INVARIANT DISTRIBUTIONS SUPPORTED ON THE NILPOTENT CONE OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with adjoint group G and $\mathcal{D}(\mathfrak{g})$ be the algebra of differential operators with polynomial coefficients on $\mathfrak{g}$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, we give the decomposition of the semisimple $\mathcal{D}(\mathfrak{g})^G$-module of invariant distributions on $\mathfrak{g}_0$ supported on the nilpotent cone.

0. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with adjoint group G. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let W be the associated Weyl group. Denote by $W^\frown$ the set of isomorphism classes of irreducible W-modules and by $\mathcal{H}(\mathfrak{h}^*)$ the graded vector space of W-harmonic polynomials on $\mathfrak{h}$. For $\chi \in W^\frown$, set

$$b(\chi) = \inf\{j \in \mathbb{N} : [\mathcal{H}^j(\mathfrak{h}^*) : \chi] \neq 0\}$$

and choose a W-submodule $V_\chi \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)$ in the class of χ. Denote by $d(\chi)$ the dimension of V_χ.

Let $S(\mathfrak{g}^*)$ be the algebra of polynomial functions on $\mathfrak{g}$ and $\mathcal{D}(\mathfrak{g})$ be the algebra of differential operators on $\mathfrak{g}$, with coefficients in $S(\mathfrak{g}^*)$. The group G acts on $\mathfrak{g}$, via the adjoint action, and hence has an induced action on $S(\mathfrak{g}^*)$, $S(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. Denote the differential of this action by $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$. Let $S_+(\mathfrak{g})^G$ and $S_+(\mathfrak{g}^*)^G$ be the set of invariant elements without constant term. Recall that $\mathbf{N}(\mathfrak{g})$, the nilpotent cone of $\mathfrak{g}$, is the variety of zeroes of the ideal $S_+(\mathfrak{g}^*)^G S(\mathfrak{g}^*)$.

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ with adjoint group $G_0 \subset G$. Denote by $\mathsf{Db}(\mathfrak{g}_0)$ the $\mathcal{D}(\mathfrak{g})$-module of distributions on $\mathfrak{g}_0$. Then, the subspace of invariant distributions $\mathsf{Db}(\mathfrak{g}_0)^{G_0} = \{T \in \mathsf{Db}(\mathfrak{g}_0) : \tau(\mathfrak{g}).T = 0\}$ is a $\mathcal{D}(\mathfrak{g})^G$-module, containing the submodule of invariant distributions supported on the nilpotent cone

$$\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil} = \big\{\Theta \in \mathsf{Db}(\mathfrak{g}_0)^{G_0} : \operatorname{Supp}\Theta \subset \mathbf{N}(\mathfrak{g}_0)\big\}$$

where $\mathbf{N}(\mathfrak{g}_0) = \mathbf{N}(\mathfrak{g}) \cap \mathfrak{g}_0$ is the nilpotent cone of $\mathfrak{g}_0$. The structure of $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$ as a vector space is well understood, see, for example, [1, 5]. Let $[\mathfrak{h}_1], \dots, [\mathfrak{h}_r]$ be the conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$. For each j, let $\varepsilon_{I,j} : W(\mathfrak{h}_j) \to \{\pm 1\}$ be the imaginary signature of the real Weyl group $W(\mathfrak{h}_j)$. Then [5, Proposition 6.1.1] there exists a vector space isomorphism

$$(*) \qquad \bigoplus_{j=1}^r S(\mathfrak{h}_{j,\mathbb{C}})^{\varepsilon_{I,j}} \xrightarrow{\sim} \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$$

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where $S(\mathfrak{h}_{j,\mathbb{C}})^{\varepsilon_{I,j}}$ is the isotypic component of type $\varepsilon_{I,j}$ in the $W(\mathfrak{h}_j)$-module $S(\mathfrak{h}_{j,\mathbb{C}})$.

One aim of this note is to give a complete description of the $\mathcal{D}(\mathfrak{g})^G$-module $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$. This description is given in terms of the simple summands of the equivariant holonomic $\mathcal{D}(\mathfrak{g})$-module

$$\mathcal{M} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G\big).$$

By [9], [18] or [13], it is known that we have a decomposition

$$\mathcal{M} = \bigoplus\nolimits_{\chi\in W^\wedge} d(\chi)\,\mathcal{M}_\chi$$

where the $\mathcal{M}_\chi$ are pairwise non-isomorphic simple $\mathcal{D}(\mathfrak{g})$-modules. Moreover, the support (in $\mathfrak{g}$) of $\mathcal{M}_\chi$ is the closure of a nilpotent orbit and $\mathcal{M}_\chi^G$ is a simple $\mathcal{D}(\mathfrak{g})^G$-module. Then we have, see Corollary 3.6:

Theorem A. *The $\mathcal{D}(\mathfrak{g})^G$-module $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$ decomposes as*

$$\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil} \cong \bigoplus\nolimits_{\chi\in W^\wedge} m_\chi \mathcal{M}_\chi^G$$

where $m_\chi = \sum_{j=1}^r \dim V_\chi^{\varepsilon_{I,j}}$.

This theorem is proved by combining the isomorphism $(*)$ and the properties, established in [18, 11, 12, 13], of the Harish-Chandra homomorphism

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W.$$

In the particular case where $\mathfrak{g}_0$ is a complex Lie algebra $\mathfrak{g}_1$ (viewed as a real Lie algebra), Theorem A was proved by N. Wallach [18]. In this case, $\mathfrak{g} \simeq \mathfrak{g}_1 \times \mathfrak{g}_1$, $W \simeq W_1 \times W_1$ where W_1 is the Weyl group of $\mathfrak{g}_1$. Then, each $\mathcal{M}_\chi$ occuring in the decomposition of $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$ is of the form $\mathcal{M}_\phi \boxtimes \mathcal{M}_\phi$ with $\chi = \phi \boxtimes \phi$, $\phi \in W_1^\wedge$, and one has $m_\chi = 1$. Hence $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil} \cong \oplus_{\phi\in W_1^\wedge} \mathcal{M}_\phi^{G_1} \boxtimes \mathcal{M}_\phi^{G_1}$ as a $\mathcal{D}(\mathfrak{g})^G$-module.

The next corollary is an easy consequence of Theorem A.

Corollary B. *Let $\chi \in W^\wedge$. Then, $\mathcal{M}_\chi \cong \mathcal{D}(\mathfrak{g}).\Theta$ for some $\Theta \in \mathsf{Db}(\mathfrak{g}_0)$ if, and only if, $V_\chi^{\varepsilon_{I,j}} \neq 0$ for some $j \in \{1,\dots,r\}$.*

In Remark 3.7, we apply this result to give examples of modules $\mathcal{M}_\chi$ which cannot be generated by a distribution on any real form of $\mathfrak{g}$.

1. Preliminary results

We retain the notation of the introduction. Denote by Δ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and fix a system Δ^+ of positive roots. Set $n = \dim\mathfrak{g}$, $\ell = \dim\mathfrak{h}$ and $\nu = \#\Delta^+$, hence $n = 2\nu + \ell$. Let π be the product of positive roots and recall that $x \in \mathfrak{g}$ is called generic if $\pi(x) \neq 0$. If $\mathfrak{a} \subset \mathfrak{g}$, we denote by $\mathfrak{a}'$ the set of generic elements in $\mathfrak{a}$.

For $q \in S(\mathfrak{g})$, let $\partial(q) \in \mathcal{D}(\mathfrak{g})$ be the corresponding differential operator with constant coefficients. Let $\{e_i\}_{1\leqslant i\leqslant n}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form κ such that $\{e_i\}_{1\leqslant i\leqslant \ell}$ is a basis of $\mathfrak{h}$. Denote by $x_i \in S(\mathfrak{g}^*)$, $1 \leqslant i \leqslant n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\mathfrak{g}$ and $\mathfrak{h}$ by $\mathsf{E}_\mathfrak{g} = \sum_{i=1}^n x_i\partial_i$ and $\mathsf{E}_\mathfrak{h} = \sum_{i=1}^\ell x_i\partial_i$.

We now give some notation and results from [11, 12, 13, 18]. Recall first that the algebra homomorphism, defined by Harish-Chandra,

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$$

extends the Chevalley isomorphisms $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$ and $S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{h}^*)^W$. The map δ is surjective and its kernel is $\mathcal{I} = (\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$. This enables one to identify, through δ, modules over $A(\mathfrak{g}) := \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$ with $\mathcal{D}(\mathfrak{h})^W$-modules.

Lemma 1.1. *Let $D \in \mathcal{D}(\mathfrak{g})^G$. Then $D = P + Q$ with $P \in \mathbb{C}\langle S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G\rangle$ and $Q \in \mathcal{I}$.*

Proof. By [11], we know that $\mathcal{D}(\mathfrak{h})^W = \mathbb{C}\langle S(\mathfrak{h})^W, S(\mathfrak{h}^*)^W\rangle$. The lemma is therefore consequence of the properties of δ previously recalled. □

Recall that the $(\mathcal{D}(\mathfrak{h})^W, W)$-module $S(\mathfrak{h}^*)$ decomposes as

$$S(\mathfrak{h}^*) \cong \bigoplus\nolimits_{\chi \in W^\wedge} V^\chi \otimes_{\mathbb{C}} V_\chi \tag{1.1}$$

where $V^\chi = \mathrm{Hom}_W(V_\chi, S(\mathfrak{h}^*))$ is a simple $\mathcal{D}(\mathfrak{h})^W$-module. Let $\{v_\chi^1, \dots, v_\chi^{d(\chi)}\}$ be a basis of V_χ, then $V^\chi \cong \mathcal{D}(\mathfrak{h})^W.v_\chi^j$ for all j and (1.1) implies that

$$S(\mathfrak{h}^*) = \bigoplus\nolimits_{\chi \in W^\wedge} \bigoplus\nolimits_{j=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W.v_\chi^j.$$

Now, set $\mathcal{N} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} S(\mathfrak{h}^*)$ and $\mathcal{N}_\chi = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} V^\chi$. We have

$$\mathcal{N} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G\big) \tag{1.2}$$

and, using (1.1),

$$\mathcal{N} = \bigoplus\nolimits_{\chi \in W^\wedge} \mathcal{N}_\chi \otimes_{\mathbb{C}} V_\chi. \tag{1.3}$$

Then each $\mathcal{N}_\chi$ is a simple (holonomic) $\mathcal{D}(\mathfrak{g})$-module [13] and, therefore, $\mathcal{N}$ is a semisimple $\mathcal{D}(\mathfrak{g})$-module (see also [9]). Let $\mathsf{C}(\mathcal{N})$ be the full subcategory of finitely generated $\mathcal{D}(\mathfrak{g})$-modules of the form $\oplus_{\chi \in W^\wedge} m_\chi \mathcal{N}_\chi$, $m_\chi \in \mathbb{N}$. From [13] we know that the category $\mathsf{C}(\mathcal{N})$ is equivalent to the category W-mod (of finite dimensional W-modules) via the functor

$$\mathrm{Sol} : \mathsf{C}(\mathcal{N}) \longrightarrow W\text{-mod}, \quad \mathrm{Sol}(N) = \mathrm{Hom}_{\mathcal{D}(\mathfrak{h})^W}(N^G, S(\mathfrak{h}^*))$$

where W acts on $\mathrm{Sol}(N)$ through its natural action on $S(\mathfrak{h}^*)$.

The Killing form κ induces a G-isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ and an algebra automorphism $\varkappa$ of $\mathcal{D}(\mathfrak{g})$, defined by $\varkappa(\partial(v)) = \kappa(v, \_)$, $\varkappa(\kappa(v, \_)) = -\partial(v)$, for all $v \in \mathfrak{g}$. Hence, in coordinates, $\varkappa(\partial_j) = x_j$, $\varkappa(x_j) = -\partial_j$. Set $i = \sqrt{-1} \in \mathbb{C}$ and denote by $\mathtt{i}$ the automorphism of $\mathcal{D}(\mathfrak{g})$ given by $\mathtt{i}(\partial_j) = -i\partial_j$, $\mathtt{i}(x_j) = ix_j$. Define then the "Fourier transformation" $F_\mathfrak{g} \in \mathrm{Aut}\,\mathcal{D}(\mathfrak{g})$ by $F_\mathfrak{g} = \mathtt{i} \circ \varkappa = \varkappa \circ \mathtt{i}^{-1}$; thus $F_\mathfrak{g}(x_j) = i\partial_j$, $F_\mathfrak{g}(\partial_j) = ix_j$. One easily checks that $\varkappa(\tau(x)) = F_\mathfrak{g}(\tau(x)) = \tau(x)$ for all $x \in \mathfrak{g}$; moreover, $\varkappa$ and $F_\mathfrak{g}$ are G-equivariant. Similarly, since κ is non degenerate and W-invariant on $\mathfrak{h}$, one can define W-equivariant automorphisms $\varkappa$ and $F_\mathfrak{h} = \mathtt{i} \circ \varkappa$ in $\mathrm{Aut}\,\mathcal{D}(\mathfrak{h})$.

Lemma 1.2. *One has $\delta \circ F_\mathfrak{g} = F_\mathfrak{h} \circ \delta$.*

Proof. A direct computation shows that $\delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P))$ when P belongs to $S(\mathfrak{g})^G$ or $S(\mathfrak{g}^*)^G$. Since δ is a homomorphism, it follows that $\delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P))$ for all $P \in \mathbb{C}\langle S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G\rangle$. Now, let $D \in \mathcal{D}(\mathfrak{g})^G$ and write $D = P + Q$ as in Lemma 1.1. Then, since $F_\mathfrak{g}(\mathcal{I}) = \mathcal{I}$, we have $\delta(F_\mathfrak{g}(D)) = \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) = F_\mathfrak{h}(\delta(D))$. □

Recall that $\mathcal{H}(\mathfrak{h}^*)$ is the vector space of W-harmonic polynomials on $\mathfrak{h}$. Hence

$$\mathcal{H}(\mathfrak{h}^*) = \{f \in S(\mathfrak{h}^*) : \partial(q).f = 0 \text{ for all } q \in S_+(\mathfrak{h})^W\}$$

and, as W-module, $\mathcal{H}(\mathfrak{h}^*)$ identifies with the regular representation of W. The vector space $\mathcal{H}(\mathfrak{h}^*)$ is a graded subspace of $S(\mathfrak{h}^*)$ and we set $\mathcal{H}^j(\mathfrak{h}^*) = S^j(\mathfrak{h}^*) \cap \mathcal{H}(\mathfrak{h}^*)$, $0 \leqslant j \leqslant \nu$. Define the harmonic elements of $S(\mathfrak{h})$ by $\mathcal{H}(\mathfrak{h}) = F_{\mathfrak{h}}(\mathcal{H}(\mathfrak{h}^*)) = \bigoplus_{j=0}^{\nu}\mathcal{H}^j(\mathfrak{h})$. (We could as well have set $\mathcal{H}(\mathfrak{h}) = \varkappa(\mathcal{H}(\mathfrak{h}^*))$, since $\mathcal{H}^j(\mathfrak{h}^*)$ is stable under $\mathtt{i}$.)

Since $V_\chi \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)$, we have $(\mathtt{E}_{\mathfrak{h}} - b(\chi)).v_\chi^j = 0$. For all $d \in L := \operatorname{ann}_{\mathcal{D}(\mathfrak{h})^W}(v_\chi^j)$, we have $[\mathtt{E}_{\mathfrak{h}} - b(\chi), d] = [\mathtt{E}_{\mathfrak{h}}, d] \in L$. It follows that $L = \bigoplus_{k\in\mathbb{Z}} L \cap \mathcal{D}^k(\mathfrak{h})^W$, where $\mathcal{D}^k(\mathfrak{h}) = \{d \in \mathcal{D}(\mathfrak{h}) : [\mathtt{E}_{\mathfrak{h}}, d] = kd\}$. Equivalently, L is stable under the $\mathbb{C}^*$-action on $\mathcal{D}(\mathfrak{h})$ given by $f \mapsto \lambda f$, $\partial(v) \mapsto \lambda^{-1}\partial(v)$, $f \in \mathfrak{h}^*$, $v \in \mathfrak{h}$. In particular, we see that $F_{\mathfrak{h}}(L) = \varkappa(L)$.

Let R be a ring and $\alpha \in \operatorname{Aut}(R)$. If M is an R-module, we define the R-module M^α to be the abelian group M with action of $a \in R$ on $x \in M$ given by $a_\bullet x = \alpha(a)x$. This applies to the modules $\mathcal{N}$, $\mathcal{N}_\chi$ and the automorphism $\alpha = {F_{\mathfrak{g}}}^{-1}$. Define

$$\mathcal{M} = \mathcal{N}^{{F_{\mathfrak{g}}}^{-1}}, \qquad \mathcal{M}_\chi = \mathcal{N}_\chi^{{F_{\mathfrak{g}}}^{-1}}.$$

Thus, from (1.2) and (1.3), we obtain

$$\mathcal{M} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G\big) \cong \bigoplus_{\chi\in W^\wedge} \mathcal{M}_\chi \otimes_{\mathbb{C}} V_\chi.$$

Remark. In [13] one defines $\mathcal{M}_\chi$ to be $\mathcal{N}_\chi^{\varkappa^{-1}}$, but the two definitions agree. Indeed, let $V^\chi \cong \mathcal{D}(\mathfrak{h})^W.v_\chi^j = \mathcal{D}(\mathfrak{h})^W/L$ be as above. Then,

$$\mathcal{N}_\chi \cong \mathcal{D}(\mathfrak{g})/J, \quad J = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G + \mathcal{D}(\mathfrak{g})\delta^{-1}(L).$$

Write $\mathcal{N}_\chi = \mathcal{D}(\mathfrak{g}).(\bar{1}\otimes_{A(\mathfrak{g})} v_\chi^j)$, where $\bar{1}$ is the canonical generator of $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. From $\delta(\mathtt{E}_{\mathfrak{g}}) = \mathtt{E}_{\mathfrak{h}} - \nu$, we get that $(\mathtt{E}_{\mathfrak{g}} - (b(\chi) - \nu)).(\bar{1}\otimes_{A(\mathfrak{g})} v_\chi^j) = 0$. It follows (as above) that J is stable under the natural $\mathbb{C}^*$-action on $\mathcal{D}(\mathfrak{g})$. Hence, $F_{\mathfrak{g}}(J) = \varkappa(J)$ and we have $\mathcal{N}_\chi^{\varkappa^{-1}} = \mathcal{N}_\chi^{{F_{\mathfrak{g}}}^{-1}}$.

We can define the category $\mathsf{C}(\mathcal{M})$ similar to $\mathsf{C}(\mathcal{N})$. We clearly have $M \in \mathsf{C}(\mathcal{M})$ if, and only if, $N = M^{F_{\mathfrak{g}}} \in \mathsf{C}(\mathcal{N})$. Moreover, by [13], this is equivalent to saying that M is a G-equivariant finitely generated $\mathcal{D}(\mathfrak{g})$-module such that $M = \mathcal{D}(\mathfrak{g})M^G$ and $\operatorname{Supp} M \subset \mathbf{N}(\mathfrak{g})$. This is also equivalent to: N is a G-equivariant finitely generated $\mathcal{D}(\mathfrak{g})$-module such that $N = \mathcal{D}(\mathfrak{g})N^G$ and N is S_+-finite (meaning that each $v \in N$ is killed by a power of $S_+(\mathfrak{g})^G$).

Recall that $\mathcal{N}_\chi^G \xrightarrow{\sim} V^\chi$ through the identification of $A(\mathfrak{g})$ with $\mathcal{D}(\mathfrak{h})^W$.

Lemma 1.3. *One has:* $\mathcal{M}_\chi^G \xrightarrow{\sim} (V^\chi)^{{F_{\mathfrak{h}}}^{-1}}$.

Proof. Write $\mathcal{N}_\chi = \mathcal{D}(\mathfrak{g})/J$. Then, $\mathcal{M}_\chi = \mathcal{D}(\mathfrak{g})/F_{\mathfrak{g}}(J)$ and $\mathcal{M}_\chi^G = \mathcal{D}(\mathfrak{g})^G/F_{\mathfrak{g}}(J^G)$. By Lemma 1.2, $\delta(F_{\mathfrak{g}}(J^G)) = F_{\mathfrak{h}}(\delta(J^G))$, therefore $\mathcal{M}_\chi^G \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^W/F_{\mathfrak{h}}(\delta(J^G))$. Since $V^\chi \cong \mathcal{D}(\mathfrak{h})^W/\delta(J^G)$, the lemma follows. □

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ with adjoint group $G_0 \subset G$. There exists a natural action of $\mathcal{D}(\mathfrak{g})$ on $\mathsf{Db}(\mathfrak{g}_0)$ defined by

$$\langle \partial(v).T, f\rangle = \langle T, -\partial(v).f\rangle, \quad \langle \xi.T, f\rangle = \langle T, \xi f\rangle$$

for all $T \in \mathsf{Db}(\mathfrak{g}_0)$, $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$, $v \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. This induces a structure of $\mathcal{D}(\mathfrak{g})^G$-module on $\mathsf{Db}(\mathfrak{g}_0)^{G_0}$. From $\mathcal{I}.\,\mathsf{Db}(\mathfrak{g}_0)^{G_0} = 0$, we obtain a natural $A(\mathfrak{g})$-module structure on $\mathsf{Db}(\mathfrak{g}_0)^{G_0}$.

Fix a basis $\{u_1, \dots, u_n\}$ of $\mathfrak{g}_0$ such that $\kappa(u_j, u_k) = \pm\delta_{jk}$ and denote by dy be the Lebesgue measure associated to this choice. Let $\mathcal{S}(\mathfrak{g}_0)$ be the Schwartz space on $\mathfrak{g}_0$. Define, as in [18, Appendix 1], the Fourier transform of $f \in \mathcal{S}(\mathfrak{g}_0)$ by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathfrak{g}_0} f(y) e^{-i\kappa(y,x)} dy$$

Let T be a tempered distribution on $\mathfrak{g}_0$. The Fourier transform of T is defined by $\langle \widehat{T}, f\rangle = \langle T, \hat{f}\rangle$ for $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$. Then we have

$$\forall\, D \in \mathcal{D}(\mathfrak{g}), \quad \forall\, T \in \mathsf{Db}(\mathfrak{g}_0), \quad \widehat{D.T} = F_{\mathfrak{g}}(D).\widehat{T}. \tag{1.4}$$

Recall [2] that $T \in \mathsf{Db}(\mathfrak{g}_0)$ is said to be homogeneous of degree d if, for all $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$, $t \in \mathbb{R}^*$, $\langle T, f_t\rangle = t^d \langle T, f\rangle$, where $f_t(v) = t^{-n} f(t^{-1}v)$. Then, a homogeneous distribution of degree d is tempered and satisfies $\mathtt{E}_{\mathfrak{g}}.T = dT$. We will need the following well known result:

Lemma 1.4. *Let $T \in \mathsf{Db}(\mathfrak{g}_0)$ be tempered and set $M = \mathcal{D}(\mathfrak{g}).T$. Then $M^{F_{\mathfrak{g}}} \cong \mathcal{D}(\mathfrak{g}).\widehat{T}$.*

Proof. By (1.4) we have $\operatorname{ann}_{\mathcal{D}(\mathfrak{g})}(\widehat{T}) = {F_{\mathfrak{g}}}^{-1}(\operatorname{ann}_{\mathcal{D}(\mathfrak{g})}(T))$. Hence the result. □

Let $\mathbf{N}(\mathfrak{g}_0)$ be the set of nilpotent elements of $\mathfrak{g}_0$. Define $\mathcal{D}(\mathfrak{g})$-submodules of $\mathsf{Db}(\mathfrak{g}_0)$ by

$$\mathsf{Db}(\mathfrak{g}_0)_{nil} = \{\Theta \in \mathsf{Db}(\mathfrak{g}_0) : \operatorname{Supp}\Theta \subset \mathbf{N}(\mathfrak{g}_0)\}$$
$$\mathsf{Db}(\mathfrak{g}_0)_{S_+} = \{T \in \mathsf{Db}(\mathfrak{g}_0) : \exists k \in \mathbb{N},\ (S_+(\mathfrak{g})^G)^k.T = 0\}.$$

The elements of $\mathsf{Db}(\mathfrak{g}_0)_{S_+}$ are called S_+-finite. Observe that $\mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0}$ and $\mathsf{Db}(\mathfrak{g}_0)_{S_+}^{G_0}$ are $\mathcal{D}(\mathfrak{g})^G$-modules. The next theorem is consequence of the results proved in [18].

Theorem 1.5. (1) $\mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0} = \{\Theta \in \mathsf{Db}(\mathfrak{g}_0)^{G_0} : \mathcal{D}(\mathfrak{g}).\Theta \in \mathsf{C}(\mathcal{M})\}$.

(2) $\mathsf{Db}(\mathfrak{g}_0)_{S_+}^{G_0} = \{T \in \mathsf{Db}(\mathfrak{g}_0)^{G_0} : \mathcal{D}(\mathfrak{g}).T \in \mathsf{C}(\mathcal{N})\}$.

(3) $\Theta \in \mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0} \iff \widehat{\Theta} \in \mathsf{Db}(\mathfrak{g}_0)_{S_+}^{G_0}$.

Proof. (1) follows from [18, Theorem 6.1], since $\mathcal{D}(\mathfrak{g}).\Theta \in \mathsf{C}(\mathcal{M})$ is equivalent to $\mathcal{D}(\mathfrak{g})^G.\Theta \cong \oplus_{\chi \in W^\wedge} m_\chi \mathcal{M}_\chi^G$.

(2) and (3) are consequences of (1) and Lemma 1.4. □

Remark 1.6. Let $T \in \mathsf{Db}(\mathfrak{g}_0)_{S_+}^{G_0}$. Recall that by the Harish-Chandra regularity theorem, T is given by

$$\langle T, f\rangle = \int_{\mathfrak{g}_0'} F_T(y) f(y) dy$$

for some analytic function F_T on $\mathfrak{g}_0'$, locally integrable on $\mathfrak{g}_0$.

2. The distributions $\Theta_{u,\Gamma}$ and $T_{p,\Gamma}$

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, with adjoint group G_0, $\mathfrak{h}_0$ a Cartan subalgebra and let H_0 be the associated Cartan subgroup. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_0$ and adopt the notation of §1. Denote by $W(\mathfrak{h}_0)$ the real Weyl group, i.e. $W(\mathfrak{h}_0) = N_{G_0}(\mathfrak{h}_0)/Z_{G_0}(\mathfrak{h}_0)$. Define

$$\Delta_R = \{\alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset \mathbb{R}\} \quad \text{(the real roots)}$$
$$\Delta_I = \{\alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset i\mathbb{R}\} \quad \text{(the imaginary roots).}$$

A root which is neither real nor imaginary is called complex. Let Δ_I^+ be a positive system of roots in Δ_I and set $\pi_I = \prod_{\alpha \in \Delta_I^+} \alpha$. Then each $w \in W(\mathfrak{h}_0)$ permutes the imaginary roots and one can define a character of $W(\mathfrak{h}_0)$, the imaginary signature, by

$$\varepsilon_I : W(\mathfrak{h}_0) \to \{\pm 1\}, \quad w.\pi_I = \varepsilon_I(w)\pi_I.$$

If V is a $W(\mathfrak{h}_0)$-module we denote by V^{ε_I} the isotypic component of type ε_I in V.

In the sequel, we adopt the notation of [5] with the minor difference that we use $e^{-i\kappa(x,y)}$ in the definition of the Fourier transform.

Let $h \in \mathfrak{h}_0'$ and $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$. Define [5, §3.1] the distribution $\mu_{G_0.h}$ by

$$\langle \mu_{G_0.h}, f\rangle = |\det \mathrm{ad}_{\mathfrak{g}_0/\mathfrak{h}_0}(h)|^{\frac{1}{2}} \int_{G_0/H_0} f(\dot{g}.h)d\dot{g}$$

Then one defines the function $J_{\mathfrak{g}_0}(f)$, or simply $J(f)$, on $\mathfrak{h}_0'$ by

$$J_{\mathfrak{g}_0}(f) = \{h \mapsto \langle \mu_{G_0.h}, f\rangle\}.$$

Set $\mathfrak{h}_0^{\mathrm{reg}} = \{h \in \mathfrak{h}_0 : \pi_I(h) \neq 0\}$ and fix a connected component Γ of $\mathfrak{h}_0^{\mathrm{reg}}$. Let $u \in S(\mathfrak{h})$; Harish-Chandra has shown, see [17, §8.1, p. 123], that one can define a tempered G_0-invariant distribution on $\mathfrak{g}_0$ by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{g}_0), \quad \langle \Theta_{u,\Gamma}, f\rangle = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u).J(f)](h). \tag{2.1}$$

Furthermore $\Theta_{u,\Gamma} \in \mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0}$ and, when $u \in S^b(\mathfrak{h})$, $\Theta_{u,\Gamma}$ is homogeneous of degree $-b-\nu-\ell$.

Now let $p \in S(\mathfrak{h}^*)$ and define $T \in \mathsf{Db}(\mathfrak{g}_0)_{S_+}^{G_0}$ by

$$T_{p,\Gamma} = \widehat{\Theta}_{F_{\mathfrak{h}}(p),\Gamma} = \Big\{ f \mapsto \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(F_{\mathfrak{h}}(p)).J(\hat{f})](h)\Big\}. \tag{2.2}$$

Then, $T_{p,\Gamma}$ is tempered and is homogeneous of degree $b-\nu$ when $p \in S^b(\mathfrak{h}^*)$.

Lemma 2.1. (1) *Let $\varphi \in S(\mathfrak{g}^*)^G$. Then, $\varphi T_{p,\Gamma} = T_{\delta(\varphi)p,\Gamma}$.*

(2) *Let $q \in S(\mathfrak{g})^G$. Then, $\partial(q).T_{p,\Gamma} = T_{\partial(\delta(q)).p,\Gamma}$.*

Proof. Set $u = F_{\mathfrak{h}}(p)$, $\phi = \delta(\varphi) \in S(\mathfrak{h}^*)^W$ and $s = \delta(q) \in S(\mathfrak{h})^W$. Let $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$.

(1) By definition, see (2.2), $\langle \varphi T_{p,\Gamma}, f\rangle = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u).J(\widehat{\varphi f})](h)$. But, [17, Lemma 3.2.7, p. 38], (1.4) and Lemma 1.2 imply that $J(\widehat{\varphi f}) = \partial(F_{\mathfrak{h}}(\phi)).J(\hat{f})$. Hence,

$$\begin{aligned}\langle \varphi T_{p,\Gamma}, f\rangle &= \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u)\partial(F_{\mathfrak{h}}(\phi)).J(\hat{f})](h) = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(F_{\mathfrak{h}}(\phi p)).J(\hat{f})](h)\\ &= \langle T_{\phi p,\Gamma}, f\rangle,\end{aligned}$$

as desired.

(2) By (1.4), $\partial(q).T_{p,\Gamma}$ is the Fourier transform of ${F_{\mathfrak{g}}}^{-1}(q)\Theta_{u,\Gamma}$, hence

$$\langle \partial(q).T_{p,\Gamma}, f\rangle = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u).J({F_{\mathfrak{g}}}^{-1}(q)\hat{f})](h).$$

Set $g = J(\hat{f})$. From [17, Lemma 3.2.7, p. 38] and Lemma 1.2 we obtain that $J({F_{\mathfrak{g}}}^{-1}(q)\hat{f}) = {F_{\mathfrak{h}}}^{-1}(s)g$. Therefore

$$\langle \partial(q).T_{p,\Gamma}, f\rangle = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u).({F_{\mathfrak{h}}}^{-1}(s)g)](h).$$

Recall (see §1) that we have chosen a coordinate system $\{x_j; e_j\}_{1\leqslant j\leqslant \ell}$. With standard notation, we write $x^\alpha = \prod_{k=1}^{\ell} x_k^{\alpha_k}$, $e^\mu = \prod_{k=1}^{\ell} e_k^{\mu_k}$ and

$$p = \textstyle\sum_{\alpha\in\mathbb{N}^\ell} p_\alpha x^\alpha, \quad s = \sum_{\mu\in\mathbb{N}^\ell} s_\mu e^\mu.$$

Set $\partial^\mu = \prod_j \partial(e_j)^{\mu_j}$; thus $\partial(s) = \sum_{\mu\in\mathbb{N}^\ell} s_\mu \partial^\mu$. Order $\mathbb{N}^\ell$ by saying that $\mu \leq \alpha$ if $\mu_j \leqslant \alpha_j$ for all j. Set $\alpha! = \prod_j \alpha_j!$ and $\binom{\alpha}{\mu} = \prod_j \binom{\alpha_j}{\mu_j}$, when $\mu\leq\alpha$. Then:

$$\partial^\mu(x^\alpha) = \begin{cases} 0 & \text{if } \mu \not\leq \alpha,\\ \frac{\alpha!}{(\alpha-\mu)!} x^{\alpha-\mu} & \text{if } \mu\leq\alpha.\end{cases}$$

Now we have $u = F_{\mathfrak{h}}(p) = \sum_\alpha p_\alpha i^{|\alpha|}\partial^\alpha$ and ${F_{\mathfrak{h}}}^{-1}(s) = \sum_\mu q_\mu i^{-|\mu|}x^\mu$. Therefore, using the Leibniz formula, we get that

$$\begin{aligned}\partial(u).({F_{\mathfrak{h}}}^{-1}(s)g) &= \textstyle\sum_\alpha p_\alpha i^{|\alpha|}\partial^\alpha({F_{\mathfrak{h}}}^{-1}(s)g)\\ &= \textstyle\sum_\alpha\sum_\mu\sum_{\beta\leq\alpha} p_\alpha s_\mu i^{|\alpha|-|\mu|}\binom{\alpha}{\beta}\partial^\beta(x^\mu)\partial^{\alpha-\beta}(g).\end{aligned}$$

But $\lim_{h\to 0}\partial^\beta(x^\mu)(h) = 0$ unless $\beta=\mu$, hence

$$\lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(u).({F_{\mathfrak{h}}}^{-1}(s)g)](h) = \sum_\alpha\sum_{\mu\leq\alpha} p_\alpha s_\mu i^{|\alpha|-|\mu|}\binom{\alpha}{\mu}\mu!\lim_{\substack{h\to 0\\ h\in\Gamma}}[\partial^{\alpha-\mu}(g)](h).$$

On the other hand, we have

$$\langle T_{\partial(s).p,\Gamma}, f\rangle = \lim_{\substack{h\to 0\\ h\in\Gamma}} [\partial(F_{\mathfrak{h}}(\partial(s).p)).g](h).$$

Since $\partial(s).p = \sum_\alpha\sum_{\mu\leq\alpha}\frac{\alpha!}{(\alpha-\mu)!}s_\mu p_\alpha x^{\alpha-\mu}$, we obtain that

$$\langle T_{\partial(s).p,\Gamma}, f\rangle = \sum_\alpha\sum_{\mu\leq\alpha}\frac{\alpha!}{(\alpha-\mu)!}s_\mu p_\alpha i^{|\alpha|-|\mu|}\lim_{\substack{h\to 0\\ h\in\Gamma}}[\partial^{\alpha-\mu}(g)](h).$$

This proves the desired equality. □

Theorem 2.2. *Let $p\in S(\mathfrak{h}^*)$ and $D\in\mathcal{D}(\mathfrak{g})^G$. Then, $D.T_{p,\Gamma} = T_{\delta(D).p,\Gamma}$.*

Proof. Since $T_{p,\Gamma}$ is G_0-invariant, we have $\mathcal{I}.T_{p,\Gamma}=0$. Let $P\in\mathbb{C}\langle S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G\rangle$; by Lemma 2.1 and an obvious induction, we obtain that $P.T_{p,\Gamma} = T_{\delta(P).p,\Gamma}$. The theorem then follows from Lemma 1.1. □

Recall, see Remark 1.6, that $\widehat{\Theta}_{u,\Gamma}\in \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+}$ is determined by a locally integrable function on $\mathfrak{g}_0$. We still denote this function by $\widehat{\Theta}_{u,\Gamma}$.

Lemma 2.3. ([5, Lemme 6.1.2]) *There exists $c_\Gamma\in\mathbb{C}^*$, such that*

$$a_{\Delta_I^+}(h)|\det\operatorname{ad}_{\mathfrak{g}_0/\mathfrak{h}_0}(h)|^{\frac{1}{2}}\widehat{\Theta}_{F_{\mathfrak{h}}(p),\Gamma}(h) = c_\Gamma p(h)$$

for all $p\in S(\mathfrak{h}^*)^{\varepsilon_I}$ and $h\in\mathfrak{h}_0^{\text{reg}}$. □

Remark. In the notation of the lemma, if $u = F_{\mathfrak{h}}(p)$, the function $\tilde{u}(ih)$ of [5] is replaced here by $p(h)$ since we are using $e^{-i\kappa(x,y)}$ in the definition of the Fourier transform.

Theorem 2.4. *Let $p \in S(\mathfrak{h}^*)^{\varepsilon_I}$. There exists a bijective map*

$$\rho : \mathcal{D}(\mathfrak{g})^G.T_{p,\Gamma} \longrightarrow \mathcal{D}(\mathfrak{h})^W.p, \quad \rho(D.T_{p,\Gamma}) = \delta(D).p$$

which, through δ, yields an isomorphism

$$\rho : A(\mathfrak{g}).T_{p,\Gamma} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^W.p$$

Proof. We first need to show that ρ is well defined. Let $D \in \mathcal{D}(\mathfrak{g})^G$; by Theorem 2.2 we have

$$(\dagger) \qquad D.T_{p,\Gamma} = T_{\delta(D).p,\Gamma} = \widehat{\Theta}_{F_{\mathfrak{h}}(\delta(D).p),\Gamma}.$$

Suppose that $D.T_{p,\Gamma} = 0$. Then, the analytic function associated to $T_{\delta(D).p,\Gamma} \in \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+}$ vanishes on $\mathfrak{h}_0^{\mathrm{reg}}$. Notice that, since $\delta(D)$ is W-invariant, $\delta(D).p \in S(\mathfrak{h}^*)^{\varepsilon_I}$. Therefore Lemma 2.3 gives $\delta(D).p = 0$ on $\mathfrak{h}_0^{\mathrm{reg}}$. Thus $\delta(D).p = 0$ on $\mathfrak{h}$ and ρ is well defined.

Now, it follows easily from ($\dagger$) that ρ is a linear bijection. Since $\mathcal{I}.T_{p,\Gamma} = 0$, the last assertion is clear. □

Recall that we denote by $V_\chi \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)$ a simple W-module in the class of $\chi \in W^\frown$.

Corollary 2.5. *Let $p \in S(\mathfrak{h}^*)^{\varepsilon_I}$ such that $\mathbb{C}W.p$ is simple. Then there exists $\chi \in W^\frown$ such that $V_\chi^{\varepsilon_I} \neq 0$. We have:*

1. $\mathcal{D}(\mathfrak{g}).T_{p,\Gamma} \xrightarrow{\sim} \mathcal{N}_\chi$ *and* $\mathcal{D}(\mathfrak{g})^G.T_{p,\Gamma} \xrightarrow{\sim} V^\chi$*;*
2. $\mathcal{D}(\mathfrak{g}).\Theta_{F_{\mathfrak{h}}(p),\Gamma} \xrightarrow{\sim} \mathcal{M}_\chi$ *and* $\mathcal{D}(\mathfrak{g})^G.\Theta_{F_{\mathfrak{h}}(p),\Gamma} \xrightarrow{\sim} (V^\chi)^{F_{\mathfrak{h}}{}^{-1}}$.

Proof. The first assertion follows from $\mathcal{H}(\mathfrak{h}^*) \cong \mathbb{C}W$. Then, 1 and 2 are consequences of $V^\chi \cong \mathcal{D}(\mathfrak{h})^W.p$, Lemma 1.3 and Theorem 2.4. □

Remark 2.6. Let $\chi \in W^\frown$ be such that $V_\chi^{\varepsilon_I} \neq 0$. It follows obviously from the previous corollary that

$$\mathcal{N}_\chi \cong \mathcal{D}(\mathfrak{g}).T_{p,\Gamma}, \qquad \mathcal{M}_\chi \cong \mathcal{D}(\mathfrak{g}).\Theta_{u,\Gamma}$$

where $0 \neq p \in V_\chi^{\varepsilon_I} \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)^{\varepsilon_I}$ and $u = F_{\mathfrak{h}}(p) \in \mathcal{H}^{b(\chi)}(\mathfrak{h})^{\varepsilon_I}$.

3. The decomposition of $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+}$ and $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$

Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and let $[\mathfrak{h}_1], \dots, [\mathfrak{h}_r]$ be the conjugacy classes of Cartan subalgebras in $\mathfrak{g}_0$. For each $j = 1, \dots, r$ we denote by

$$\mathfrak{h}_{j,\mathbb{C}} = \mathfrak{h}_j \otimes_{\mathbb{R}} \mathbb{C}, \quad W_j = W(\mathfrak{g}, \mathfrak{h}_{j,\mathbb{C}}), \quad \Delta^+_{I,j} \text{ a set of positive imaginary roots,}$$
$$\varepsilon_{I,j} : W(\mathfrak{h}_j) = W(G_0, \mathfrak{h}_j) \to \{\pm 1\} \text{ the imaginary signature associated to } \mathfrak{h}_j.$$

For each j we fix a connected component Γ_j of $\mathfrak{h}_j^{\mathrm{reg}}$. The results of §2 then apply to $\mathfrak{h}_0 = \mathfrak{h}_j$, $\Gamma = \Gamma_j$ etc.

Remark 3.1. Recall that the $\mathfrak{h}_{j,\mathbb{C}}$ are G-conjugate. Therefore, if $1 \leqslant j, k \leqslant r$, the algebras $\mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}$ and $\mathcal{D}(\mathfrak{h}_{k,\mathbb{C}})^{W_k}$ are naturally isomorphic. Denote this isomorphism by γ_{jk} and let δ_j be the Harish-Chandra isomorphism from $A(\mathfrak{g})$ onto $\mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}$. One can check that $\delta_k = \gamma_{jk} \circ \delta_j$. Therefore, we can choose an "abstract" Cartan subalgebra $\mathfrak{h}$ and identify δ_j with the homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$, where $W = W(G, \mathfrak{h})$. Then, if $\chi \in W^\wedge$, we have an irreducible W-module $V_\chi \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)$ and a simple $\mathcal{D}(\mathfrak{h})^W$-module V^χ.

For each $\chi \in W^\wedge$, choose a simple W-module $V_{\chi,j} \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*_{j,\mathbb{C}})$, $V_{\chi,j} \cong V_\chi$. Write $V_{\chi,j} = V_{\chi,j}^{\varepsilon_{I,j}} \oplus E_{\chi,j}$ with $E_{\chi,j}$ stable under $W(\mathfrak{h}_j)$. Let $\{v^k_{\chi,j}\}_{1 \leqslant k \leqslant d(\chi)}$ be a basis of $V_{\chi,j}$ such that

$$V_{\chi,j}^{\varepsilon_{I,j}} = \bigoplus_{k=1}^{n_j(\chi)} \mathbb{C} v^k_{\chi,j}, \quad E_{\chi,j} = \bigoplus_{k=n_j(\chi)+1}^{d(\chi)} \mathbb{C} v^k_{\chi,j}$$

(hence $n_j(\chi) = \dim V_\chi^{\varepsilon_{I,j}}$).

Lemma 3.2. *The $\mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}$-module $S(\mathfrak{h}^*_{j,\mathbb{C}})^{\varepsilon_{I,j}}$ decomposes as*

$$S(\mathfrak{h}^*_{j,\mathbb{C}})^{\varepsilon_{I,j}} = \bigoplus_{\chi \in W^\wedge} \bigoplus_{k=1}^{n_j(\chi)} \mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}.v^k_{\chi,j}$$

with $\mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}.v^k_{\chi,j} \cong V^\chi$.

Proof. Clearly, we can drop the index j and write $\mathfrak{h}_0 = \mathfrak{h}_j$, $\mathfrak{h} = \mathfrak{h}_{j,\mathbb{C}}$, $v^k_\chi = v^k_{\chi,j}$ etc. Since $\mathcal{D}(\mathfrak{h})^W.v^k_\chi \subset S(\mathfrak{h}^*)^{\varepsilon_I}$ for $1 \leqslant k \leqslant n(\chi) = \dim V_\chi^{\varepsilon_I}$, one has

$$S(\mathfrak{h}^*)^{\varepsilon_I} \supset \bigoplus_{\chi \in W^\wedge} \bigoplus_{k=1}^{n(\chi)} \mathcal{D}(\mathfrak{h})^W.v^k_\chi.$$

Recall from §1 that $S(\mathfrak{h}^*) = \oplus_\chi S(\mathfrak{h}^*)[\chi]$ with $S(\mathfrak{h}^*)[\chi] = \oplus_{k=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W.v^k_\chi$. Write $S(\mathfrak{h}^*)[\chi] = E_1 \oplus E_2$, where $E_1 = \oplus_{k=1}^{n(\chi)} \mathcal{D}(\mathfrak{h})^W.v^k_\chi$ and $E_2 = \oplus_{k=n(\chi)+1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W.v^k_\chi$. Notice that E_1, E_2 are stable under $W(\mathfrak{h}_0)$ and that we have $S(\mathfrak{h}^*)[\chi]^{\varepsilon_I} = E_1 \oplus E_2^{\varepsilon_I}$.

We now show that $E_2^{\varepsilon_I} = 0$. This will prove that

$$S(\mathfrak{h}^*)^{\varepsilon_I} = \bigoplus_{\chi \in W^\wedge} \bigoplus_{k=1}^{n(\chi)} \mathcal{D}(\mathfrak{h})^W.v^k_\chi.$$

Let $D \in \mathcal{D}(\mathfrak{h})^W$ and $v \in V_\chi$. Notice first that if $D.v \neq 0$, the operator D yields an isomorphism of W-modules $V_\chi \xrightarrow{\sim} D.V_\chi$. Therefore, if $V_\chi = \oplus_k S_k$ with S_k irreducible $W(\mathfrak{h}_0)$-module, we get that $D.V_\chi = \oplus_k D.S_k$, $D.S_k \cong S_k$. It follows that if $v \in E_\chi$ (the $W(\mathfrak{h}_0)$-stable complement of $V_\chi^{\varepsilon_I}$) then $D.v \in D.E_\chi$ with $D.E_\chi \cap S(\mathfrak{h}^*)^{\varepsilon_I} = 0$. Let $p = \sum_{k=n(\chi)+1}^{d(\chi)} D_k.v^k_\chi \in E_2$. Then, $\mathbb{C}W(\mathfrak{h}_0).p \subset \sum_{k>n(\chi)} \mathbb{C}W(\mathfrak{h}_0).(D_k.v^k_\chi)$ and, by the previous remarks, $\big(\mathbb{C}W(\mathfrak{h}_0).(D_k.v^k_\chi)\big)^{\varepsilon_I} = 0$. Thus $(\mathbb{C}W(\mathfrak{h}_0).p)^{\varepsilon_I} = 0$, which shows that $E_2^{\varepsilon_I} = 0$. □

Recall the following result:

Proposition 3.3. ([5, Proposition 6.1.1]) (1) *The linear map*

$$\mathbf{T} : \bigoplus_{j=1}^r S(\mathfrak{h}^*_{j,\mathbb{C}})^{\varepsilon_{I,j}} \longrightarrow \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+}, \quad \mathbf{T}(p_1, \dots, p_r) = \sum_{j=1}^r T_{p_j,\Gamma_j}$$

is an isomorphism of vector spaces.

(2) *The map* $\mathbf{T}$ *induces an isomorphism:*

$$\bigoplus_{j=1}^r \mathcal{H}(\mathfrak{h}^*_{j,\mathbb{C}})^{\varepsilon_{I,j}} \xrightarrow{\sim} \{T \in \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+} : S_+(\mathfrak{g})^G.T = 0\}.$$

Proof. (2) follows from the proof of [5, Proposition 6.1.1]. □

Theorem 3.4. *Set* $\mathbf{T}(\mathfrak{h}_j) = \sum_{p \in S(\mathfrak{h}_{j,\mathbb{C}}^*)^{\varepsilon_{I,j}}} \mathbb{C}T_{p,\Gamma_j}$. *Then we have the following decomposition of* $\mathcal{D}(\mathfrak{g})^G$*-modules:*

$$\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+} = \bigoplus_{j=1}^r \mathbf{T}(\mathfrak{h}_j)$$

with

$$\mathbf{T}(\mathfrak{h}_j) = \bigoplus_{\chi \in W^\wedge} \bigoplus_{k=1}^{n_j(\chi)} \mathcal{D}(\mathfrak{g})^G.T_{v^k_{\chi,j},\Gamma_j}$$

and $\mathcal{D}(\mathfrak{g})^G.T_{v^k_{\chi,j},\Gamma_j} \cong \mathcal{N}^G_\chi$.

Proof. The decomposition of $\mathbf{T}(\mathfrak{h}_j)$, as a $\mathcal{D}(\mathfrak{g})^G$-module, is consequence of Theorem 2.4, Lemma 3.2 (using the isomorphism $\delta_j : A(\mathfrak{g}) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{j,\mathbb{C}})^{W_j}$) and Proposition 3.3. The decomposition of $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+}$ follows from Proposition 3.3. □

Using the Fourier transform, we obtain the following:

Corollary 3.5. *The* $\mathcal{D}(\mathfrak{g})^G$*-module* $\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil}$ *decomposes as*

$$\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil} = \bigoplus_{j=1}^r \bigoplus_{\chi \in W^\wedge} \bigoplus_{k=1}^{n_j(\chi)} \mathcal{D}(\mathfrak{g})^G.\Theta_{F_\mathfrak{h}{}^{-1}(v^k_{\chi,j}),\Gamma_j}$$

with $\mathcal{D}(\mathfrak{g})^G.\Theta_{F_\mathfrak{h}{}^{-1}(v^k_{\chi,j}),\Gamma_j} \cong \mathcal{M}^G_\chi$. □

The next corollary follows from Theorem 3.4 and Corollary 3.5.

Corollary 3.6. *We have:*

$$\mathsf{Db}(\mathfrak{g}_0)^{G_0}_{S_+} \cong \bigoplus_{\chi \in W^\wedge} m_\chi \mathcal{N}^G_\chi, \qquad \mathsf{Db}(\mathfrak{g}_0)^{G_0}_{nil} \cong \bigoplus_{\chi \in W^\wedge} m_\chi \mathcal{M}^G_\chi$$

where $m_\chi = \sum_{j=1}^r \dim V_\chi^{\varepsilon_{I,j}}$. □

Remark 3.7. Let $\chi \in W^\wedge$. It is not always possible to "realize" the modules $\mathcal{N}_\chi$ and $\mathcal{M}_\chi$ as $\mathcal{D}(\mathfrak{g}).T$ for some $T \in \mathsf{Db}(\mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. By the previous results, this statement is equivalent to the existence of a Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{g}_0$ such that $V_\chi^{\varepsilon_{I,j}} \neq 0$. D. Renard has observed that, using the results of W. Rossmann [15], this can be translated to a question about centralizers of nilpotent elements. Fix a real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$ with adjoint group $G_\mathbb{R}$. If $x \in \mathfrak{g}_\mathbb{R}$ is nilpotent one defines a subgroup of the component group $A(G.x)$ (see §4 for notation) by

$$A(G_\mathbb{R}.x) = G^x_\mathbb{R} / G^x_\mathbb{R} \cap (G^x)^0.$$

Recall that $\chi \in W^\wedge$ can be written $\sigma(\mathbf{O}, \psi)$ via the Springer correspondence, where $\mathbf{O} \subset \mathfrak{g}$ is a nilpotent orbit and $\psi : A(\mathbf{O}) \to \mathrm{GL}(E)$ is an irreducible representation. Then, by [15, Corollary 3.2 & Theorem 3.3], there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_\mathbb{R}$ such that $V_\chi^{\varepsilon_I} \neq 0$ if, and only if, there exists a nilpotent element $x \in \mathfrak{g}_\mathbb{R}$ such that $\mathbf{O} = G.x$ and $E^{A(G_\mathbb{R}.x)} \neq 0$.

Let $\mathfrak{g} = \mathfrak{sp}(\ell, \mathbb{C})$ and let $\phi \in W^\wedge$ be the long sign character, i.e. $V_\phi = \mathbb{C}\pi_l$ where π_l is the product of the long roots. Then, see [6, §13.3], $\phi = \sigma(\mathbf{O}, \psi)$ where $\mathbf{O} = G.x$ is the subregular nilpotent orbit with partition $[2\ell - 2, 2]$ and ψ is the non trivial character of $A(\mathbf{O}) \cong \{\pm 1\}$. The real forms of $\mathfrak{g}$ are $\mathfrak{sp}(\ell, \mathbb{R})$ and the $\mathfrak{sp}(p, q)$, $p + q = \ell$. Assume now that $\ell \geqslant 3$. By the classification of nilpotent orbits in $\mathfrak{sp}(p, q)$, see [7, Theorem 9.2.5], we know that $\mathbf{O} \cap \mathfrak{sp}(p, q) = \emptyset$. Hence, by Rossmann's results, $V_\phi^{\varepsilon_{I,j}} = 0$ for each Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{sp}(p, q)$. On the other hand, if $G_\mathbb{R}$ is the adjoint group of $\mathfrak{sp}(\ell, \mathbb{R})$, one can show that $A(G_\mathbb{R}.x) = A(G.x)$. Thus, with the above notation, $E^{A(G_\mathbb{R}.x)} = 0$ and it follows that $V_\phi^{\varepsilon_{I,j}} = 0$ for each Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{sp}(\ell, \mathbb{R})$. For instance, when $\mathfrak{g} = \mathfrak{sp}(3, \mathbb{R})$ there are six conjugacy classes of Cartan subalgebras and one can directly verify (without using [15]) that $V_\phi^{\varepsilon_{I,j}} = 0$ for $j = 1, \dots, 6$. We thank D. Renard for showing this computation to us. $\square$

Let $x \in \mathbf{N}(\mathfrak{g}_0)$ and denote by β_x the Liouville (Kostant-Kirillov) measure on $G_0.x$. By [14] one can define $\Theta_x \in \mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0}$ by $\langle \Theta_x, f\rangle = \int_{G_0.x} f d\beta_x$ for all $f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$. Set $\mathbf{O} = G.x$. Then, see [9], [10] or [18], Θ_x is homogeneous of degree $\lambda_{\mathbf{O}} = \frac{1}{2}\dim \mathbf{O} - \dim \mathfrak{g}$ and satisfies

$$\mathcal{D}(\mathfrak{g}).\Theta_x \cong \mathcal{M}_{\chi_{\mathbf{O}}} \tag{3.1}$$

for some $\chi_{\mathbf{O}} \in W\hat{}$ such that $\lambda_{\mathbf{O}} = \nu - n - b(\chi_{\mathbf{O}})$.

Corollary 3.8. *There exists $j \in \{1, \dots, r\}$ and $u \in {F_{\mathfrak{h}}}^{-1}(V_{\chi_{\mathbf{O}},j})^{\varepsilon_{I,j}}$ such that*

$$\mathcal{D}(\mathfrak{g})^G.\Theta_x \cong \mathcal{D}(\mathfrak{g})^G.\Theta_{u,\Gamma_j}.$$

Proof. Since $\mathcal{D}(\mathfrak{g})^G.\Theta_x \cong \mathcal{M}_{\chi_{\mathbf{O}}}^G$ is a simple submodule of $\mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0}$, the claim follows from Corollary 3.5. $\square$

Remark 3.9. It is proved in [1], see also [5], that Θ_x can be written as $\sum_{j=1}^r \Theta_{a_j,\Gamma_j}$ with $a_j \in \mathcal{H}^{b(\chi_{\mathbf{O}})}(\mathfrak{h}_{j,\mathbb{C}})^{\varepsilon_{I,j}}$. It is easily seen that we may assume $\mathbb{C}W.a_j \cong V_{\chi_{\mathbf{O}}}$ for all j such that $a_j \neq 0$. W. Rossmann [15] has given conditions to ensure that $\Theta_x = \Theta_{a_j,\Gamma_j}$ for some j.

4. Example: the complex case

We assume in this section that $\mathfrak{g}_0 = \mathfrak{g}_1^{\mathbb{R}}$ is a complex semisimple Lie algebra, $\mathfrak{g}_1$, viewed as a real Lie algebra. Then, $\mathfrak{g}$ can be identified with $\mathfrak{g}_1 \times \mathfrak{g}_1$ and $\mathfrak{g}_0$ with the diagonal $\{(a,a) \in \mathfrak{g}_1 \times \mathfrak{g}_1\}$. Let $\mathfrak{h}_1$ be a Cartan subalgebra of $\mathfrak{g}_1$. Recall the following well known facts, see [17] or [18]:

- $\mathfrak{h}_0 = \{(a,a) : a \in \mathfrak{h}_1\}$ is a Cartan subalgebra of $\mathfrak{h}_0$ and $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}_1 \times \mathfrak{h}_1$;
- $W(\mathfrak{g},\mathfrak{h}) = W_1 \times W_1$, where $W_1 = W(\mathfrak{g}_1, \mathfrak{h}_1)$, and $W(\mathfrak{h}_0) = \{(w,w) \in W\}$ is isomorphic to W_1;
- there is a unique conjugacy class $[\mathfrak{h}_0]$ of Cartan subalgebras and $\mathfrak{h}_0'$ is connected;
- the roots in $\Delta(\mathfrak{g},\mathfrak{h})$ are complex and, therefore, $\varepsilon_I = 1$;
- the irreducible representations of W are of the form $\chi = \phi \boxtimes \mu$, $\phi, \mu \in W_1\hat{}$;
- one has $\phi = \phi^*$ for all $\phi \in W_1\hat{}$, where ϕ^* is the dual representation.

Observe that $\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}_1) \boxtimes \mathcal{D}(\mathfrak{g}_1)$ and $\mathcal{D}(\mathfrak{g})^G = \mathcal{D}(\mathfrak{g}_1)^{G_1} \boxtimes \mathcal{D}(\mathfrak{g}_1)^{G_1}$.

Lemma 4.1. *Let $\chi \in W\hat{}$. Then, the simple $\mathcal{D}(\mathfrak{g})$-module $\mathcal{M}_\chi$ is of the form $\mathcal{M}_\phi \boxtimes\ \mathcal{M}_\mu$ for some $\phi, \mu \in W_1\hat{}$.*

Proof. The claim follows easily from the definition of the category $\mathsf{C}(\mathcal{M})$ and the decomposition of the W-module $S(\mathfrak{h}^*) = S(\mathfrak{h}_1^*) \boxtimes S(\mathfrak{h}_1^*)$. $\square$

Corollary 4.2. ([18, Theorem 6.11]) *We have*

$$\mathsf{Db}(\mathfrak{g}_0)_{nil}^{G_0} \cong \textstyle\bigoplus_{\phi \in W_1\hat{}} \mathcal{M}_\phi^{G_1} \boxtimes \mathcal{M}_\phi^{G_1}$$

as a $\mathcal{D}(\mathfrak{g})^G$-module.

Proof. Let $\chi = \phi \boxtimes \mu \in W\hat{}$. Then, $V_\chi^{\varepsilon_I} = (V_\phi \boxtimes V_\mu)^{W_1} \neq 0$ if, and only if, $\phi = \mu$ and therefore $n(\chi) = 1$. The assertion now follows from Corollary 3.5. $\square$

Recall the following general results from [13]. Since the module $\mathcal{M}_\chi$ is irreducible and G-equivariant, its support is the closure of a nilpotent orbit $\mathbf{O} = G.x$. Furthermore, if $\imath : \mathbf{O} \hookrightarrow \mathfrak{g}$ is the inclusion, $\mathcal{M}_\chi$ is uniquely determined by its (D-module) inverse image $\mathcal{L}_\chi := \imath^! \mathcal{M}_\chi$. The $\mathcal{D}_{\mathbf{O}}$-module $\mathcal{L}_\chi$ is an irreducible integrable connection associated to an irreducible representation ψ of the component group $A(\mathbf{O}) := G^x/(G^x)^0$ (where $(G^x)^0$ is the connected component of the centralizer G^x). Therefore, since χ is uniquely determined by $\mathbf{O}$ and ψ, we set $\chi = \sigma(\mathbf{O}, \psi)$.

In our situation, i.e. in the complex case, we have $\mathbf{O} = \mathbf{O}_1^1 \times \mathbf{O}_1^2$ with $\mathbf{O}_1^j$ nilpotent orbits in $\mathfrak{g}_1$ for $j = 1, 2$. Then, $\chi = \sigma(\mathbf{O}, \psi) = \phi_1 \boxtimes \phi_2$, $\mathcal{L}_\chi = \mathcal{L}_{\phi_1} \boxtimes \mathcal{L}_{\phi_2}$, $\phi_j = \sigma(\mathbf{O}_1^j, \psi_j)$, $\psi = \psi_1 \boxtimes \psi_2$. Note that $b(\chi) = b(\phi_1) + b(\phi_2)$ and $\lambda_{\mathbf{O}} = \lambda_{\mathbf{O}_1^1} + \lambda_{\mathbf{O}_1^2}$.

Let $x \in \mathbf{N}(\mathfrak{g}_0)$; set $x = (x_1, x_1)$, $x_1 \in \mathbf{N}(\mathfrak{g}_1)$, $\mathbf{O}_1 = G_1.x_1$, $\mathbf{O} = G.x = \mathbf{O}_1 \times \mathbf{O}_1$. The inclusion $\imath : \mathbf{O} \hookrightarrow \mathfrak{g}$ is equal to $\imath_1 \times \imath_1$, where $\imath_1 : \mathbf{O}_1 \hookrightarrow \mathfrak{g}_1$. By (3.1) and Corollary 4.2 there exist $\chi \in W^\frown$, $\chi_1 \in W_1^\frown$ such that $\chi = \chi_1 \boxtimes \chi_1$ and $\mathcal{D}(\mathfrak{g}).\Theta_x \cong \mathcal{M}_{\chi_1} \boxtimes \mathcal{M}_{\chi_1}$.

It is known (Harish-Chandra) that $\Theta_x = \Theta_{u,\mathfrak{h}_0'}$ for some $u \in S(\mathfrak{h}_1) \boxtimes S(\mathfrak{h}_1)$. The following result has been proved by various authors; see [2, 3] (when $\mathbf{O}_1$ is "special"), [8], [9], [16].

Theorem 4.3. *One has* $\chi_1 = \sigma(\mathbf{O}_1, \mathrm{triv})$*, and there exists* $p \in (V_{\chi_1} \boxtimes V_{\chi_1})^{W_1}$ *such that* $\Theta_x = \Theta_{F_{\mathfrak{h}}(p),\mathfrak{h}_0'}$.

Proof. Recall from [9] or [10] that $\chi = \chi_1 \boxtimes \chi_1 = \sigma(\mathbf{O}, \mathrm{triv})$. This means that

$$\mathcal{L}_\chi = \mathcal{L}_{\chi_1} \boxtimes \mathcal{L}_{\chi_1} = \mathcal{O}_{\mathbf{O}} = \mathcal{O}_{\mathbf{O}_1} \boxtimes \mathcal{O}_{\mathbf{O}_1}$$

(where we denote by $\mathcal{O}_X$ the structural sheaf of an algebraic variety X). This yields $\mathcal{L}_{\chi_1} = \mathcal{O}_{\mathbf{O}_1}$ and $\chi_1 = \sigma(\mathbf{O}_1, \mathrm{triv})$.

Set $T_x = \widehat{\Theta}_x$; then $\mathcal{D}(\mathfrak{g}).T_x = \mathcal{N}_{\chi_1} \boxtimes \mathcal{N}_{\chi_1}$ (see Lemma 1.4). Since $S_+(\mathfrak{g}^*)^G.\Theta_x = 0$ we have $S_+(\mathfrak{g})^G.T_x = 0$. It follows from Proposition 3.3(2) that we can write $T_x = T_{p,\mathfrak{h}_0'}$ for some $p \in (\mathcal{H}(\mathfrak{h}_1^*) \boxtimes \mathcal{H}(\mathfrak{h}_1^*))^{W_1}$ or, equivalently, $\Theta_x = \Theta_{F_{\mathfrak{h}}(p),\mathfrak{h}_0'}$. Now, by Theorem 2.4, $\mathcal{D}(\mathfrak{h})^W.p = V^{\chi_1} \boxtimes V^{\chi_1}$ and therefore $\mathbb{C}W.p \cong V_{\chi_1} \boxtimes V_{\chi_1}$. Moreover, $T_x = T_{p,\mathfrak{h}_0'}$ is homogeneous of degree $b(\chi_{\mathbf{O}}) - 2\nu = 2b(\chi_1) - 2\nu = \deg p - 2\nu$. Thus $\deg p = 2b(\chi_1)$ and, by definition of V_{χ_1}, $p \in (V_{\chi_1} \boxtimes V_{\chi_1})^{W_1}$. □

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