THE QUANTUM COORDINATE RING OF THE SPECIAL LINEAR GROUP

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ABSTRACT. *We prove that, even under the multiparameter definition of* [**AST**]*, the quantum coordinate ring* $\mathcal{O}_{\mathbf{q}}(SL_n(k))$ *of the special linear group* $SL_n(k)$ *satisfies most of the standard ring-theoretic properties of the classical coordinate ring* $\mathcal{O}(SL_n(k))$.

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THE RESULTS.

Fix a field k. Let $\mathcal{O}_{\mathbf{q}} = \mathcal{O}_{\mathbf{q}}(SL_n(k))$ be the (multiparameter) quantum coordinate ring of the special linear group $SL_n(k)$ and let $\mathcal{M}_{\mathbf{q}} = \mathcal{O}_{\mathbf{q}}(M_n(k))$ be the corresponding quantum coordinate ring of all $n \times n$ matrices, as defined in [**AST**]. (The definition of these and other concepts used in this introduction are given in the next section). By definition, $\mathcal{O}_{\mathbf{q}} = \mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$, where $\Delta_{\mathbf{q}}$ is a central element in $\mathcal{M}_{\mathbf{q}}$ called the "quantum determinate". One would like to assert that the standard properties of the classical coordinate ring $\mathcal{O}(SL_n(k))$, for example integrality and finite global homological dimension, also hold for $\mathcal{O}_{\mathbf{q}}$. This is particularly true since it is easy to show that these properties do hold for $\mathcal{M}_{\mathbf{q}}$ (this follows from the fact that, as is proved in [**AST**, pp.890-1], $\mathcal{M}_{\mathbf{q}}$ is an iterated Ore extension of k in the sense of [**Co**, Section 12.2]). However, it is typically hard (and in the abstract impossible) to show that such properties pass to factor rings. The main aim of this note is to make the following observation, giving a different method for obtaining $\mathcal{O}_{\mathbf{q}}$ from $\mathcal{M}_{\mathbf{q}}$:

PROPOSITION. *Set $\mathcal{O}_{\mathbf{q}}(GL_n(k)) = \mathcal{O}_{\mathbf{q}}(M_n(k))[\Delta_{\mathbf{q}}^{-1}]$ and let z be a central indeterminate. Then*

$$\mathcal{O}_{\mathbf{q}}(SL_n(k)) \otimes_k k[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(GL_n(k)).$$

One should interpret this result as a "quantum" analogue of the well known fact that $SL_n(k) \times k^* \cong GL_n(k)$. Once stated, this Proposition is almost trivial to prove. Its significance, however, is that many desirable properties pass from a ring to a central localization. Thus, for example, the Proposition allows one to prove:

COROLLARY. *(i) $\mathcal{O}_{\mathbf{q}}$ is a domain and a maximal order in its division ring of fractions.*
(ii) $GKdim(\mathcal{O}_{\mathbf{q}}) = gldim(\mathcal{O}_{\mathbf{q}}) = n^2 - 1$.
(iii) $\mathcal{O}_{\mathbf{q}}$ is Auslander regular and CM.
(iv) $K_0(\mathcal{O}_{\mathbf{q}}) = \mathbb{Z}$.

At least for the standard 1-parameter version of $\mathcal{O}_{\mathbf{q}}$, the fact that $\mathcal{O}_{\mathbf{q}}$ is a domain can be proved in several other ways, but all of them seem to require considerably more knowledge about the structure of $\mathcal{O}_{\mathbf{q}}$. For example, for most values of the quantum parameter and for any classical group G, a proof that $\mathcal{O}_q(G)$ is a domain can be obtained by combining [**En**] and [**Jo**, Lemme 9.11], while for $G = SL_n$ it also follows from the appendix to [**APW**]. Also, D. Jordan and the authors have shown independently that $\mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}})$ is a domain, from which it follows that $\mathcal{O}_{\mathbf{q}}$ is a domain.

THE PROOFS.

Given a ring C, write C^* for the set of units of C. Let $\mathbf{q} = \{\lambda,\, q_{ij} : 1 \leq i,j \leq n\} \subset k^*$ be fixed, non-zero scalars that satisfy $\lambda \neq -1$ and $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$, for all $1 \leq i,j \leq n$. Define $\mathcal{O}_{\mathbf{q}}(k^{(n)})$ to be the k-algebra with generators $\{x_i : 1 \leq i \leq n\}$ and relations $x_j x_i = q_{ji} x_i x_j$, for all $1 \leq i,j \leq n$. Define $\mathbf{p} = \{\lambda, p_{ij}\}$ by $p_{ij} = \lambda^{-1} q_{ij}$, for all $i > j$, and, as before, $p_{ji} = p_{ij}^{-1}$ and $p_{ii} = 1$. Then $\mathcal{M}_{\mathbf{q}} = \mathcal{O}_{\mathbf{q}}(M_n(k))$ is defined to be the universal bi-algebra having $\mathcal{O}_{\mathbf{q}}(k^{(n)})$ as a left comodule algebra and $\mathcal{O}_{\mathbf{p}}(k^{(n)})$ as a right comodule algebra, in the sense of [**Ma**, Section 5]. Thus, $\mathcal{M}_{\mathbf{q}}$ is the k-algebra with generators $\{x_{ij} : 1 \leq i,j \leq n\}$ and relations defined by [**AST**, Equ. (8)]. The precise definition of these relations is not important, here, except that they are of the following form:

$$x_{ij}x_{\ell m} = \begin{cases} \alpha_{ij\ell m} x_{\ell m} x_{ij} + (\lambda - 1)\lambda^{-1} q_{im} x_{\ell j} x_{im} & \text{if } i > \ell \text{ and } j > m \\ \alpha_{ij\ell m} x_{\ell m} x_{ij} & \text{otherwise} \end{cases} \tag{1}$$

for some $\alpha_{ij\ell m} \in k^*$. We remark that the restrictions on the scalars $\mathbf{q}$ and $\mathbf{p}$ given above are precisely what is required for $\mathcal{M}_{\mathbf{q}}$ to have the same Hilbert series as a polynomial ring in n^2 variables (see [**AST**, Theorem 1]). The universal argument of [**Ma**, Section 8] shows that there exists a *quantum determinant*

$$\Delta_{\mathbf{q}} = \sum_{\pi \in S_n} \alpha_\pi x_{1,\pi(1)} x_{2,\pi(2)} \cdots x_{n,\pi(n)}, \tag{2}$$

where the $\alpha_\pi \in k^*$ are certain scalars and are defined in [**AST**, Equ. (15)]. By [**AST**, Theorem 3],

$$\Delta_{\mathbf{q}} \text{ is central} \iff \lambda^j \prod_{m=1}^{n} q_{jm} = \lambda^k \prod_{m=1}^{n} q_{km} \text{ for all } j,k. \tag{3}$$

Thus, $\mathcal{O}_{\mathbf{q}}(SL_n(k)) = \mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$ is defined precisely when (3) holds. For k sufficiently large, this gives a $\binom{n-1}{2} + 1$ parameter family of deformations of $\mathcal{O}(SL_n(k))$ (see [**GGS**, Section 14]). If $\mathcal{S} = \{\Delta_{\mathbf{q}}^r : r \geq 1\}$, we define $\mathcal{O}_{\mathbf{q}}(GL_n(k)) = (\mathcal{M}_{\mathbf{q}})_{\mathcal{S}} = \mathcal{M}_{\mathbf{q}}[\Delta_{\mathbf{q}}^{-1}]$.

Finally, if C is a commutative k-algebra, we define $\mathcal{O}_{\mathbf{q}}(C) = \mathcal{O}_{\mathbf{q}}(SL_n(k)) \otimes_k C$ and $\mathcal{M}_{\mathbf{q}}(C) = \mathcal{O}_{\mathbf{q}}(M_n(k)) \otimes_k C$. The results of this paper actually hold if k is taken to be any Noetherian, commutative domain (in which case it is unnecessary to define $\mathcal{O}_{\mathbf{q}}(C)$) but, in order to prove this, one first needs to prove the corresponding generalization of [**AST**].

The standard, one parameter quantum coordinate ring $\mathcal{O}_q(SL_n(k))$ of $SL_n(k)$, as for example defined in [**Ma**] or [**PW**], is obtained by taking $\lambda = q^2$ and $q_{ij} = q$ for all $i > j$.

PROOF OF THE PROPOSITION. Let C be a commutative k-algebra and $\mu \in C^*$. Then, as an C-algebra, $\mathcal{M}_{\mathbf{q}}(C) = \mathcal{O}_{\mathbf{q}}(M_n(k)) \otimes_k C$ is still defined by the relations given in (1). The important point to note about these relations is that they are homogeneous in the set of variables $\{x_{1j} : 1 \leq j \leq n\}$. In other words, if a given relation from (1) has r occurrences of elements from the set $\{x_{1j}\}$ occurring in one monomial, then every monomial in that relation has r occurrences of elements from $\{x_{1j}\}$. Thus, there is an C-algebra automorphism σ_μ of $\mathcal{M}_{\mathbf{q}}(C)$ defined by

$$\sigma_\mu(x_{1j}) = \mu^{-1}x_{1j} \text{ and } \sigma_\mu(x_{ij}) = x_{ij}, \qquad \text{for all } 1 \leq j \leq n \text{ and } 2 \leq i \leq n.$$

By the description of $\Delta_{\mathbf{q}}$ in (2), one sees that $\sigma_\mu(\Delta_{\mathbf{q}}) = \mu^{-1}\Delta_{\mathbf{q}}$. Now assume that $C = k[z, z^{-1}]$, for an indeterminate z. Then:

$$\begin{aligned} \mathcal{O}_{\mathbf{q}}(SL_n(k))[z, z^{-1}] &\cong \mathcal{O}_{\mathbf{q}}(SL_n(k)) \otimes_k k[z, z^{-1}] &&\cong \mathcal{M}_{\mathbf{q}} \otimes_k k[z, z^{-1}]/(\Delta_{\mathbf{q}} - 1) \\ &\overset{\sigma_z}{\cong} \mathcal{M}_{\mathbf{q}} \otimes_k k[z, z^{-1}]/(\Delta_{\mathbf{q}} - z) &&\cong \mathcal{M}_{\mathbf{q}}[\Delta_{\mathbf{q}}^{-1}]. \end{aligned}$$

Thus, $\mathcal{O}_{\mathbf{q}}(SL_n(k))[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(GL_n(k))$. $\square$

Another way of viewing this result is as follows: Under the isomorphism, $\mathcal{O}_{\mathbf{q}}[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(GL_n(k))$, the element z maps to $\Delta_{\mathbf{q}}$, and so, by inverting $\mathcal{C} = k[z]^*$, respectively $\mathcal{D} = k[\Delta_{\mathbf{q}}]^*$, we obtain $\mathcal{O}_{\mathbf{q}}(SL_n(k(z))) \cong (\mathcal{M}_{\mathbf{q}})_{\mathcal{D}}$.

Let M be a finitely generated module over a Noetherian k-algebra A. Then the Gelfand-Kirillov and homological dimensions of M will be denoted by $GKdim(M)$, respectively $hd(M)$. The global homological dimension of A will be written $gldim(A)$. If the injective dimensions of ${}_AA$ and A_A are finite, then they are equal, by [**Za**, Lemma A], and this integer will be denoted by $injdim(A)$. If $injdim(A) < \infty$, then A is called *Auslander-Gorenstein* if A satisfies the following condition: For any integers $0 \leq i < j$ and finitely generated (right) A-module M, one has $\mathrm{Ext}^i_A(N, A) = 0$ for all (left) A-submodules N of $\mathrm{Ext}^j_A(M, A)$. If A is an Auslander-Gorenstein ring of finite global dimension, then A is called *Auslander-regular*. Set $j(M) = \min\{j : \mathrm{Ext}^j(M, A) \neq 0\}$. The ring A is *CM* if $j(M) + GKdim(M) = Gkdim(A)$ holds for all finitely generated A-modules M.

Before proving the Corollary, we need the following result that provides some more-or-less well known facts about these conditions.

Lemma. *Suppose that R is a Noetherian ring that is Auslander-regular and CM. Let $S = R[x; \sigma, \delta]$ be an Ore extension, in the sense of* [**Co**, *Section 12.2*]. *Then:*

(i) S is Auslander-regular.

(ii) Assume that $R = \bigoplus_{i\geq 0} R_i$ is a connected graded k-algebra (thus $R_0 = k$) such that $\sigma(R_i) \subseteq R_i$ for each $i \geq 0$. Then S is CM.

(iii) Let f be a central, regular element of R. Then R/fR is Auslander-Gorenstein and CM.

Proof: (i) This follows from [**Ek**, Theorem 4.2].

(ii) Filter S by degree in x and note that the corresponding graded ring $\mathrm{gr}(S)$ is isomorphic to $R[y;\sigma]$. The hypotheses on R ensure that $\mathrm{gr}(S)$ has the structure of a connected graded ring, defined by $\mathrm{gr}(S)_n = \bigoplus_{i+j=n} R_i y^j$. Moreover, y is a normal, homogeneous element in $\mathrm{gr}(S)$ and $gr(S)/y\,\mathrm{gr}(S) \cong R$. Hence, by [**Lv**, Theorem 3.6], $\mathrm{gr}(S)$ is (graded) CM and, by part (i) $\mathrm{gr}(S)$ is Auslander-regular. If M is a finitely generated S-module, give M a good filtration and consider the associated graded $\mathrm{gr}(S)$-module $\mathrm{gr}(M)$. Then, by [**Bj**, Theorem 4.3], $j_S(M) = j_{\mathrm{gr}(S)}(\mathrm{gr}(M))$ while, by [**MS**, Theorem 1.3], $GKdim_S(M) = GKdim_{\mathrm{gr}(S)}(\mathrm{gr}(M))$. Thus,

$$j_S(M) = GKdim(\mathrm{gr}(S)) - GKdim(\mathrm{gr}(M)) = GKdim(S) - GKdim(M)$$

and S is CM.

(iii) To avoid triviality, assume that f is not a unit. Set $\overline{R} = R/fR$. Since $j_R(\overline{R}) = 1$, the CM condition implies that $GKdim(\overline{R}) = GKdim(R) - 1$. Let M be a finitely generated $\overline{R}$-module. By the Rees Lemma [**Ro**, Theorem 9.37], $\mathrm{Ext}^j_R(M,R) = \mathrm{Ext}^{j-1}_{\overline{R}}(M,\overline{R})$, for each $j \geq 1$. It follows that $\overline{R}$ is Auslander-Gorenstein and CM. $\square$

The extra conditions in part (ii) of the Lemma are necessary since, in general, $GKdim(M) \neq GKdim(\mathrm{gr}(M))$. For example, suppose that $R = \mathbb{C}[z, z^{-1}, y]$, where z and y are central indeterminates, and σ is the $\mathbb{C}$-automorphism of R defined by $\sigma(z) = z$ but $\sigma(y) = zy$. Then, let $S = R[x;\sigma,0]$ and set $M = S/(x-1)S$ and $N = S/xS$ (thus, $N = \mathrm{gr}(M)$ in the notation of the proof of part (ii) of the Lemma). It follows from [**MS**, Proposition 3.4] that $j(M) = j(N) = 1$ but $GKdim(M) = 3 > 2 = GKdim(N)$.

PROOF OF THE COROLLARY. Order the generators x_{ij} of $\mathcal{M}_{\mathbf{q}}$ lexicographically and consider the corresponding chain of rings

$$k\langle x_{11}\rangle \subset k\langle x_{11}\rangle\langle x_{12}\rangle \subset \cdots \subset \mathcal{M}_{\mathbf{q}}. \qquad (\dagger)$$

If $R \subset S = R\langle x\rangle$ is a successive pair of rings from this chain, then [**AST**, pp.890-1] shows that there exists a k-algebra automorphism τ and a τ-derivation δ of R such that S is isomorphic to the Ore extension $R[x;\tau,\delta]$. Thus $\mathcal{M}_{\mathbf{q}}$ is an iterated Ore extension.

(i) By [**MR**, Theorem 1.2.9], $\mathcal{M}_{\mathbf{q}}$ is a Noetherian domain. Since $\mathcal{O}_{\mathbf{q}} = \mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$, certainly $\mathcal{O}_{\mathbf{q}}$ is Noetherian. By the Proposition, $\mathcal{O}_{\mathbf{q}}[z, z^{-1}] \cong (\mathcal{M}_{\mathbf{q}})_{\mathcal{S}}$ is a domain, and hence so is its subring $\mathcal{O}_{\mathbf{q}}$. By [**MaR**, Proposition V.2.5], $\mathcal{M}_{\mathbf{q}}$ is a maximal order in its division ring of fractions D; that is, if $\mathcal{M}_{\mathbf{q}} \subseteq T \subseteq D$, for some ring T such that $aTb \subseteq \mathcal{M}_{\mathbf{q}}$ for some non-zero elements $a, b \in \mathcal{M}_{\mathbf{q}}$, then $T = \mathcal{M}_{\mathbf{q}}$. By [**MaR**, Proposition IV.2.1], $(\mathcal{M}_{\mathbf{q}})_{\mathcal{S}}$ is also a maximal order in D. It follows easily from the Proposition that $\mathcal{O}_{\mathbf{q}}$ is a maximal order. This proves part (i) of the corollary.

(ii) and (iii) By [**AST**, Proposition 2, and its proof], $GKdim(\mathcal{M}_{\mathbf{q}}) = gldim(\mathcal{M}_{\mathbf{q}}) = n^2$. Thus, by the Proposition and [**MR**, Theorem 7.5.3(iv)],

$$gldim(\mathcal{O}_{\mathbf{q}}) = gldim(\mathcal{O}_{\mathbf{q}}[z, z^{-1}]) - 1 = gldim((\mathcal{M}_{\mathbf{q}})_{\mathcal{S}}) - 1 \leq gldim(\mathcal{M}_{\mathbf{q}}) - 1 = n^2 - 1.$$

By (1), $\mathcal{M}_{\mathbf{q}}$ has the structure of a connected graded ring, by giving each x_{ij} degree one. If $R \subset S = R[x; \sigma, \delta]$ are a pair of successive rings in the chain (†), then this induces, on R, the structure of connected graded ring $R = \bigoplus_{i \geq 0} R_i$ and implies that $\sigma(R_j) \subseteq R_j$, for each j. Thus, by part (ii) of the Lemma and induction, $\mathcal{M}_{\mathbf{q}}$ is Auslander-regular and CM. Now regard $\mathcal{O}_{\mathbf{q}}$ as $\mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$. Then, part (iii) of the Lemma implies that $\mathcal{O}_{\mathbf{q}}$ is Auslander-Gorenstein and CM, with $GKdim(\mathcal{O}_{\mathbf{q}}) = GKdim(\mathcal{M}_{\mathbf{q}}) - 1 = n^2 - 1$. Since $gldim(\mathcal{O}_{\mathbf{q}}) < \infty$, this implies that $\mathcal{O}_{\mathbf{q}}$ is Auslander regular.

It remains to show that $gldim(\mathcal{O}_{\mathbf{q}}) \geq n^2 - 1$. Consider the factor ring

$$A = \mathcal{O}_{\mathbf{q}}/(x_{ij} : i \neq j,\, x_{\ell\ell} - 1 : \ell \neq 1).$$

The description of the relations of $\mathcal{O}_{\mathbf{q}}$ in (1) and (2) imply that $A \cong k[x_{11}]/(x_{11} - \gamma)$, for some $\gamma \in k^*$. Therefore, both A and $\mathcal{O}_{\mathbf{q}}$ have a 1-dimensional module S. (This also follows from [**AST**, Theorem 3].) But, by the CM condition, $j(S) = GKdim(\mathcal{O}_{\mathbf{q}}) - GKdim(S) = n^2 - 1$. Thus, $gldim(\mathcal{O}_{\mathbf{q}}) \geq n^2 - 1$. This completes the proof of parts (ii) and (iii) of the Corollary.

(iv) By the proof of (ii), $gldim(\mathcal{M}_{\mathbf{q}}) < \infty$. Thus, by [**MR**, Corollary 12.3.6 and Theorem 12.6.13], $K_0(\mathcal{M}_{\mathbf{q}}) = \mathbb{Z}$. Therefore, by the Proposition and [**MR**, Proposition 12.1.12], $K_0(\mathcal{O}_{\mathbf{q}}[z, z^{-1}]) = K_0((\mathcal{M}_{\mathbf{q}})_{\mathcal{S}}) = \mathbb{Z}$. By [**MR**, Corollary 12.3.6], this implies that $K_0(\mathcal{O}_{\mathbf{q}}) = \mathbb{Z}$. □

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