n_3 + n_4 = 0$.

§5. Point modules.

The first result in this section, namely (5.2), shows that if $p \in E \cup \mathcal{S}$, then the corresponding point module is $M(p) \cong A/Au + Av + Aw$ where $u, v, w \in A_1$ are linear forms such that $\mathcal{V}(u, v, w) = \{p\}$. It is obvious from the definition of $M(p)$ that there is a surjective map $A/Au + Av + Aw \to M(p)$, since $M(p) = \bigoplus_{j=0}^{\infty} \mathbb{C}e_j$ and $x \in A_1$ acts via $x.e_j = x(p - j\tau)e_{j+1}$. The main result in this section is that if $p, q \in E$, then there is a short exact sequence $0 \to M(p + \tau, q - \tau) \to M(p, q) \to M(p) \to 0$.

Lemma 5.1. Let $p \in E$, and let $W \subset A_1$ be the subspace vanishing at p. Then $J \subset AW$ whence $A/AW \cong B/BW$.

Proof. Consider $A_1 \otimes A_1$ as bihomogeneous forms on $\mathbb{P}^3 \times \mathbb{P}^3$. Then $\mathcal{V}(I_2 + A_1 \otimes W) = \mathcal{V}(I_2) \cap \mathcal{V}(A_1 \otimes W) = \Gamma \cap (\mathbb{P}^3 \times \{p\}) = (p^\sigma, p)$. Since $I_2 + A_1 \otimes W$ vanishes at a unique point of $\mathbb{P}^3 \times \mathbb{P}^3$, it is of codimension ≤ 1 in $A_1 \otimes A_1$. Hence any function vanishing at (p^σ, p) lies in $I_2 + A_1 \otimes W$. In particular, as $J_2(\Delta_\tau) = 0$, $J_2 \subset I_2 + A_1 \otimes W$. ■

Remark. The lemma does not apply to $e_i \in \mathcal{S}$, because J_2 does not vanish at (e_i, e_i). In fact, if $e_i \in \mathcal{S}$, then $M(e_i)$ is not a B-module. Hence there are two cases in the proof of the following proposition, depending on whether or not $p \in E$.

Proposition 5.2. Let $p \in E \cup \mathcal{S}$. Then the point module corresponding to p is $M(p) = A/AW$ where $W \subset A_1$ is the subspace vanishing at p.

Proof. Let $p \in E$. There is a surjective map $A/AW \to M(p)$, so it suffices to show that $dim(A_{n+1}/A_nW) = 1$ for all $n \geq 1$ (we did the case $n = 1$ in (5.1)). Since $A/AW \cong B/BW$ it is enough to show that $dim(B_{n+1}/B_nW) = 1$ for all $n \geq 1$.

Let $i : E \to \mathbb{P}^3$ be the inclusion, and let $\mathcal{L} = i^*\mathcal{O}(1)$. Define $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*(\mathcal{L}) \otimes \ldots \otimes (\sigma^{n-1})^*(\mathcal{L})$. By [21,§3], it follows that B_{n+1}/B_nW is the cokernel of the map $H^0(E, \mathcal{L}_n) \otimes W \to H^0(E, \mathcal{L}_{n+1})$ which is the restriction of the map

$$H^0(E, \mathcal{L}_n) \otimes H^0(E, \mathcal{L}) \to H^0(E, \mathcal{L}_n) \otimes H^0(E, \mathcal{L}^{\sigma^n}) \to H^0(E, \mathcal{L}_{n+1}).$$

Note that $W = H^0(E, \mathcal{L}(-p))$. Hence we wish to show that the cokernel of the map

$$H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}(-p)) \to H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}(-p)^{\sigma^n}) \to H^0(\mathcal{L}_n \otimes \mathcal{L}(-p)^{\sigma^n}) \to H^0(E, \mathcal{L}_{n+1})$$

is of dimension 1.

Apply $\mathcal{L}_n \otimes -$ to the sequence $0 \to \mathcal{L}(-p)^{\sigma^n} \to \mathcal{L}^{\sigma^n} \to \mathcal{L}^{\sigma^n}/\mathcal{L}(-p)^{\sigma^n} \to 0$ and take cohomology. Since $\deg(\mathcal{L}_n \otimes \mathcal{L}(-p)^{\sigma^n}) > 0$ (actually $= 4n + 3$), $H^1(\mathcal{L}_n \otimes \mathcal{L}(-p)^{\sigma^n}) = 0$ and we have an exact sequence

$$0 \to H^0(\mathcal{L}_n \otimes \mathcal{L}(-p)^{\sigma^n}) \to H^0(\mathcal{L}_{n+1}) \to H^0(\mathcal{L}_n \otimes \mathcal{L}^{\sigma^n}/\mathcal{L}(-p)^{\sigma^n}) \to 0.$$

These spaces have dimensions $4n+3$, $4n+4$, and 1 respectively. Hence it suffices to prove that the cokernel of the map $H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}(-p)^{\sigma^n}) \to H^0(\mathcal{L}_n \otimes \mathcal{L}(-p)^{\sigma^n})$ is zero. Apply [13, Theorem 2(b)] with $\mathcal{F} = \mathcal{L}_n, L = \mathcal{L}(-p)^{\sigma^n}$ and $i = 0$, to conclude that this cokernel is indeed zero (the hypothesis of Mumford's Theorem only needs to be checked for $i = 1$, and by degree arguments it is satisfied).

Now suppose that $p = e_i \in \mathcal{S}$. Let $W_i := \mathbb{C}x_0 \oplus \cdots \oplus \widehat{\mathbb{C}x_i} \oplus \cdots \oplus \mathbb{C}x_3$ be the subspace of A_1 vanishing at e_i. A careful examination of the defining relations of A shows that AW_i is a 2-sided ideal, and that $A/AW_i \cong \mathbb{C}[x_i]$. In particular, there is a surjective map $\mathbb{C}[x_i] \longrightarrow M(e_i)$, and because the Hilbert series are the same this map is an isomorphism. Thus, $A/AW_i \cong M(e_i)$ as required. ■

A more elementary proof of (5.2) follows from (2.13).

Lemma 5.3. Let $p, q \in E$. Then there is a short exact sequence

$$0 \longrightarrow K \longrightarrow M(p,q) \longrightarrow M(p) \longrightarrow 0$$

where the kernel K is a shifted line module.

Proof. Let $u, v, w \in A_1$ be such that $\ell_{pq} = \mathcal{V}(u,v)$, and $\mathcal{V}(u,v,w) = \{p\}$. Thus $M(p) \cong A/Au + Av + Aw$, and $M(p,q) \cong A/Au + Av$. This shows the existence of a surjective map $M(p,q) \to M(p)$ with kernel $K \cong A\overline{w}$, the submodule of $A/Au + Av$ generated by the image of w. Its Hilbert series is $(1-t)^{-2} - (1-t)^{-1} = t(1-t)^{-2}$, so $A.\overline{w}$ is a shifted line module. ■

Remarks. 1. If $p + q \notin E_2$, then ℓ_{pq} meets $E \cup \mathcal{S}$ only at p and q, so the only point modules which are quotients of $M(p,q)$ are $M(p)$ and $M(q)$. However, if $p + q = \omega_i \in E_2$, then ℓ_{pq} passes through the point $e_i \in \mathcal{S}$, so $M(e_i)$ is a quotient of $M(p,q)$. This describes all the ways in which a point module can arise as a quotient of a line module.

2. Our notation obscures the fact that there is only one surjection (up to scalar multiples) from $M(p,p)$ to $M(p)$. This is implicit in the proof of (5.3): p determines $\mathbb{C}u + \mathbb{C}v + \mathbb{C}w$, the subspace of A_1 vanishing at p, so if $\ell_{pp} = \mathcal{V}(u,v)$, then the kernel of any map $M(p,p) \to M(p)$ must contain $\mathbb{C}\overline{w} = (\mathbb{C}u + \mathbb{C}v + \mathbb{C}w)/(\mathbb{C}u + \mathbb{C}v)$.

3. Next we determine the line module K in (5.3). This is done in (5.5), and (5.7) describes the kernel of the map $M(p, \omega_i - p) \to M(e_i)$.

Lemma 5.4. Let $p, q \in E$. The elements of E_2 may be labelled $E_2 = \{\omega_i, \omega_j, \omega_k, \omega_\ell\}$ in such a way that

(a) $p + q \notin \{\omega_i, \omega_j\}$, $\quad \omega_i + \omega_j \notin \{p - q, p - q - 2\tau\}$ and

(b) there exists $0 \neq u \in A_1$ such that $(u)_0 = p + q + (\omega_i - p) + (\omega_i - q)$ and $u(e_j) \neq 0$.

Proof. If $p + q \notin E_2$ then (a) is easily satisfied. On the other hand if $p + q = \omega_\ell$ then $\{\omega_i + \omega_j, \omega_i + \omega_k, \omega_j + \omega_k\}$ consists of 3 distinct elements, so one of these is not in $\{p - q, p - q - 2\tau\}$. Again (a) holds.

Now pick $0 \neq u \in A_1$ with the zero locus as in the statement of the lemma. Now $u(e_j) = 0$ if and only if either $p + q = \omega_j$ or $p + (\omega_i - q) = \omega_j$. Since these possibilities are excluded by (a), (b) is also true. $\blacksquare$

Theorem 5.5. Let $p, q \in E$. Then there is a short exact sequence

$$0 \longrightarrow M(p+\tau, q-\tau)[-1] \longrightarrow M(p,q) \longrightarrow M(p) \longrightarrow 0.$$

Proof. Choose $\omega_i, \omega_j \in E_2$ and $u \in A_1$ as in (5.4). Thus $(u)_0 = (p) + (q) + (\omega_i - p) + (\omega_i - q)$. Define $0 \neq v, w \in A_1$ by $(v)_0 = (p) + (q) + (\omega_j - p) + (\omega_j - q)$ and $(w)_0 = (p) + (\omega_i - p) + (\omega_j - p) + (\omega_i + \omega_j + p)$. The careful choice of ω_i and ω_j ensures that u, v, w are linearly independent. Therefore $\mathcal{V}(u, v, w) = \{p\}$, and the secant lines $\mathcal{V}(u, v) = \ell_{pq}$, $\mathcal{V}(u, w) = \ell_{p,\omega_i - p}$ and $\mathcal{V}(v, w) = \ell_{p,\omega_j - p}$ are pairwise distinct. By (5.3), $A\overline{w} \subset A/Au + Av$ is the line module to be determined. By (4.4) there exist non-zero elements $x, y, b, b' \in A_1$ such that $xw - bu = yw - b'v = 0$. Furthermore, since $div(\frac{u}{w}) = (q) + (\omega_i - q) - (\omega_j - p) - (\omega_i + \omega_j + p)$ we must have $(x)_0 = (q - \tau) + (\omega_i - q - \tau) + (p + \tau) + (\omega_i - p + \tau)$. Similarly $(y)_0 = (q-\tau) + (\omega_j - q - \tau) + (p+\tau) + (\omega_j - p + \tau)$. Notice that x and y are linearly independent: if not then $(x)_0 = (y)_0$ so $(\omega_i - q - \tau) + (\omega_i - p + \tau) = (\omega_j - q - \tau) + (\omega_j - p + \tau)$, whence $\omega_i - q - \tau = \omega_j - p + \tau$ contradicting our choice of ω_i and ω_j. Therefore $A\overline{w} \cong A/Ax + Ay$ and $\mathcal{V}(x, y) = \ell_{p+\tau, q-\tau}$. $\blacksquare$

The next result should be compared with [14, Theorem 3.2].

Proposition 5.6. Suppose that $p - q \notin \mathbb{Z} \cdot 2\tau$. Then $M(p, q)$ has a basis $\{e_{ij} \mid (i, j) \in \mathbb{N}^2\}$ with the property that
(a) $M(p,q)_n$ has basis $\{e_{ij} \mid i + j = n\}$;
(b) $A.e_{ij} \cong M(p + (j - i)\tau, q + (i - j)\tau)$;
(c) $A.e_{ij}$ has basis $\{e_{i+k, j+\ell} \mid k, \ell \geq 0\}$;
(d) if $x \in A_1$, then $x.e_{ij} \in \mathbb{C}e_{i+1,j} \oplus \mathbb{C}e_{i,j+1}$;
(e) if $x \in A_1$, then $x.e_{ij} \in \mathbb{C}e_{i+1,j}$ if and only if $x(q + (i - j)\tau) = 0$, and $x.e_{ij} \in \mathbb{C}e_{i,j+1}$ if and only if $x(p + (j - i)\tau) = 0$.

Proof. Write $M = M(p, q)$, and pick $0 \neq e_{00} \in M_0$. Clearly (a) is true for $n = 0$, and (b) is true for $i = j = 0$. The truth of (c) and (d) will follow from the way in which the basis is constructed

Since $p \neq q$, there are short sequences $0 \to K(p) \to M(p, q) \to M(p) \to 0$ and $0 \to K(q) \to M(p, q) \to M(q) \to 0$ where, by (5.4), $K(p) = Ae_{01} \cong M(p + \tau, q - \tau)$ and $K(q) = Ae_{10} \cong M(p - \tau, q + \tau)$, for some $e_{01}, e_{10} \in M_1$. The elements e_{01} and e_{10} are linearly independent, because if not, then $M(p) \cong M/Ae_{01} = M/Ae_{10} \cong M(q)$, whence $p = q$ (which contradicts the hypothesis on p and q). Since $dim(M_1) = 2$, it follows that $\{e_{01}, e_{10}\}$ is a basis for M_1. This proves (a) for $n = 1$, and (b) for $i + j = 1$.

Since $p-q \notin \mathbb{Z}.2\tau$, $p+\tau \neq q-\tau$, so the previous paragraph may be applied to $Ae_{01} \cong M(p+\tau, q-\tau)$. Hence there exist $e_{02}, e_{11} \in M_2$ such that $A_1.e_{01} = \mathbb{C}e_{02} \oplus \mathbb{C}e_{11}$, and $Ae_{02} \cong M(p+2\tau, q-2\tau)$ and $Ae_{11} \cong M(p,q)$. Similarly, there exist $e'_{11}, e_{20} \in M_2$ such that $A_1.e_{10} = \mathbb{C}e_{20} \oplus \mathbb{C}e'_{11}$, and $Ae_{20} \cong M(p-2\tau, q+2\tau)$ and $Ae'_{11} \cong M(p,q)$.

We will now show that $\mathbb{C}e_{11} = \mathbb{C}e'_{11}$. By (6.1) there exists a central element $\Omega \in A_2$ such that $\Omega.M(p,q) \neq 0$. Then $0 \neq \Omega.e_{00} \in M_2$. Since a point module associated to a point of E is a B-module, $\Omega.M(p) = \Omega.M(q) = 0$. Hence $\Omega.e_{00} \in Ae_{01} \cap Ae_{10}$. Let $u, v \in A_1$ be such that $\ell_{pq} = \mathcal{V}(u,v)$. Then $u.e_{00} = v.e_{00} = 0$. Since $Ae_{11} \cong M(p,q)$, we also have $v.e_{11} = u.e_{11} = 0$. On the other hand, if $u.e_{02} = ve_{02} = 0$, then $\ell_{p+2\tau,q-2\tau} = \mathcal{V}(u,v) = \ell_{pq}$; this forces $\{p+2\tau, q-2\tau\} = \{p,q\}$ which is impossible, since $\tau \notin E_2$ and $p-q \notin \mathbb{Z}.2\tau$. Hence $\mathbb{C}e_{11}$ is the unique 1-dimensional subspace of $A_1.e_{01}$ which is annihilated by u and v. But, since Ω is central, $\Omega.e_{00}$ is also killed by u and v, so $\mathbb{C}e_{11} = \mathbb{C}\Omega e_{00}$. Similarly, $\mathbb{C}e'_{11} = \mathbb{C}\Omega e_{00}$. Thus $\mathbb{C}e_{11} = \mathbb{C}e'_{11}$, and we can take $e'_{11} = e_{11}$ in the previous paragraph.

Now $M_2 = A_2.e_{00} = A_1e_{01} + A_1e_{10} = \mathbb{C}e_{02} + \mathbb{C}e_{11} + \mathbb{C}e_{20}$. Since $dim(M_2) = 3$, $\{e_{02}, e_{11}, e_{20}\}$ is a basis for M_2. Hence (a) is true for $n=2$, and (b) is true for $i+j=2$.

We proceed by induction. Suppose we have obtained e_{ij} for all $i+j \leq n$, that (a) is true for all $m \leq n$, and that (b) is true for all $i+j \leq n$. We apply the earlier arguments to $A_1.e_{i,n-i}$. This gives elements $e_{i+1,n-i}$ and $e'_{i,n-i+1}$ in M_{n+1} such that $A_1.e_{i,n-i} = \mathbb{C}e_{i+1,n-i} + \mathbb{C}e'_{i,n-i+1}$. The previous argument applied to $A_1e_{i,n-i}$ and $A_1e_{i-1,n-i+1}$ shows that we can take $e_{i,n-i+1} = e'_{i,n-i+1}$. Hence we have elements $e_{i,n-i+1}(0 \leq i \leq n+1)$, such that $A.e_{i,n-i+1} \cong M(p+(n+1-2i)\tau, q-(n+1-2i)\tau)$ and $A_1.e_{i,n-i} = \mathbb{C}e_{i+1,n-i} + \mathbb{C}e_{i,n-i+1}$. (Notice that we needed to use the fact that $p-q \notin \mathbb{Z}\cdot 2\tau$). Since $M_{n+1} = A_1.M_n$, it follows that $\{e_{0,n+1}, e_{1,n} \cdots, e_{n,1}, e_{n+1,0}\}$ spans M_{n+1}, and since $dim(M_{n+1}) = n+2$, it is actually a basis. This proves (a) for $n+1$, and (b) for $i+j = n+1$. Hence (a) and (b) follow by induction. ■

Remark. If $p-q \in \mathbb{Z}\cdot 2\tau$, then $M(p,q)$ contains a submodule isomorphic to $M(p',p')$ for some p', and the arguments we have just used fail - indeed there is only one submodule of $M(p',p')$ such that the quotient is isomorphic to $M(p')$, so we are unable to obtain e_{01} and e_{10} as in the proof of the proposition.

The basis has the property that if $\Omega \in A_2$ is central, then $\Omega.e_{ij} \in \mathbb{C}e_{i+1,j+1}$ for all i,j (since Ω annihilates every point module $M(p)$ with $p \in E$).

Theorem 5.7. Let $p, q \in E$ be such that $p+q = \omega_i \in E_2$. Then there is a short exact sequence

$$0 \to M(p-\tau, q-\tau)[-1] \to M(p,q) \to M(e_i) \to 0.$$

Proof. Let $u, v, w \in A_1$ be such that $\mathcal{V}(u,v) = \ell_{pq}$ and $\mathcal{V}(u,v,w) = \{e_i\}$. By (5.3), the kernel of the surjection $M(p,q) \to M(e_i)$ is $K \cong A.\overline{w}$ contained in $A/Au + Av$. Thus the kernel is isomorphic to $M(\ell)$ where ℓ is the line $\mathcal{V}(x,y)$, and $x, y \in A_1$ satisfy $x.w, y.w \in A_1u + A_1v$.

We can choose u, v, w such that $\mathcal{V}(u,v), \mathcal{V}(u,w)$ and $\mathcal{V}(v,w)$ are three distinct secant lines of E. For example, choose u, v, w such that $(u)_0 = p+q+r+s$, $(v)_0 = p+q+r'+s'$ and $(w)_0 = r+s+r'+s'$ with r, s, r', s' in general position subject to $r+s = r'+s' = \omega_i$. Now apply (4.4) to these three secant lines. This gives $x, y, b, b' \in A_1$ such that $xw - bu = yw - b'v = 0$. Since w is not zero at p or q but both u and v are, it follows that $(x)_0$ and $(y)_0$ both contain $(p-\tau)+(q-\tau)$. Thus $\mathcal{V}(x,y) = \ell_{p-\tau,q-\tau}$ as required. ■

We now study the finite dimensional simple quotients of point modules. First consider a graded algebra $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$ generated by A_1, with $\dim(A_1) < \infty$.

By [2] the point modules for A are parametrised by a space Γ which is an inverse limit of projective varieties. Furthermore, there is a map $\sigma{:}\Gamma \to \Gamma$ such that if $p \in \Gamma$, and $M(p)$ is the corresponding point module, then $M(p^{-\sigma})$ is isomorphic to the submodule $M(p)^{\geq 1} = \bigoplus_{j\geq 1} M(p)_j$ of $M(p)$. Thus $M(p)^{\geq k} \cong M(p^{\sigma^{-k}})$.

Fix a basis $x_0, \ldots, x_{n-1}$ for A_1, and consider these as a system of homogeneous coordinates on $\mathbb{P}(A_1^*)$. Suppose (as is the case for the Sklyanin algebra, and the 3-generated 3-dimensional regular algebras of [1]) that there is an embedding $\Gamma \to \mathbb{P}(A_1^*)$, such that σ is an automorphism of Γ, and for each $p \in \Gamma$, $M(p)_0$ and $M(p)_1$ have bases e_0 and e_1 with the property that $x_i.e_0 = x_i(p)e_1$ for all i. Then $M(p)$ has a basis $e_0, e_1, \ldots$ such that $x.e_j = x(p^{\sigma^{-j}})e_{j+1}$ for all $x \in A_1$, and all j. This assumption applies in (5.8) and (5.9).

The point modules for the Sklyanin algebra are parametrised by $E \cup \mathcal{S}$ [21, §2], and the automorphism is given by $\sigma(p) = p+\tau$ if $p \in E$, and $\sigma(e_i) = e_i$. Hence if $p \in E$ then $M(p)$ has basis e_j such that $x.e_j = x(p - j\tau)e_{j+1}$ for all $x \in A_1$.

Lemma 5.8. Let $M = M(p) = Ae_0$ be a point module as above. Let $a \in \mathbb{N}$.

(a) Let N be a submodule of M which does not contain any e_i. If $\dim(M/N) = a$, then $p^{\sigma^b} = p$ for some $b \in \{1, \ldots, a\}$; in particular $|\langle\sigma\rangle \cdot p|$ divides b.

(b) Suppose that $|\langle\sigma\rangle \cdot p| = a$, and N is a submodule of M such that $\dim(M/N) = a$. Suppose that $N' \supsetneqq N$ is a submodule of M. If $N' \neq M$ then N' contains some e_j.

(c) Suppose that $|\langle\sigma\rangle \cdot p| = a$. For each $0 \neq \lambda \in \mathbb{C}$, $N = A(e_0 - \lambda e_a)$ is a proper submodule of M with basis $\{e_i - \lambda e_{i+a} \mid i \geq 0\}$. Furthermore, M/N is simple with basis $\{\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{a-1}\}$.

(d) If $|\langle\sigma\rangle \cdot p| = \infty$, then the only simple quotient of $M(p)$ is the trivial module.

Proof. (a) Choose b minimal such that N contains an element of the form $e = \sum_{j=0}^{b} \lambda_j e_{i+j}$ with $\lambda_0\lambda_b \neq 0$. Clearly $1 \leq b \leq a$. Let $x_0, x_1, \ldots, x_{n-1}$ be a basis for A_1. For each k, N contains $x_k \cdot e = \sum_{0\leq j\leq b} \lambda_j x_k(p^{\sigma^{i+j}})e_{i+j+1}$. Consider the $2 \times n$ matrix, the rows of which are the homogeneous coordinates of the points p^{σ^i} and $p^{\sigma^{i+b}}$, namely

$$X = (x_k(p^{\sigma^{i+j}}))_{j=0,b;0\leq k\leq n-1}.$$

If $\mathit{rank}(X) = 2$, then $x_k(p^{\sigma^i})x_\ell(p^{\sigma^{i+b}}) - x_\ell(p^{\sigma^i})x_k(p^{\sigma^{i+b}}) \neq 0$ for some k and ℓ. Hence N contains the element $(x_\ell(p^{\sigma^{i+b}})x_k - x_k(p^{\sigma^{i+b}})x_\ell) \cdot e$ which is a linear combination of

$e_{i+1},\dots,e_{i+b+1}$ with the coefficient of e_{i+1} non-zero, and the coefficient of e_{i+b+1} equal to zero. This contradicts the minimality of b, so we conclude that $rank(X)=1$. Hence the rows are scalar multiples of each other. Therefore $p^{\sigma^i}=p^{\sigma^{i+b}}$, whence $p=p^{\sigma^b}$.

(b) Suppose that N' does not contain any e_j. Then $\dim(M/N')=d<a$ and (a) applies to M/N'. That is, $a=|\langle\sigma\rangle\cdot p|$ divides b for some $b\in\{1,...,d\}$. This is impossible.

(c) Define $\varphi: M(p)\to M(p)^{\geq a}$ by $\varphi(e_i)=e_{i+a}$. Because $p^{\sigma^a}=p$, φ is A-linear. Thus $N=\operatorname{Im}(1-\lambda\varphi)$ has basis $\{(1-\lambda\varphi)(e_i)\mid i\geq 0\}=\{e_i-\lambda e_{i+a}\mid i\geq 0\}$, and M/N has basis as claimed.

Suppose that $N'\supsetneqq N$ is a submodule. If N' contains some e_j, choose the minimal such j. If $j\geq a$, then $e_{j-a}-\lambda e_j\in N\subseteq N'$ implies $e_{j-a}\in N'$. Hence $j<a$. But then $e_a\in N'$, whence $e_0\in N'$, so $N'=M$. Hence by (b) M/N is simple.

(d) Suppose that M/N is a simple quotient of M. If N contains e_j, then M/N is trivial, since $M/Ae_j=M/M^{\geq j}$ has a composition series all of whose factors are the trivial module. Hence if M/N is a non-trivial simple quotient, then (a) applies, and this contradicts the fact that $|\langle\sigma\rangle\cdot p|=\infty$. ■

Proposition 5.9. Let $M(p)$ be a point module as above. The following are equivalent:
(a) $p^\sigma=p$;
(b) $M(p)$ has at least one non-trivial 1-dimensional quotient module;
(c) for all $\lambda\in\mathbb{C}^\times$, $M(p)/A(e_0-\lambda e_1)$ is a 1-dimensional A-module;
(d) $\operatorname{Ann}(e_0)$ is a 2-sided ideal, and $A/\operatorname{Ann}(e_0)$ is a polynomial ring in 1 variable.

Proof. (c) $\implies$ (b) is obvious, and (a) $\iff$ (c) by (5.8c).

(b) $\iff$ (a) Let $M(p)/N$ be a non-trivial 1-dimensional module. Since every composition factor of $M(p)/Ae_i$ is the trivial module, $e_i\notin N$. Hence $p=p^\sigma$ by (5.8a).

(d) $\implies$ (b) The Hilbert series of $M(p)\cong A/\operatorname{Ann}(e_0)$ is $(1-t)^{-1}$, so $M(p)\cong\mathbb{C}[X]$ has plenty of 1-dimensional quotients.

(a) $\implies$ (d) We have $M(p)=M(p^{\sigma^k})\cong M(p)^{\geq k}$, so $\operatorname{Ann}(e_0)$ annihilates every e_k. Thus $\operatorname{Ann}(e_0)=\operatorname{Ann}M(p)$ is a 2-sided ideal, and $A/\operatorname{Ann}(e_0)\cong\mathbb{C}[X]$, the only ring with Hilbert series $(1-t)^{-1}$. ■

We now return to the Sklyanin algebra, and observe that (5.9) applies to the 4 points e_i, since $e_i^\sigma=e_i$.

Proposition 5.10. Let $\{i,j,k,\ell\}=\{0,1,2,3\}$. For each i, define $I_i:=Ax_j+Ax_k+Ax_\ell$. Then $M(e_i)\cong A/I_i$, and $I_i=\operatorname{Ann}M(e_i)$.

Proof. It follows at once from the defining relations that I_i is a two-sided ideal, and that $A/I_i\cong\mathbb{C}[x_i]$. It is obvious that $I_i\subset\operatorname{Ann}(M(e_i)_0)$, so the result follows. ■

Theorem 5.11. If τ is of infinite order, then $\operatorname{Ann}M(p)=\langle\Omega_1,\Omega_2\rangle$ for all $p\in E$.

Proof. Set $J = \mathrm{Ann} M(p)$. Since $M(p)$ is critical, J is a prime ideal (this well known result can be proved by combining [9, Prop. 5.6] with the argument in [22, Prop. 3.9]). By the remarks after [21, Corollary 2.8], $J \supset \langle \Omega_1, \Omega_2 \rangle$. Suppose the result is false.

Then A/J is prime of GK-dimension 1, so by [19], it is a finite module over its center. By Goldie's Theorem A/J has a simple artinian ring of fractions, Q say. Since $M(p)$ is a faithful A/J-module, $Q \otimes_{A/J} M(p) \neq 0$, whence Q embeds in a direct sum of copies of this. Therefore A/J embeds in a (finite) direct sum of copies of $M(p)$. In particular, a finite dimensional simple A/J-module is also a quotient of $M(p)$. However, by (5.8d), the only such module is the trivial module. Hence A/J has a unique finite dimensional simple module. This is absurd since $\dim(A/J) = \infty$, and A/J is a finite module over its center (for example, it contradicts [20]). ■

§6. Annihilators of Line Modules.

Write $Z_2 = \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$ for the linear span of the two central elements in A_2 described in [21, §3.9], and in Section 1 above.

We summarize the results in this section. Theorem 6.3 proves that if $p, q \in E$, then there exists a unique (up to scalar multiple) non-zero element of Z_2 which annihilates $M(p,q)$. We write $\Omega(p,q)$ for this element. It is best to think of $\Omega(p,q)$ as an element of the projective space $\mathbb{P}(Z_2) \cong \mathbb{P}^1$, and we shall usually do this. Next (6.5) and (6.6) show that $\Omega(p,q)$ depends only on $p+q$, so we prefer to write $\Omega(p,q) = \Omega(p+q)$. It is then proved that $\Omega(r) = \Omega(r')$ if and only if either $r = r'$ or $r + r' = -2\tau$. Finally (6.13) proves that every element of $\mathbb{P}(Z_2)$ is of the form $\Omega(r)$; that is every element of Z_2 annihilates some line module. It is also shown that if τ is of infinite order then $AnnM(p,q)$ is generated by $\Omega(p+q)$.

Lemma 6.1. If M is a line module, then $\mathrm{Ann}_{Z_2}M$ is a 1-dimensional subspace of Z_2. That is, M is annihilated by a unique (up to scalar multiples) non-zero homogeneous central element of degree 2.

Proof. Let $M = A/Au + Av$. Since $\mathcal{V}(u,v)$ is a secant line, (4.4) shows that there exists $0 \neq a, b \in A$, such that $av - bu$ is a non-zero central element of A annihilating M.

Now we show that M cannot be annihilated by both Ω_1 and Ω_2. If it were, then M would be a B-module. However, B is a domain of GK-dimension 2, so any cyclic B-module of GK-dimension 2 must be isomorphic to B itself. But B and M have different Hilbert series, so $M \not\cong B$. ■

Proposition 6.2. Let $0 \neq \Omega \in Z_2$. Then $\overline{A} = A/\langle\Omega\rangle$ is a domain.

Proof. Suppose not. Then there are non-zero homogeneous elements $a, b \in A$ such that $ab \in \langle\Omega\rangle$, but $a \notin \langle\Omega\rangle$ and $b \notin \langle\Omega\rangle$. Choose such a and b, such that $\deg(a) + \deg(b)$ is minimal. Choose Ω' such that $\mathbb{C}\Omega \oplus \mathbb{C}\Omega' = \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$.

Claim: The image of Ω' in $\overline{A}$ is regular. Proof: If not then $\Omega' x = \Omega y$ for some x, y with $x \notin \langle\Omega\rangle$. Write $\Omega' = \lambda_1\Omega_1 + \lambda_2\Omega_2$, and $\Omega = \mu_1\Omega_1 + \mu_2\Omega_2$. Thus $(\lambda_1\Omega_1 + \lambda_2\Omega_2)x = (\mu_1\Omega_1 + \mu_2\Omega_2)y \Longrightarrow (\lambda_1 x - \mu_1 y)\Omega_1 = (\mu_2 y - \lambda_2 x)\Omega_2 \Longrightarrow \mu_2 y - \lambda_2 x = \Omega_1 z$, and $\lambda_1 x - \mu_1 y = \Omega_2 z$ for some $z \in A$, since (Ω_1, Ω_2) is a regular sequence in A. In particular $\Omega' z = (\lambda_1\Omega_1 + \lambda_2\Omega_2)z = \lambda_1(\mu_2 y - \lambda_2 x) + \lambda_2(\lambda_1 x - \mu_1 y) = (\lambda_1\mu_2 - \mu_1\lambda_2)y$. Since Ω and Ω' are linearly independent, $\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}$ is invertible, whence $y = \lambda\Omega' z$ for some $0 \neq \lambda \in \mathbb{C}$. Hence $\Omega' x = \lambda\Omega\Omega' z$ so $x = \lambda\Omega z \in \langle\Omega\rangle$. This contradiction proves the claim.□

Since $A/\langle\Omega, \Omega'\rangle = A/\langle\Omega_1, \Omega_2\rangle$ is a domain, either $a \in \langle\Omega, \Omega'\rangle$ or $b \in \langle\Omega, \Omega'\rangle$. Suppose that $a = \Omega c + \Omega' d$. By cancelling off high order terms, we can assume that $\deg(c) = \deg(d) = \deg(a) - 2$. Now $\Omega cb + \Omega' db = ab \in \langle\Omega\rangle$, so $\Omega' db \in \langle\Omega\rangle$. By the previous paragraph, $db \in \langle\Omega\rangle$. But $\deg(d) + \deg(b) < \deg(a) + \deg(b)$, and $b \notin \langle\Omega\rangle$, so by choice of a, b it follows that $d \in \langle\Omega\rangle$. Therefore, a $(= \Omega c + \Omega' d)$ is also in $\langle\Omega\rangle$. This again is a contradiction, so $a \notin \langle \Omega, \Omega' \rangle$. Therefore $b \in \langle \Omega, \Omega' \rangle$. A similar argument shows that $b \in \langle \Omega \rangle$. This too is a contradiction, and we are forced to conclude that no such a and b exist. Thus $\overline{A}$ is a domain. ■

Remark. We will prove in (6.12) that the center of $A/\langle \Omega \rangle$ is a polynomial ring in one variable when τ is of infinite order.

Theorem 6.3. Suppose that τ is of infinite order. Then the annihilator of a line module is generated by a non-zero homogeneous central element of degree 2.

Proof. Let $p, q \in E$, and set $I = \mathrm{Ann}M(p,q)$. Since $M(p)$ is a quotient of $M(p,q)$, and $\mathrm{Ann}M(p) = \langle \Omega_1, \Omega_2 \rangle$ by (5.9) we have $I \subset \langle \Omega_1, \Omega_2 \rangle$. By (6.1), there exists $0 \neq \Omega \in A_2$ central, such that $\Omega \in I$. By the uniqueness of Ω, we have $I \neq \langle \Omega_1, \Omega_2 \rangle$. However, $A/\langle \Omega \rangle$ is prime of GK-dimension 3, and since I is also prime (because $M(p,q)$ is critical), we conclude that $\langle \Omega \rangle = I$. ■

Write $\Omega(p,q)$ for "the" element determined by (6.3). Thus when τ is of infinite order $\mathrm{Ann}M(p,q) = \langle \Omega(p,q) \rangle$. The rest of this section is devoted to getting more precise information about $\Omega(p,q)$. In particular, it will be shown that it depends only on $p+q$, so will be denoted by $\Omega(p+q)$ once that has been done.

Define non-zero central elements $\Omega(\omega_i)$, $i = 0,1,2,3$, in A_2 as follows:

$$\begin{aligned}
\Omega(\omega_0) &= (1+\gamma)x_1^2 + (1+\alpha\gamma)x_2^2 + (1-\alpha)x_3^2 \\
\Omega(\omega_1) &= (1+\gamma)x_0^2 + (\alpha\gamma-\gamma)x_2^2 + (\alpha+\gamma)x_3^2 \\
\Omega(\omega_2) &= (1+\alpha\gamma)x_0^2 + (\gamma-\alpha\gamma)x_1^2 - (\alpha+\alpha\gamma)x_3^2 \\
\Omega(\omega_3) &= (1-\alpha)x_0^2 + (\alpha+\gamma)x_1^2 + (\alpha+\alpha\gamma)x_2^2.
\end{aligned}$$

These are the only degree two central elements of "rank 3" i.e. which are linear combinations of only three of the x_j^2.

Proposition 6.4. If ℓ is a secant line passing through e_i, then $\Omega(\omega_i)$ annihilates the line module $M(\ell)$. In particular, all line modules corresponding to lines on the singular quadric $Q(w_i)$, are annihilated by the central element $\Omega(\omega_i)$.

Proof. Write $\ell = \mathcal{V}(u,v)$, and let $av - bu \in A_2$ be a non-zero central element annihilating $M(\ell)$. Since $av-bu$ is central it is a linear combination of x_j^2 with $j = 0,1,2,3$. Say $av-bu = \sum \mu_j x_j^2$. Then $a\otimes v - b\otimes u - \sum \mu_j x_j \otimes x_j \in I_2$. Since $e_i \in \ell$, $u(e_i) = v(e_i) = 0$. Therefore, $a \otimes v$ and $b \otimes u$ both vanish at (e_i, e_i), as do all the elements of I_2. Therefore $(\sum \mu_j x_j \otimes x_j)(e_i, e_i) = 0$, so $\mu_i = 0$. But there is a unique (up to scalar multiple) non-zero central element in A_2, whose coefficient of x_i^2 is zero, namely $\Omega(\omega_i)$. Hence $av-bu = \Omega(\omega_i)$, up to a scalar multiple. ■

After (6.4), one can say, somewhat inaccurately, that the singular quadrics correspond to the singular central elements.

Proposition 6.5. Let ℓ and ℓ' be lines lying on a common quadric containing E, and suppose that $\ell \cap \ell' = \emptyset$. Then there exists a non-zero central element $\Omega \in A_2$ which annihilates both $M(\ell)$ and $M(\ell')$.

Proof. Write $\ell \cap E = \{p, q\}$ and $\ell' \cap E = \{p', q'\}$. By (3.4d), Q is smooth, so by (3.10c) $p + q = p' + q'$. Choose any $x, y, z, w \in E$ such that $\{x, y\} \neq \{z, w\}$ and $x + y + p + q = z + w + p + q = 0$. Then there exist linear forms $u, v, u', v' \in A_1$ such that

$$\begin{aligned}(u)_0 &= (x) + (y) + (p) + (q)\\ (v)_0 &= (z) + (w) + (p) + (q)\\ (u')_0 &= (x) + (y) + (p') + (q')\\ (v')_0 &= (z) + (w) + (p') + (q')\end{aligned}$$

Thus $\ell = \ell_{pq} = \mathcal{V}(u, v)$ and $\ell' = \ell_{p'q'} = \mathcal{V}(u', v')$.

Choose $r \in E$, such that $r \notin \{p, q\} + E_2$, and such that r is not a zero of u, v, u' or v'. Replace u' and v' by non-zero scalar multiples of themselves such that $(u - u')(r) = (v - v')(r) = 0$. Then $(u - u')_0 = (x) + (y) + (r) + (\hat{r})$ and $(v - v')_0 = (z) + (w) + (r) + (\hat{r})$ for some $\hat{r} \in E$. In particular, $\mathcal{V}(u - u', v - v') = \ell_{r\hat{r}}$ is secant line of E, and since $r + \hat{r} = -(x + y) = p + q$, this line also lies on Q by (3.10d).

Apply (4.4) to $\ell_{r\hat{r}}$. There exist $0 \neq a, b \in A_1$ such that $(a \otimes (v - v') - b \otimes (u - u'))(\Gamma) = 0$. In fact, we must have $(a)_0 = (x - \tau) + (y - \tau) + (r + \tau) + (\hat{r} + \tau)$, and $(b)_0 = (z - \tau) + (w - \tau) + (r + \tau) + (\hat{r} + \tau)$. In particular, $av - bu = av' - bu'$ in A.

Now apply (4.2) to $\mathcal{V}(u, v) = \ell_{pq}$. There exists $\lambda \in \mathbb{C}$ such that $(a \otimes v - \lambda b \otimes u)(\Delta_\tau) = 0$. Similarly there exists $\lambda' \in \mathbb{C}$ such that $(a \otimes v' - \lambda' b \otimes u')(\Delta_\tau) = 0$. Hence $b \otimes ((\lambda - 1)u - (\lambda' - 1)u')$ vanishes on Δ_τ. Thus $(\lambda - 1)u - (\lambda' - 1)u'$ vanishes on E, and since E spans $\mathbb{P}^3$, $(\lambda - 1)u = (\lambda' - 1)u'$. However, $(u)_0 \neq (u')_0$ since $\ell \neq \ell'$. Therefore $\lambda = \lambda' = 1$.

Therefore $(a \otimes v - b \otimes u)(\Delta_\tau) = (a \otimes v' - b \otimes u')(\Delta_\tau) = 0$, so $av - bu$ is central in A. Since $\mathcal{V}(u, v) = \ell_{pq}$ and $r + \tau \notin \{p + \tau, q + \tau\}$ it follows from (4.4) that $av - bu \neq 0$. Therefore $av - bu$ is a non-zero central element of A. Hence the result. ■

Corollary 6.6. Let $z \in E$. Then there exists a non-zero element, $\Omega(z)$ say, in Z_2, such that $\Omega(z).M(p, q) = 0$ whenever $p + q = z$. In particular, $\Omega(p + q)$ annihilates $M(p, q)$.

Proof. Suppose that $z = p + q = r + s$. Set $\ell = \ell_{pq}$ and $\ell' = \ell_{rs}$. If $z = \omega_i$, then ℓ and ℓ' both pass through e_i, so (6.4) shows that there exists $0 \neq \Omega \in Z_2$ such that $\Omega.M(\ell) = \Omega.M(\ell') = 0$. If $z \in E_2$, then ℓ and ℓ' lie on the smooth quadric $Q(z)$, and by (3.10) do not intersect. Hence by (6.5), there exists $0 \neq \Omega \in Z_2$ such that $\Omega.M(\ell) = \Omega.M(\ell') = 0$. The notational definitions give $\Omega(p + q).M(p, q) = 0$. ■

We now turn to the problem of deciding exactly when two line modules are annihilated by the same homogeneous degree two central element. The result we prove in (6.9d) is that $M(p,q)$ and $M(p',q')$ have the same homogeneous degree two central annihilator if and only if either $p+q=p'+q'$ or $(p+q)+(p'+q')=-2\tau$. The proof begins with a lemma which contains most of the technicalities.

Lemma 6.7. Let $r,r'\in E$. Pick $\omega_i\in E_2$ such that $r\neq\omega_i$ and $r'\neq\omega_i$. Suppose that $r+r'\neq 0$. Let $p\in E$ be such that $2p\notin\{2r+r', r+2r', r+r'+\omega_i, r-2\tau, r'-2\tau\}$. Define

$$
\begin{aligned}
q=x'&=r-p\\
x=q'&=r'-p\\
y=y'&=p-(r+r')\\
w=w'&=\omega_i-p\\
z&=p+\omega_i-r\\
z'&=p+\omega_i-r'.
\end{aligned}
$$

Then

(i) $w\notin\{x,y,q\}$
(ii) $y\notin\{p,q,x,z\}$
(iii) $z'\notin\{q,y\}$
(iv) $q-\tau\notin\{p+\tau,q+\tau\}$
(v) $q'-\tau\notin\{p+\tau,q'+\tau\}$
(vi) $0=p+q+x+y=p+q'+x'+y'=p+q+w+z=p+q'+w'+z'$.

Proof. All the following conclusions contradict the hypotheses

(i) $w=x\Longrightarrow r'=\omega_i$.
$w=y\Longrightarrow 2p=r+r'+\omega_i$.
$w=q\Longrightarrow r=\omega_i$.
(ii) $y=p\Longrightarrow r+r'=0$.
$y=q\Longrightarrow 2p=2r+r'$.
$y=z\Longrightarrow r'=\omega_i$.
$y=x\Longrightarrow 2p=2r'+r$.
(iii) $z'=q\Longrightarrow 2p=r+r'+\omega_i$.
$z'=y\Longrightarrow r=w_i$.
(iv) $q-\tau=p+\tau\Longrightarrow 2p=r-2\tau$.
$q-\tau=q+\tau\Longrightarrow 2\tau=0$.
(v) $q'-\tau=p+\tau\Longrightarrow 2p=r'-2\tau$.
$q'-\tau=q'+\tau\Longrightarrow 2\tau=0$.
(vi) This is straightforward. ■

Proposition 6.8. Let $r,r'\in E$. Suppose that $r\neq r'$ and $r+r'\neq 0$. Then $\Omega(r)=\Omega(r')$ (up to scalar multiple) $\Longleftrightarrow r+r'=-2\tau$.

Proof. Choose $\omega_i, p, q, x, y, w, z, q', x', y', w', z' \in E$ as in (6.7). Thus $\Omega(r) = \Omega(p+q)$ and $\Omega(r') = \Omega(p+q')$.

By (6.7(vi)) we can pick $u = u', v, v' \in A_1$ such that

$$\begin{aligned}(u)_0 &= p+q+x+y\\(v)_0 &= p+q+w+z\\(u')_0 &= p+q'+x'+y'\\(v')_0 &= p+x+w'+z'.\end{aligned}$$

Each of these linear forms is determined up to a scalar multiple. By (6.7(i)), the planes $u=0$ and $v=0$ are distinct. Thus $\ell = \mathcal{V}(u,v)$ and $\ell' = \mathcal{V}(u',v')$ so $\Omega(r) \in A_1 v + A_1 u$ and $\Omega(r') \in A_1 v' + A_1 u'$.

By (6.7(ii)), $v(y) \neq 0$ and by (6.7(i),(ii),(iii)), $v'(y) \neq 0$. Hence we can, and do, pick v and v' such that $(v-v')(y) = 0$. Therefore u and $v-v'$ have two (distinct) common zeroes on E, namely y and p. However $v(e_i) = v'(e_i) = 0$ since $p+w = p'+w' = \omega_i$. Therefore $(v-v')(e_i) = 0$. In contrast $u(e_i) \neq 0$ because $w = \omega_i - p \notin \{x,y,q\}$. Therefore $\mathcal{V}(u, v-v') = \ell_{py}$ is a secant line of E.

Let $a = a' \in A_1$ be chosen such that $(a)_0 = (a')_0 = (x-\tau)+(y-\tau)+(q-\tau)+(p+3\tau)$. By (6.7) $q-\tau \notin \{p+\tau, q+\tau\}$ so by (4.2) and (4.4) applied to $\mathcal{V}(u,v) = \ell_{pq}$, there exists $b \in A_1$ such that $\Omega(r) = av - bu$. Similarly there exists $b' \in A_1$ such that $\Omega(r') = a'v' - b'u'$.

Therefore $\Omega(r) - \Omega(r') = a(v-v') - (b-b')u$, since $u = u'$, so $\Omega(r) = \Omega(r')$ if and only if $(a \otimes (v-v') - (b-b') \otimes u)(\Gamma) = 0$. However $(a \otimes (v-v') - (b-b') \otimes u)(\Delta_\tau) = 0$, since $av - bu$ and $a'v' - b'u'$ are both central. Consideration of $(a)_0$, together with (4.4) applied to $\mathcal{V}(u,v) = \ell_{py}$, shows that $(a \otimes (v-v') - (b-b') \otimes u)(\Gamma) = 0$ if and only if $y - \tau \in \{p+\tau, y+\tau\}$. Since $\tau \notin E_2$, it follows that $av - bu = a'v' - b'u'$ if and only if $-2\tau = p - y$. Equivalently $\Omega(r) = \Omega(r')$ if and only if $r + r' = -2\tau$. ■

Corollary 6.9. Let $r, r' \in E$.

(a) If $r + r' = -2\tau$, then $\Omega(r) = \Omega(r')$.

(b) If $r \in E \backslash E_2$, then $\Omega(r) \neq \Omega(-r)$.

(c) If $\Omega(r) = \Omega(r')$, then either $r = r'$, or $r + r' = -2\tau$.

(d) $\Omega(r) = \Omega(r')$ if and only if either $r = r'$ or $r + r' = -2\tau$.

Proof. (a) The result is true if $r = r'$, so suppose that $r \neq r'$. Since $\tau \notin E_2$, $r + r' \neq 0$, so (6.8) applies. Hence $\Omega(r) = \Omega(r')$.

(b) Set $s = -2\tau - r$, and $s' = -2\tau + r$. Thus $s + s' = -4\tau \neq 0$, and $s \neq s'$ because $\tau \notin E_2$, so by (6.8) $\Omega(s) \neq \Omega(s')$. However, by (a) $\Omega(s) = \Omega(r)$ and $\Omega(s') = \Omega(-r)$, hence the result.

(c) Suppose that $\Omega(r) = \Omega(r')$ and $r \neq r'$. If $r + r' \neq 0$, then (6.8) applies, proving that $r + r' = -2\tau$. On the other hand, if $r + r' = 0$, then $r \in E_2$ by (b). This in turn implies that $r = r'$, a contradiction.

(d) This follows from (a) and (c). ■

Theorem 6.10. Let $p, q, p', q' \in E$. If τ is of infinite order then $AnnM(p,q) = AnnM(p',q')$ if and only if either $p+q = p'+q'$ or $(p+q)+(p'+q') = -2\tau$.

Proof. This is a consequence of (6.3) and (6.9d). ■

For completeness we also describe the annihilator of a plane module when τ is of infinite order.

Proposition 6.11. Suppose that τ is of infinite order. Let $M(p,q,r,s)$ be a plane module. Then $AnnM(p,q,r,s) = 0$.

Proof. Set $M = M(p,q,r,s)$, and $J = Ann(M)$. Let $I = \langle \Omega(p+q) \rangle$. Then $J \subseteq I$, since $M(p,q)$ is a quotient of M. Since A/J is prime, and $GKdim(A/I) = 3$, either $J = 0$ or $J = I$. If $J = 0$ we are finished, so suppose that $J = I$. By (6.2) A/J is a domain. But M is a quotient of A/J of GK-dimension 3, so $A/J \cong M$. However, $A/J = A/\Omega A$ with $\Omega \in A_2$, so $H_{A/J}(t) = (1-t^2)$. $H_A(t) \neq (1-t)^{-3} = H_M(t)$. This contradiction shows that $J = 0$. ■

The next result answers a question of Sklyanin [17, p.269].

Proposition 6.12. Suppose that τ is of infinite order. Then the center of A is $Z(A) = \mathbb{C}[\Omega_1, \Omega_2]$.

Proof. Consider $B = A/\langle \Omega_1, \Omega_2 \rangle$. By [21], B is a noetherian domain of GK-dimension 2. Hence if the center of B is strictly larger than $\mathbb{C}$, then B satisfies a polynomial identity by [20]. Hence by [9,§10.3] $GKdim(A/\mathrm{Ann}M(p)) = GKdim M(p) = 1$, which contradicts (5.11). Hence the center of A is contained in $\mathbb{C} + \langle \Omega_1, \Omega_2 \rangle$.

First we show that the center of $\overline{A} = A/\langle \Omega_1 \rangle$ is $\mathbb{C}[\Omega_2]$. Suppose not. Pick a homogeneous $w \in \overline{A}$ of minimal degree such that w is central, but $w \notin \mathbb{C}[\Omega_2]$. By the previous paragraph the center of $\overline{A}$ is contained in $\mathbb{C} + \langle \Omega_2 \rangle$, so we can write $w = \Omega_2 w'$. But $\overline{A}$ is a domain by (6.2), so w' is also central in $\overline{A}$. However, $\deg(w') < \deg(w)$, so $w' \in \mathbb{C}[\Omega_2]$. Hence $w \in \mathbb{C}[\Omega_2]$ also.

Suppose the result is false. Pick a homogeneous central element z of minimal degree such that $z \notin \mathbb{C}[\Omega_1, \Omega_2]$. Since $z \in \langle \Omega_1, \Omega_2 \rangle$ we can write $z = \Omega_1 a_1 + \Omega_2 a_2$. Moreover, since (Ω_1, Ω_2) is a regular sequence on A, we may choose a_1, a_2 such that $\deg(z) = \max\{\deg(a_1), \deg(a_2)\}$. The image of z is central in $A/\langle \Omega_1 \rangle$, so $a_2 \in \mathbb{C}[\Omega_2] + \langle \Omega_1 \rangle$. Hence we can write $a_2 = b_2 + \Omega_1 c_2$ where $b_2 \in \mathbb{C}[\Omega_2]$. Similarly, we can write $a_1 = b_1 + \Omega_2 c_1$ where $b_1 \in \mathbb{C}[\Omega_1]$. Therefore $z = \Omega_1 b_1 + \Omega_2 b_2 + \Omega_1 \Omega_2 (c_1 + c_2)$. But $\Omega_i b_i \in \mathbb{C}[\Omega_i]$ is central, and since A is a domain, it follows that $c_1 + c_2$ is central. However, $\deg(a_i) > \deg(c_i)$, so $\deg(c_1 + c_2) < \deg(z)$. Therefore $c_1 + c_2 \in \mathbb{C}[\Omega_1, \Omega_2]$. Hence $z \in \mathbb{C}[\Omega_1, \Omega_2]$ also, which contradicts our choice of z. ■

Our final result, which completes the circle of ideas in this section, shows that every non-zero element of Z_2 annihilates some line module. The proof is rather unsatisfactory, because it does not show the real reason this happens. It would be good to find another proof. One possibility is to describe the action of A on a general $M(p,q)$ in an explicit way using the basis in (5.6). That would then allow one to explicitly calculate the action of the central elements Ω_i on the generator e_{00}, and thus determine the annihilator $\Omega(p+q)$.

The key technical points in the proof of (6.13) are as follows. Firstly, the Grassmannian of d-planes in the vector space W is denoted by $G(d,W)$. The map $G(d,W) \to G(dimW - d, W^*)$ given by $U \mapsto U^\perp$ is an isomorphism of varieties. If $m \in \mathbb{N}$, and V is a fixed subspace of W, and Y is the subset of $G(d,W)$ consisting of those subspaces U such that $dim(U \cap V) = m$, then the map $Y \to G(m,V)$ given by $U \mapsto U \cap V$ is continuous. If $m \in \mathbb{N}$, and Y is the subspace of $G(d,W) \times G(d,W)$ consisting of pairs (U_1, U_2) such that $dim(U_1 + U_2) = m$, then the map $Y \to G(m,W)$ given by $(U_1,U_2) \mapsto U_1 + U_2$ is continuous.

We will denote the linear span of points $p,q,r \in \mathbb{P}^3$ by $\overline{pqr}$.

Theorem 6.13. Every element of Z_2 annihilates some line module.

Proof. We must show that the map $E \to \mathbb{P}(Z_2) \cong \mathbb{P}^1$ given by $z \mapsto \Omega(z)$ is surjective. Define an equivalence relation on E by $z \sim z'$ if either $z = z'$ or $z + z' = -2\tau$. The projection from e_0 to a general hyperplane in $\mathbb{P}^3$ sends the singular quadric Q_0, and hence E, to a smooth conic. By (3.4) and (3.6), the fibres are $\pm z$. Identifying the conic with $\mathbb{P}^1$, gives a morphism $g : E \to \mathbb{P}^1$. Hence the map $z \mapsto g(z+\tau)$ is a morphism whose fibres are precisely the equivalence classes. Thus $E/\!\sim\, \cong \mathbb{P}^1$.

By (6.10) there is an injective map $f : E/\!\sim\, \to \mathbb{P}(Z_2)$ given by $f(z) = \Omega(z)$. Since we are working over $\mathbb{C}$, $\mathbb{P}^1$ is homeomorphic to the 2-sphere S^2, so we have an injective map $f : S^2 \to S^2$ which we wish to show is surjective. If f is continuous, then f must be surjective. To see this, first recall that a continuous bijective map from a compact space to a Hausdorff space is a homeomorphism. If f is not surjective, then its image is contained in a disc $S^2 - \{p\}$ so there would be a copy of S^2 inside the disc. This is impossible. Hence the rest of the proof is devoted to showing that $f : E/\!\sim\, \to \mathbb{P}(Z_2)$ is continuous.

Let $z, z' \in E$. Our goal is to show that if z' is 'close to z' then $\Omega(z')$ is 'close' to $\Omega(z)$. Fix some $p,q,r,s \in E$ in general position, such that $p+q = z$. Fix $u, v \in A_1$ such that $\mathbb{C}u = \overline{pqr}^\perp$, $\mathbb{C}v = \overline{pqs}^\perp$ and $\mathcal{V}(u,v) = \ell_{pq}$. Since the addition law on E is continuous, there exist $p', q' \in E$ which are close to p and q respectively, and $p' + q' = z'$. It follows that the lines $\mathbb{C}u' = \overline{p'q'r}^\perp$ and $\mathbb{C}v' = \overline{p'q's}^\perp$ are close to $\mathbb{C}u$ and $\mathbb{C}v$ respectively in $\mathbb{P}(A_1)$.

Since $\mathcal{V}(u,v)$ and $\mathcal{V}(u',v')$ are secant lines $dim(A_1u + A_1v) = dim(A_1u' + A_1v') = 7$. Therefore $A_1u + A_1v$ and $A_1u' + A_1v'$ are close to each other in $G(7, A_2)$. The fact that these are secant lines also implies $Z_2 \cap (A_1u + A_1v)$ and $Z_2 \cap (A_1u' + A_1v')$ are lines in $\mathbb{P}(Z_2)$ which are close to each other. But these lines are precisely $\Omega(z)$ and $\Omega(z')$ respectively. Hence the result. ■

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