HIGHER SYMMETRIES OF POWERS OF THE LAPLACIAN AND RINGS OF DIFFERENTIAL OPERATORS

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Abstract. We study the interplay between the minimal representations of the orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$ and the *algebra of symmetries* $\mathscr{S}(\square^r)$ of powers of the Laplacian $\square$ on $\mathbb{C}^n$. The connection is made through the construction of a highest weight representation of $\mathfrak{g}$ via the ring of differential operators $\mathcal{D}(X)$ on the singular scheme $X = (\mathtt{F}^r = 0) \subset \mathbb{C}^n$, for $\mathtt{F} = \sum_{j=1}^n X_i^2 \in \mathbb{C}[X_1, \dots, X_n]$.

In particular we prove that $U(\mathfrak{g})/K_r \cong \mathscr{S}(\square^r) \cong \mathcal{D}(X)$ for a certain primitive ideal K_r. Interestingly, if (and only if) n is even with $r \geq \frac{n}{2}$ then both $\mathscr{S}(\square^r)$ and its natural module $\mathcal{A} = \mathbb{C}[\frac{\partial}{\partial X_n}, \dots, \frac{\partial}{\partial X_n}]/(\square^r)$ have a finite dimensional factor. The same holds for the $\mathcal{D}(X)$-module $\mathcal{O}(X)$.

We also study higher dimensional analogues $M_r = \{x \in A : \square^r(x) = 0\}$ of the module of harmonic elements in $A = \mathbb{C}[X_1, \dots, X_n]$ and of the space of "harmonic densities". In both cases we obtain a minimal $\mathfrak{g}$-representation that is closely related to the $\mathfrak{g}$-modules $\mathcal{O}(X)$ and $\mathcal{A}$.

Essentially all these results have real analogues, with the Laplacian replaced by the d'Alembertian $\square_p$ on the pseudo-Euclidean space $\mathbb{R}^{p,q}$ and $\mathfrak{g}$ replaced by the real Lie algebra $\mathfrak{so}(p+1, q+1)$.

Contents

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1. Introduction

In this paper we explain a close relationship between two extensively studied subjects: the construction of "minimal representations" of the orthogonal Lie algebra $\mathfrak{so}(p+1,q+1)$ and the "algebra of symmetries" of powers of the Laplacian on the pseudo-Euclidean space $\mathbb{R}^{p,q}$. The connection is made through the construction of a highest weight representation of $\mathfrak{so}(n+2,\mathbb{C})$ via differential operators on a singular subscheme of $\mathbb{C}^n$. This approach, and its relation to Howe duality, goes back to [LSS, LS] (see also [Jos4]).

Let $\mathbb{R}[X_1,\dots,X_n]$ be a polynomial ring in $n \geq 3$ variables over the real numbers, with ring of differential operators (also known as the Weyl algebra) $A_n(\mathbb{R}) = \mathbb{R}[X_j,\partial_{X_k}; 1 \leq j,k \leq n]$, where $\partial_{X_k} = \frac{\partial}{\partial X_k}$. An important technique for the group-theoretical analysis of the differential equations associated to a differential operator $\Theta \in A_n(\mathbb{R})$ is to determine the higher symmetries of Θ. Here, an operator $P \in A_n(\mathbb{R})$ is a *(higher) symmetry* of Θ provided $\Theta P = P'\Theta$ for some second operator $P' \in A_n(\mathbb{R})$. The operator P is a trivial symmetry when $P \in A_n(\mathbb{R})\Theta$ and factoring the space of symmetries by the trivial ones gives the *algebra of symmetries* $\mathscr{S}(\Theta)$ (see §8 for more details). These symmetries are important to differential equations through exact integrability and separation of variables (see for example [BG, BKM, Mil]) and have further applications in general relativity and the theory of higher spin fields (see [Eas] and the references therein).

Finding these symmetries is, however, nontrivial and has been the focus of considerable research. In this paper we are concerned with the case when $\Theta = \Box_p^r$ is some power of the d'Alembertian $\Box_p = \sum_{j=1}^p \partial_{X_j}^2 - \sum_{j=p+1}^{p+q} \partial_{X_j}^2$ defined on the pseudo-Euclidean space $\mathbb{R}^{p,q}$, where $n = p+q$, equipped with the metric $\mathtt{F}_p = \sum_{j=1}^p X_j^2 - \sum_{j=p+1}^n X_j^2$. In this case the algebra of symmetries is known and, moreover, forms a factor algebra of the enveloping algebra $U(\mathfrak{so}(p+1,q+1))$ of the real orthogonal Lie algebra $\mathfrak{so}(p+1,q+1)$; see, [Eas] for the first definitive study for $r=1$ and then [EL, GS, BG, Mic] for more general results proved using a variety of methods. Earlier results, albeit without full proofs, appear in [ShS].

We give a more algebraic approach to this problem which also permits a much more detailed description of the $\mathfrak{so}(p+1,q+1)$-module structure of the space of symmetries, which in turn has some interesting consequences. In particular, in Theorem 9.8 and Corollary 9.10 we prove:

Theorem 1.1. (1) *For an appropriate primitive ideal K_r there exists an isomorphism*

$$\frac{U(\mathfrak{so}(p+1,q+1))}{K_r} \xrightarrow{\sim} \mathscr{S}(\Box_p^r),$$

where $n = p+q$. In particular, $\mathscr{S}(\Box_p^r)$ is a finitely generated, prime noetherian ring of Goldie rank r (see Remark 1.4).

(2) *Consider $\mathcal{A} = \mathbb{R}[\partial_{X_1},\dots,\partial_{X_n}]/(\Box_p^r)$ under its natural $\mathscr{S}(\Box_p^r)$-module structure. If n is even with $r \geq \frac{n}{2}$, then $\mathcal{A}$ has a unique proper factor $\mathscr{S}(\Box_p^r)$-module L, which is finite dimensional. Otherwise $\mathcal{A}$ is an irreducible $\mathscr{S}(\Box_p^r)$-module.*

Remark 1.2. Theorem 9.8 also give explicit sets of generators for the algebra $\mathscr{S}(\Box_p^r)$. For the weights of the corresponding highest weight $\mathfrak{so}(n+2,\mathbb{C})$-module $\mathcal{A}_{\mathbb{C}} = \mathcal{A}\otimes_{\mathbb{R}}\mathbb{C}$ and its factor $L_{\mathbb{C}}$, see Theorems 1.5 and 4.13.

The significance of the finite dimensional module L in this theorem (and of its annihilator, which will be an ideal of finite codimension in $\mathscr{S}(\Box_p^r)$) needs further study.

Let us now describe our approach. First, results for the real algebras follow easily for the corresponding results for complex algebras and so, for the rest of the introduction we concentrate on the latter case; thus we work with the polynomial ring $A = \mathbb{C}[X_1, \dots, X_n]$ and Weyl algebra $A_n(\mathbb{C}) = \mathbb{C}[X_j, \partial_{X_k}; 1 \le j, k \le n]$ and consider the complex algebra of symmetries $\mathscr{S}(\Delta_1^r)$ for the Laplacian $\Delta_1 = \sum_{j=1}^n \partial_{X_j}^2$. Secondly, let $\mathcal{F}$ be the Fourier transform of the Weyl algebra $A_n = A_n(\mathbb{C})$ which interchanges the X_j and ∂_{X_j}. This maps the Laplacian Δ_1 onto $\mathtt{F} = \sum_{j=1}^n X_j^2$. A quick examination of the definitions shows that $\mathcal{F}$ gives an anti-isomorphism from $\mathscr{S}(\Delta_1^r)$ onto $\mathbb{I}(\mathtt{F}^r A_n)/\mathtt{F}^r A_n$, where $\mathbb{I}(\mathtt{F}^r A_n) = \{\theta \in A_n : \theta \mathtt{F}^r A_n \subseteq \mathtt{F}^r A_n\}$ is the *idealizer* of $\mathtt{F}^r A_n$. Moreover, it is standard that $\mathbb{I}(\mathtt{F}^r A_n)/\mathtt{F}^r A_n \cong \mathcal{D}(R_r)$, the ring of differential operators on the commutative ring $R_r = A/\mathtt{F}^r A$ (see Section 3).

Thus, we need to understand the ring $\mathcal{D}(R_r)$ and its canonical module R_r. A remarkable property of the algebra R_1 is that it carries a representation of the orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$, and the starting point of this paper is a result proved in [LSS, Theorem 4.6]: there exists a maximal ideal J_1, called the Joseph ideal, in the enveloping algebra $U(\mathfrak{g})$ such that $\mathcal{D}(R_1) \cong U(\mathfrak{g})/J_1$. Via the Fourier transform, this also gives an alternative proof of Eastwood's result [Eas] on $\mathscr{S}(\Box_p)$.

Now suppose that $r \ge 1$. The algebra R_r also carries a representation of $\mathfrak{g}$ (see Section 4) and the main aim of this paper is to use this to describe $\mathcal{D}(R_r)$:

Theorem 1.3 (see Theorems 7.6 and 8.6)**.** *There exist a primitive ideal $J_r \subset U(\mathfrak{g})$ and isomorphisms*

$$U(\mathfrak{g})/J_r \xrightarrow{\sim} \mathcal{D}(R_r) \xrightarrow{\sim} \mathscr{S}(\Delta_1^r).$$

Remark 1.4. (1) The primitive ideal $J_r \subset U(\mathfrak{g})$ from Theorem 1.3 is also the annihilator of the $\mathfrak{g}$-module R_r. This representation of $\mathfrak{g}$ is *minimal* in the sense that the associated variety of J_r is the closure of the minimal non-zero nilpotent orbit $\mathbf{O}_{\min}$ in $\mathfrak{g}$. At least for $n > 3$, the Joseph ideal J_1 is the unique completely prime ideal with this associated variety, which explains why it necessarily appears in this theorem. In contrast, for $r > 1$ the ideal J_r is not completely prime; in fact $U(\mathfrak{g})/J_r$ has Goldie rank r, in the sense that its simple artinian ring of fractions is an $r \times r$ matrix ring over a division ring.

(2) The ideal K_r of Theorem 1.1 is the intersection $K_r = J_r \cap U(\mathfrak{so}(p+1, q+1))$.

(3) Results like Theorems 1.1 and 1.3 are very sensitive to the precise operator Δ_1 or $\Box_p$. For example, if $\mathtt{F}$ is replaced by $\varphi = \sum X_j^3 \in A = \mathbb{C}[X_1, X_2, X_3]$ then $\mathcal{D}(A/\varphi A)$ is neither finitely generated nor noetherian, and even has an infinite ascending chain of ideals [BGG]. By applying the Fourier transform $\mathcal{F}$, the same properties hold for the algebra of symmetries $\mathscr{S}(\Theta)$, for $\Theta = \sum_{j=1}^3 \partial_{X_j}^3 \in A_3$.

The idea behind the proof of Theorem 1.3 is to relate $\mathcal{D}(R_r)$ to $\mathcal{D}(R_1)$ and hence to reduce this theorem to the known case $U(\mathfrak{g})/J_1 \cong \mathcal{D}(R_1)$. This is achieved by showing that the set of differential operators $\mathcal{D}(R_r, R_1)$ from R_r to R_1 (which naturally relates $\mathcal{D}(R_r)$ to $\mathcal{D}(R_1)$) equals the module of $\mathfrak{g}$-*finite vectors* $\mathcal{L}(R_r, R_1)$ that connects $U(\mathfrak{g})/J_r$ to $U(\mathfrak{g})/J_1$. Showing that, in fact, $\mathcal{D}(R_r, R_1) = \mathcal{L}(R_r, R_1)$ is the key step in the proof of the theorem, since it reduces the problem to understanding algebras in the category $\mathcal{O}$ of highest weight $\mathfrak{g}$-modules; see Sections 5 and 6 in particular. (We remark that, here and elsewhere, we use the Lie-theoretic notation from [Bou2, Bou3] and [Jan], although the relevant terms are also defined in the body of the paper.)

Along the way we also obtain a detailed understanding of the $\mathfrak{g}$-module structure of R_r, and this forms the next main result.

Theorem 1.5 (see Theorem 4.13). *The $\mathfrak{g}$-module $N(\lambda) = R_r$ is a highest weight module, with highest weight $\lambda = (\frac{n}{2} - r)\varpi_1$ where ϖ_1 is the first fundamental weight of $\mathfrak{g}$. Furthermore:*

(i) *if n is even with $r < \frac{n}{2}$, or if n is odd, then $N(\lambda) = L(\lambda)$ is irreducible;*

(ii) *if n is even with $r \geq \frac{n}{2}$, then $N(\lambda)$ has an irreducible socle S isomorphic to $L(\mu)$, while the quotient $N(\lambda)/S = L(\lambda)$ is irreducible and finite dimensional. (The formula for μ is given in (4.14).)*

One significant consequence of this proof is that

$$\mathcal{D}(R_r) \textit{ is a maximal order in its simple ring of fractions}$$

(this concept is the natural noncommutative analogue of being integrally closed; see also Definition 5.6).

The space of harmonic polynomials $H = \{p \in A : \Delta_1(p) = 0\}$, and its real analogues, are fundamental objects with many applications; see, for example, [HSS, KO] for applications to minimal representations of the conformal group $\mathrm{O}(p+1, q+1)$ and [Bek1, Bek3] for applications in physics. Given that we have been interested in symmetries of powers of the Laplacian, it is therefore natural to consider solutions of the r-th power of the Laplacian Δ_1^r. In other words, we are interested in the space of "harmonics of level r", defined as

$$M_r = \{f \in A \,:\, \Delta_1^r(f) = 0\}.$$

This is the topic of Section 10 where we show that M_r is a $\mathfrak{g}$-module, indeed an $\mathscr{S}(\Delta_1^r)$-module, that is closely related to the $\mathcal{D}(R_r)$-module R_r through the isomorphism of Theorem 1.3:

Theorem 1.6 (see Corollary 10.12). (i) *In the category $\mathcal{O}$ of $\mathfrak{g}$-modules, M_r is isomorphic to the dual $N(\lambda)^\vee$ of $N(\lambda) = R_r$.*

(ii) *Consequently, if n is even with $r < \frac{n}{2}$ or if n is odd, then $M_r \cong N(\lambda) \cong L(\lambda)$ is simple.*

(iii) *If n is even with $r \geq \frac{n}{2}$, then M_r has an irreducible finite dimensional socle $E \cong L(\lambda)$. The quotient $M_r/E \cong L(\mu)$ is an irreducible highest weight module.*

Remark 1.7. As was true for Theorem 1.1, there are also analogues of this result for solutions of powers of the d'Alembertian $\square_p$ and these are described in Corollary 10.13.

One should observe that the action of the Lie algebra $\mathfrak{g}$ on R_r and M_r is not given by linear vector fields. For instance, one needs differential operators P_j of order 2 for R_r and their Fourier transforms $\mathcal{F}(P_j)$ for the action on M_r; see (3.7) and Theorem 9.8. This is similar to the action on the minimal representation of $\mathfrak{so}(p+1, q+1)$ in its Schrödinger model, as described in [KM].

The $\mathfrak{g}$-module M_1 of harmonic polynomials is also an incarnation of the scalar singleton module introduced by Dirac through the ambient method ([Dir1, Dir2, EG, Bek3]). In this approach one studies the Laplacian on $\mathbb{R}^n$ via a conformal compactification of $\mathbb{R}^n$ which appears as a projective quadric $\mathcal{Q} = \{\mathtt{Q} = 0\} \subset \mathbb{RP}^{n+1}$ together with the action of the Laplacian from $\mathbb{R}^{n+2}$ on densities of a particular weight. Similar questions arise for densities on the "generalised light cone" $\{\mathtt{Q}^r = 0\}$ and, by [EG] and

[GJMS], the ambient method also works here; in this case for densities of weight $-\frac{n}{2}+r$. This produces an $\mathfrak{so}(p+1,q+1)$-module, which in the case of the Minkowski space time corresponds to the higher-order singleton as defined in [BG].

A more detailed review of this technique is given in the final Section 11, where we give an algebraic version of the ambient construction and relate it to the $\mathfrak{g}$-modules R_r and M_r described above. Roughly speaking, in our setting the generalised light cone $\{\mathtt{Q}^r = 0\}$ is replaced by a factor $B/\mathtt{Q}^r B$ of a polynomial ring B in $(n+2)$ variables, equipped with an appropriate Laplacian $\Delta \in \mathcal{D}(B)$, while the densities are replaced by homogeneous polynomials in a finite extension $S/\mathtt{Q}^r S$ of $B/\mathtt{Q}^r B$.

As such, the Laplacian Δ acts on the space $(S/\mathtt{Q}^r S)(-\frac{n}{2}+r)$ of densities of weight $-\frac{n}{2}+r$ and gives the $\mathfrak{g}$-module of *harmonic densities*

$$N_\lambda = \left\{\bar{f} \in (S/\mathtt{Q}^r S)(-\tfrac{n}{2}+r) \,:\, \Delta(f) = 0\right\}$$

(see Definition 11.7 and Proposition 11.9). Under this notation and using the ideas of the ambient method, we are able to relate all the earlier constructions by proving:

Theorem 1.8 (see Corollary 11.13)**.** *There are $\mathfrak{g}$-module isomorphisms $N_\lambda \cong M_r \cong N(\lambda)^\vee$.*

We remark that one consequence of this result is that the algebra of symmetries $\mathscr{S}(\Box_{n-1})$ is isomorphic to the "on-shell higher-spin algebra" of [Bek1, §3.1.3, Corollary 3]. See Remark 11.14 for the details.

2. Notation

Fix an integer $n \geq 3$ and set $N = n+2$. In this section we fix some notation about the complex simple Lie algebra $\mathfrak{g} = \mathfrak{so}(n+2,\mathbb{C})$. The rank of $\mathfrak{g}$ will be denoted by $\operatorname{rk}\mathfrak{g} = \ell = \ell'+1$; thus, $\mathfrak{g}$ is of type B_ℓ (with $\ell = \frac{n}{2}+\frac{1}{2}$) when n is odd and of type D_ℓ (with $\ell = \frac{n}{2}+1$) when n is even. Note that $\mathfrak{g} \cong \mathfrak{sl}(4,\mathbb{C})$ when $n = 4$.

A convenient presentation of $\mathfrak{g}$ is given by derivations on the polynomial algebra in N variables. Thus, let $U_{\pm 1}, \dots, U_{\pm\ell}$ be 2ℓ indeterminates over $\mathbb{C}$. If n is odd we let U_0 denote another indeterminate and set $U_0 = 0$ when n is even; thus in each case $B = \mathbb{C}[U_{\pm 1}, \dots, U_{\pm\ell}, U_0]$ is a polynomial ring in N variables. Write $\partial_{U_j} = \frac{\partial}{\partial U_j}$ with the convention that $\partial_{U_0} = 0$ if n is even.

As in [GW, Corollaries 1.2.7 & 1.2.9] we identify $\mathfrak{g}$ with the Lie subalgebra of elements in $\mathfrak{gl}(N,\mathbb{C}) = \operatorname{End}\mathbb{C}^N$ which preserve the quadratic form:

$$\mathtt{Q} = \sum_{j=1}^{\ell} U_j U_{-j} + \frac{1}{2}U_0^2.$$

Define differential operators on B by

$$\mathtt{E} = U_0\partial_{U_0} + \sum_{j=1}^{\ell}(U_j\partial_{U_j} + U_{-j}\partial_{U_{-j}}), \qquad \Delta = \partial_{U_0}^2 + 2\sum_{j=1}^{\ell}\partial_{U_j}\partial_{U_{-j}} \tag{2.1}$$

and derivations of B by setting:

$$E_{ij} = U_i\partial_{U_j} - U_{-j}\partial_{U_{-i}} \qquad \text{for } i,j = 0,\pm 1,\dots,\pm\ell. \tag{2.2}$$

Observe that $E_{-i,-j} = -E_{j,i}$, while $E_{-i,i} = 0$ and $E_{ij}(\mathtt{Q}) = 0$. The following result is classical.

Proposition 2.3. (1) *The $\mathbb{C}$-vector space spanned by the derivations E_{ij} is a subalgebra of the Lie algebra* $(\mathrm{End}_{\mathbb{C}} B, [\ ,\ ])$ *isomorphic to the orthogonal Lie algebra* $\mathfrak{so}(n+2, \mathbb{C})$.
(2) *The Lie subalgebra $\mathfrak{sp}$ of* $\big(\mathrm{End}_{\mathbb{C}} B, [\ ,.]\big)$ *spanned by Δ, $\mathtt{Q}$ and $-\big(\mathtt{E} + \frac{N}{2}\big)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. Under the identification of $\mathfrak{gl}(N, \mathbb{C})$ with the space of vector fields spanned by $\{U_i \partial_{U_j}\}$, part (1) follows from [GW, §2.3.1, p. 70], while part (2) is in [How1, §6]. □

We can and therefore will identify $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$ with the Lie algebra spanned by the $\{E_{ij}\}$. Notice that each of the subalgebras $\mathfrak{sp}$ and $\mathfrak{g}$ is contained in the commutant of the other; this is an infinitesimal version of the existence of the dual pair $(\mathrm{O}(n+2, \mathbb{C}), \mathrm{Sp}(2, \mathbb{C}))$, see [How1, §6] or [How2, §3].

The space $\mathfrak{h} = \bigoplus_{j=1}^{\ell} \mathbb{C} E_{jj}$ is a Cartan subalgebra of $\mathfrak{g}$ and we set $\varepsilon_j = E_{jj}^* \in \mathfrak{h}^*$. Let Φ denote the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ and write $\mathfrak{g}^\alpha$ for the space of root vectors of weight $\alpha \in \mathfrak{h}^*$. The set Φ is then given by

$$\{\pm(\varepsilon_a \pm \varepsilon_b) \,:\, 1 \le a < b \le \ell\} \bigsqcup \{\pm\varepsilon_b \,:\ \ 1 \le b \le \ell\} \quad \text{when } n \text{ is odd}$$

and $\{\pm(\varepsilon_a \pm \varepsilon_b) \,:\, 1 \le a < b \le \ell\}$ when n is even. We choose positive roots by setting

$$\Phi^+ = \{\varepsilon_1 \pm \varepsilon_{b+1}\}_{1 \le b \le \ell'} \bigsqcup \{\varepsilon_{a+1} \pm \varepsilon_{b+1}\}_{1 \le a < b \le \ell'} \bigsqcup \{\varepsilon_1\} \bigsqcup \{\varepsilon_{b+1}\}_{1 \le b \le \ell'} \quad \text{if } n \text{ is odd}$$

and $\Phi^+ = \{\varepsilon_1 \pm \varepsilon_{b+1}\}_{1 \le b \le \ell'} \bigsqcup \{\varepsilon_{a+1} \pm \varepsilon_{a+1}\}_{1 \le a < b \le \ell'}$ if n is even. For more details, see [GW, §2.3.1]. In general we will use [Bou2, Bou3] as our basic reference for Lie-theoretic concepts. In particular the set Φ^+ is taken from [Bou2, Planches II et IV] and we fix the same basis $\mathsf{B} = \{\alpha_1, \dots, \alpha_\ell\}$ as described there. Set $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha$ and $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$, which is a Borel subalgebra of $\mathfrak{g}$. A more refined *Chevalley system* in $\mathfrak{g}$ is then given in the next proposition, see [Bou3, Chap. VIII, § 13] for further details.

Proposition 2.4. *The following set $\{y_\alpha \in \mathfrak{g}^\alpha\}_{\alpha \in \Phi}$ of root vectors is a Chevalley system in $\mathfrak{g}$ (recall that $[y_\alpha, y_{-\alpha}] = h_\alpha$):*

(a) *$y_{\varepsilon_a - \varepsilon_b} = E_{ab}$, $y_{\varepsilon_a + \varepsilon_b} = E_{a,-b}$, $1 \le a < b \le \ell$, while $y_{\varepsilon_b} = \sqrt{2} E_{b,0}$, $1 \le b \le \ell$, if n is odd;*
(b) *$y_{-(\varepsilon_a - \varepsilon_b)} = E_{ba}$, $y_{-(\varepsilon_a + \varepsilon_b)} = E_{-b,a}$, $1 \le a < b \le \ell$, while $y_{-\varepsilon_b} = \sqrt{2} E_{0,b}$, $1 \le b \le \ell$, if n is odd;*
(c) *$h_{\varepsilon_j \pm \varepsilon_k} = E_{jj} \pm E_{kk}$, $1 \le j < k \le \ell$ while $h_{\varepsilon_j} = 2E_{jj}$, $1 \le j \le \ell$ if n is odd.*

In particular, $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with

$$\mathfrak{n}^+ = \bigoplus_{1 \le i < j \le \ell} \mathbb{C} E_{ij} \oplus \bigoplus_{0 \le i < j \le \ell} \mathbb{C} E_{i,-j}, \quad \textit{and} \quad \mathfrak{n}^- = \bigoplus_{1 \le i < j \le \ell} \mathbb{C} E_{ji} \oplus \bigoplus_{0 \le i < j \le \ell} \mathbb{C} E_{-j,i}. \qquad \square$$

The subspace $\mathfrak{k} \subset \mathrm{Der}_{\mathbb{C}}(A)$ generated by the derivations E_{ij} with $i, j \in \{0, \pm 2, \dots, \pm\ell\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}(n, \mathbb{C})$ and is of type $\mathsf{B}_{\ell-1} = \mathsf{B}_{\ell'}$ (n odd) or $\mathsf{D}_{\ell-1} = \mathsf{D}_{\ell'}$ (n even). Furthermore, $\mathfrak{m} = \mathbb{C} E_{11} \oplus \mathfrak{k} \subset \mathfrak{g} \cong \mathfrak{so}(2, \mathbb{C}) \times \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{g}$ decomposes as:

$$\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{p}, \quad \text{for} \quad \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}^+ \text{ and } \mathfrak{k} = [\mathfrak{m}, \mathfrak{m}],$$

where $\mathfrak{r}^- = \bigoplus_{p \ne \pm 1} \mathbb{C} E_{p1} \subset \mathfrak{n}^-$ and $\mathfrak{r}^+ = \bigoplus_{p \ne \pm 1} \mathbb{C} E_{1p} \subset \mathfrak{n}^+$. Here $\mathfrak{p}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with abelian nilradical $\mathfrak{r}^+ \cong \mathbb{C}^n$ and Levi subalgebra $\mathfrak{m}$; for more details, see [LSS, §3]. In the notation of [LSS, Table 3.1], $\mathfrak{p}$ is the parabolic $\mathfrak{p}_1$ (of types B_ℓ, respectively D_ℓ when n is odd, respectively even) except that, when $n = 4$, $\mathfrak{p}$ is the parabolic $\mathfrak{p}_2$ of type A_3.

Let $\varpi_1, \dots, \varpi_\ell$ denote the fundamental weights of $\mathfrak{g}$ and $\mathsf{P} = \bigoplus_{j=1}^{\ell} \mathbb{Z}\varpi_j$ the lattice of weights. Set $\mathbb{N}^* = \mathbb{N} \smallsetminus \{0\}$ and $\mathsf{P}_{++} = \bigoplus_{j=1}^{\ell} \mathbb{N}^*\varpi_j \subset \mathsf{P}_+ = \bigoplus_{j=1}^{\ell} \mathbb{N}\varpi_j$. In order to quote results from [Jan], we will often need to shift by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = \sum_{j=1}^{\ell} \varpi_j$ and we define

$$\mathsf{P}^{++} = -\rho + \mathsf{P}_{++} = \{\mu \in \mathsf{P} : \langle \mu + \rho\,, \alpha^\vee\rangle > 0,\ \alpha \in \mathsf{B}\} = \{\mu \in \mathsf{P} : \langle \mu\,, \alpha^\vee\rangle \geq 0,\ \alpha \in \mathsf{B}\}$$
$$\mathsf{P}^{+} = -\rho + \mathsf{P}_{+} = \{\mu \in \mathsf{P} : \langle \mu + \rho\,, \alpha^\vee\rangle \geq 0,\ \alpha \in \mathsf{B}\} = \{\mu \in \mathsf{P} : \langle \mu\,, \alpha^\vee\rangle \geq -1,\ \alpha \in \mathsf{B}\}.$$

The definition of a Verma module $M(\mu)$ and its unique simple quotient $L(\mu)$, for $\mu \in \mathfrak{h}^*$, will be relative to our given triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that $L(\mu)$ is a finite dimensional simple module if and only if $\mu \in \mathsf{P}^{++}$. Finally, we let $\mathcal{O}$ denote the category of highest weight modules as defined, for example, in [Jan, 4.3].

3. Differential operators

We continue the discussion of the Lie algebra $\mathfrak{so}(n+2, \mathbb{C})$ and its presentation as differential operators, most especially those on the ring $\mathbb{C}[X_1, \dots, X_n]/(\mathtt{F}^r)$ from the introduction. It is worth emphasising that, in contrast to the last section, here we start with a polynomial ring in n rather than $n+2$ variables. Both approaches will be needed in the paper but the present approach is more subtle since $\mathfrak{so}(n+2, \mathbb{C})$ can no longer be presented as an algebra of derivations on $\mathbb{C}[X_1, \dots, X_n]$; one needs to add certain second order operators. Many of the results in this section come from [Lev, KM].

For the moment, let $\mathbb{K}$ be any field of characteristic 0 and write $\mathcal{D}(R)$ for the ring of $\mathbb{K}$-linear differential operators on a commutative $\mathbb{K}$-algebra R, as defined, for example, in [EGA]. More generally, given R-modules M and N, define $\mathcal{D}(M, N) = \bigcup_{k\geq 0} \mathcal{D}_k(M, N)$, where $\mathcal{D}_{-1}(M, N) = 0$ and, for $k \geq 0$,

$$\mathcal{D}_k(M, N) \;=\; \big\{\theta \in \operatorname{Hom}_{\mathbb{C}}(M, N) : [x, \theta] \in \mathcal{D}_{k-1}(M, N) \text{ for all } x \in R\big\}. \tag{3.1}$$

The elements of $\mathcal{D}_k(M, N)$ are called *differential operators of order at most* k. Set $\mathcal{D}(M) = \mathcal{D}(M, M)$.

Fix an integer $n \geq 3$ and set $A = \mathbb{K}[X_1, \dots, X_n]$ for commuting indeterminates X_j. Recall that $\mathcal{D}(A)$ identifies with the n-th Weyl algebra $A_n(\mathbb{K}) = \mathbb{K}[X_1, \dots, X_n, \partial_{X_1}, \dots, \partial_{X_n}]$ where, as before, $\partial_{X_i} = \frac{\partial}{\partial X_i}$. Given a factor ring $R = A/\mathfrak{a}$, the *idealiser* of $\mathfrak{a}\mathcal{D}(A)$ is defined to be

$$\mathbb{I}_{\mathcal{D}(A)}\big(\mathfrak{a}\mathcal{D}(A)\big) = \{P \in \mathcal{D}(A) \,:\, P\mathfrak{a}\mathcal{D}(A) \subseteq \mathfrak{a}\mathcal{D}(A).\} \tag{3.2}$$

As noted in [SmS, Proposition 1.6], this provides the useful description of $\mathcal{D}(R)$:

$$\mathcal{D}(R) \;\cong\; \mathbb{I}_{\mathcal{D}(A)}\big(\mathfrak{a}\mathcal{D}(A)\big)/\mathfrak{a}\mathcal{D}(A). \tag{3.3}$$

Let r be a positive integer and set

$$\mathtt{F} = \sum_{i=1}^{n} X_i^2 \in A, \qquad \text{and} \qquad R = R_r = A/\mathtt{F}^r A. \tag{3.4}$$

Define the *Laplacian* $\Delta_1 \in \mathcal{D}(A)$, the *Euler operator* $\mathtt{E}_1 \in \mathcal{D}(A)$ and derivations D_{ij} for $1 \leq i, j \leq n$ by

$$\Delta_1 = \frac{1}{2}\sum_{i=1}^{n} \partial_{X_i}^2, \qquad \mathtt{E}_1 = \sum_{i=1}^{n} X_i\partial_{X_i}, \qquad D_{ij} = -D_{ji} = X_i\partial_{X_j} - X_j\partial_{X_i}. \tag{3.5}$$

(Note the minor differences between these operators and the Euler $\mathtt{E}$ and Laplacian Δ from (2.1).) The next lemma follows from straightforward calculations; see for example [Lev, Proposition 1.2.2].

Lemma 3.6. *For $1 \le i, j, k, l \le n$ one has:*

(i) $[D_{kl}, X_i] = \delta_{il} X_k - \delta_{ik} X_l$ *while* $[D_{ij}, \mathtt{F}] = D_{ij}(\mathtt{F}) = 0$;

(ii) $[D_{ij}, D_{kl}] = \delta_{jk} D_{il} + \delta_{jl} D_{ki} + \delta_{ik} D_{lj} + \delta_{il} D_{jk}$. □

Let $d \in \mathbb{K}$ and define second order differential operators in $\mathcal{D}(A)$ by

$$P_j(d) = X_j \Delta_1 - (\mathtt{E}_1 + d)\partial_{X_j} = X_j \Delta_1 - \partial_{X_j}(\mathtt{E}_1 + d - 1), \quad \text{for} \quad 1 \le j \le n. \tag{3.7}$$

Direct computations (see, in particular, [Lev, Proposition 1.2.2 and Théorème 2.1.3] or [KM, Theorem 2.4.1]) yield:

Proposition 3.8. (1) *For $1 \le i, j, k \le n$, the elements $P_j(d)$ satisfy:*

(a) $[\mathtt{E}_1 + d, P_j(d)] = -P_j(d)$ *and* $[P_j(d), \partial_{X_k}] = \partial_{X_j}\partial_{X_k} - \delta_{j,k}\Delta_1$;

(b) $[X_i, P_j(d)] = D_{i,j} + \delta_{i,j}(\mathtt{E}_1 + d)$, *and* $[D_{k,l}, P_j(d)] = \delta_{l,j} P_k(d) - \delta_{k,j} P_l(d)$;

(c) $[P_i(d), P_j(d)] = 0$.

(2) *Assume that $\mathbb{K}$ is algebraically closed. The subspace $\widetilde{\mathfrak{g}}_d$ of $\big(\mathcal{D}(A), [\ ,\ ]\big)$ spanned by the elements X_i, $\mathtt{E}_1 + d$, D_{ij} and $P_j(d)$ (for $1 \le i, j \le n$) is isomorphic to the Lie algebra $\mathfrak{so}(n+2, \mathbb{K})$.*

(3) *The subspace of $\widetilde{\mathfrak{g}}_d$ spanned by the D_{ij} is isomorphic to $\mathfrak{so}(n, \mathbb{K})$.* □

Notation 3.9. For the rest of the section assume (for simplicity) that $\mathbb{K} = \mathbb{C}$ and fix $r \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Recall from Section 2 that the (abstract) Lie algebra $\mathfrak{so}(n+2, \mathbb{C})$ is denoted by $\mathfrak{g}$. Let $\widetilde{\psi} = \widetilde{\psi}_d : \mathfrak{g} \to \widetilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$ be an isomorphism defined by Proposition 3.8(2); an explicit map can be found in Corollary 4.4.

We now want to choose $d \in \mathbb{C}$ so that $P_j(d)(\mathtt{F}^r A) \subseteq \mathtt{F}^r A$, and hence so that $P_j(d) \in \mathbb{I}_{\mathcal{D}(A)}\big(\mathtt{F}^r \mathcal{D}(A)\big)$.

Lemma 3.10. *Let $d \in \mathbb{C}$ and $1 \le j \le n$. Then for any $k \ge 1$ and function u of class $\mathcal{C}^2$ on $\mathbb{R}^n$ or $\mathbb{C}^n$ we have*

$$P_j(d)(u\mathtt{F}^k) = P_j(2k + d)(u)\mathtt{F}^k + k\big(n - 2(k + d)\big)X_j u \mathtt{F}^{k-1}. \tag{3.11}$$

Fix $r \in \mathbb{N}^*$ and set $d = d(r) = \frac{n}{2} - r$. Then for any $u \in A$ one has:

$$P_j(d)(u\mathtt{F}^r) = P_j(2r + d)(u)\mathtt{F}^r. \tag{3.12}$$

Proof. We first show that, for $k \ge 0$,

$$\Delta_1(u\mathtt{F}^k) = \Delta_1(u)\mathtt{F}^k + 2k\mathtt{E}_1(u)\mathtt{F}^{k-1} + k(n + 2(k-1))u\mathtt{F}^{k-1}. \tag{3.13}$$

To see this note that $\partial_{X_j}(u\mathtt{F}^k) = \partial_{X_j}(u)\mathtt{F}^k + 2kX_j u\mathtt{F}^{k-1}$ and hence that

$$\partial^2_{X_j}(u\mathtt{F}^k) = \partial^2_{X_j}(u)\mathtt{F}^k + 2kX_j\partial_{X_j}(u)\mathtt{F}^{k-1} + 2k\left(u\mathtt{F}^{k-1} + X_j\partial_{X_j}(u)\mathtt{F}^{k-1} + 2(k-1)X_j^2 u\mathtt{F}^{k-2}\right).$$

Equation (3.13) follows by summing over j and (3.11) then follows by an elementary computation. Taking $d = \frac{n}{2} - r$ and $k = r$ in (3.11) immediately gives (3.12). □

Corollary 3.14. *Assume that $r \in \mathbb{N}^*$ and set $d = \frac{n}{2} - r$. Set $R = R_r = A/(\mathtt{F}^r)$ and let $\widetilde{\mathfrak{g}}_d$ the Lie subalgebra of $\mathcal{D}(A)$ defined by Proposition 3.8. Then the elements of $\widetilde{\mathfrak{g}}_d$ induce differential operators on R_r. This therefore defines an algebra morphism $\psi_r : U(\mathfrak{g}) \longrightarrow \mathcal{D}(R)$.*

Proof. Set $\mathcal{B} = \mathbb{I}_{\mathcal{D}(A)}\big(\mathtt{F}^r\mathcal{D}(A)\big)$ in the notation of (3.2). Combining Lemma 3.6 and Lemma 3.10 shows that $\widetilde{\mathfrak{g}}_d \subset \mathcal{B}$; equivalently, by (3.3), the elements of $\widetilde{\mathfrak{g}}_d$ induce differential operators on R_r. Now apply Proposition 3.8(2). $\square$

Remark 3.15. Write $\operatorname{Ker}\psi_r = J_r$; thus $U(\mathfrak{g})/J_r \xrightarrow{\sim} \operatorname{Im}(\psi_r) \subset \mathcal{D}(R)$. It is easily seen that $\psi_r(\xi) \neq 0$ for all $\xi \in \mathfrak{g}$. We therefore can, and frequently will, identify $\mathfrak{g}$ with $\mathfrak{g}_r = \psi_r(\mathfrak{g}) \subset \mathcal{D}(R)$. In this case the morphism constructed in Corollary 3.14 will be denoted $\psi_r : U(\mathfrak{g}) \to \mathcal{D}(R)$.

4. The $\mathfrak{so}(n+2,\mathbb{C})$-module structure of $A/(\mathtt{F}^r)$

We keep the notation of the last section, especially (3.4), Notation 3.9 and Remark 3.15. Set $d = \frac{n}{2} - r$, where $n \geq 3$ and $r \geq 1$ and write $P_j = P_j(d)$ in the notation of (3.7). Clearly R is a module over $\mathfrak{g} = \mathfrak{so}(n+2,\mathbb{C})$ via the morphism ψ_r and the aim of the section is to examine this module structure. In particular we show that R is either a simple $\mathfrak{g}$-module or has a unique finite dimensional factor module; the latter occurring if and only if n is even with $d \leq 0$. This proves Theorem 1.6 from the introduction.

We first show that the $\mathfrak{g}$-module R is a highest weight module for our choice of Cartan and Borel subalgebras of $\mathfrak{g}$. The action of an element $u \in U(\mathfrak{g})$ on $f \in R$ will be denoted by $u.f = \psi_r(u)(f)$. For the rest of this section we will identify $\mathfrak{g}$ with $\mathfrak{g}_r \subset \mathcal{D}(R)$ through ψ_r, or, equivalently, $\mathfrak{g}$ with $\widetilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$ through $\widetilde{\psi}_d$. We begin by making this identification more precise (see Lemma 4.3 and Corollary 4.4).

Let $i = \sqrt{-1} \in \mathbb{C}$ and define elements of $\widetilde{\mathfrak{g}}_d$ by

$$I(d) = \mathtt{E}_1 + d, \qquad h_1 = -I(d), \qquad \text{and} \qquad h_{j+1} = iD_{j,j+\ell'}, \;\; 1 \leq j \leq \ell'.$$

Thus $\bigoplus_{j=1}^{\ell} \mathbb{C}h_j \equiv \mathfrak{h}$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}_d \equiv \mathfrak{g}$.

Next, observe that the approaches of the last two sections are related by the changes of variables

$$U_{a+1} = X_a - iX_{a+\ell'}, \;\; U_{-(a+1)} = X_a + iX_{a+\ell'}, \text{ for } 1 \leq a \leq \ell'; \text{ and } U_0 = \sqrt{2}X_n \text{ when } n \text{ is odd.} \quad (4.1)$$

In this notation, $\mathtt{F} = \sum_{j=2}^{\ell} U_jU_{-j} + \frac{1}{2}U_0^2$, and $\Delta_1 = 2\sum_{j=2}^{\ell} \partial_{U_j}\partial_{U_{-j}} + \partial_{U_0}^2$ while $\mathtt{E}_1 = U_0\partial_{U_0} + \sum_{\pm k=2}^{\ell} U_k\partial_{U_k}$. As $\mathfrak{r}^- = \bigoplus_{j=1}^{n} \mathbb{C}X_j$ we can and will identify $A = \mathbb{C}[X_1,\dots,X_n]$ with $S(\mathfrak{r}^-)$, or equivalently with the algebra of polynomial functions on $\mathfrak{r}^+$.

We will also need to define a Chevalley system inside $\widetilde{\mathfrak{g}}_d$. Set:

$$V_{a+1} = \frac{1}{2}(P_a - iP_{a+\ell'}), \;\; V_{-(a+1)} = \frac{1}{2}(P_a + iP_{a+\ell'}), \text{ for } 1 \leq a \leq \ell'; \text{ and } V_0 = \frac{1}{\sqrt{2}}P_n \text{ when } n \text{ is odd.}$$

Observe that $\oplus_j\mathbb{C}P_j = \oplus_j\mathbb{C}V_j$ and it is easy to check that

$$V_j = \frac{1}{2}U_j\Delta_1 - I(d)\partial_{U_{-j}}, \quad \text{for } j \in \{0,\pm 2,\dots,\pm\ell\}. \quad (4.2)$$

The next result follows from straightforward computations.

Lemma 4.3. *The following set $\{x_\alpha\}_{\alpha\in\Phi}$ of root vectors is a Chevalley system in $\widetilde{\mathfrak{g}}_d$ or $\mathfrak{g}_r$ (recall that $[x_\alpha, x_{-\alpha}] = h_\alpha$):*

(i) $x_{\varepsilon_{a+1}\pm\varepsilon_{b+1}} = U_{a+1}\partial_{U_{\mp(b+1)}} - U_{\pm(b+1)}\partial_{U_{-(a+1)}}$, $1\le a<b\le \ell'$;

(ii) $x_{-(\varepsilon_{a+1}\pm\varepsilon_{b+1})} = U_{\mp(b+1)}\partial_{U_{a+1}} - U_{-(a+1)}\partial_{U_{\pm(b+1)}}$, $1\le a<b\le \ell'$;

(iii) $x_{\varepsilon_1\pm\varepsilon_{a+1}} = V_{\pm(a+1)}$, $1\le a\le \ell'$, *while* $x_{\varepsilon_1}=\sqrt{2}V_0$ *if* n *is odd;*

(iv) $x_{-(\varepsilon_1\pm\varepsilon_{a+1})} = U_{\mp(a+1)}$, $1\le a\le \ell'$, *while* $x_{-\varepsilon_1}=\sqrt{2}U_0$ *if* n *is odd;*

(v) $x_{\varepsilon_{b+1}} = \sqrt{2}\big(U_{b+1}\partial_{U_0} - U_0\partial_{U_{-(b+1)}}\big)$, $x_{-\varepsilon_{b+1}} = \sqrt{2}\big(U_0\partial_{U_{b+1}} - U_{-(b+1)}\partial_{U_0}\big)$, $1\le b\le \ell'$, *if* n *is odd;*

(vi) *Then* $h_{\varepsilon_j\pm\varepsilon_k} = h_j\pm h_k$, $1\le j<k\le \ell$, *while* $h_{\varepsilon_j}=2h_j$, $1\le j\le \ell$ *if* n *is odd.*

In particular, one can identify $\mathfrak{r}^+$ *with* $\oplus_j\mathbb{C}P_j=\oplus_j\mathbb{C}V_j$. $\square$

Corollary 4.4. *Retain the notation of Proposition 2.4 and Lemma 4.3. The map* $\widetilde{\psi}_d$ *from* $\mathfrak{g}\subset\mathcal{D}(B)$ *onto* $\widetilde{\mathfrak{g}}_d\subset\mathcal{D}(A)$*, given by* $\widetilde{\psi}_d(y_\alpha)=x_\alpha$ *for* $\alpha\in\Phi$*, is an isomorphism of Lie algebras.* $\square$

Remark 4.5. Recall that a Chevalley system in a semi-simple Lie algebra defines a Chevalley anti-involution [Jan, 2.1]. Let ϑ, respectively ϑ_d, be the anti-involution of $\mathfrak{g}$, respectively $\widetilde{\mathfrak{g}}_d$ defined by $\{y_\alpha\}_\alpha$, respectively $\{x_\alpha\}_\alpha$. Then, by definition, the isomorphism $\widetilde{\psi}_d$ satisfies $\widetilde{\psi}_d\circ\vartheta=\vartheta_d\circ\widetilde{\psi}_d$.

Let $G=\mathrm{SO}(n+2,\mathbb{C})\supset M$ be connected algebraic groups such that $\mathrm{Lie}(G)=\mathfrak{g}\supset\mathrm{Lie}(M)=\mathfrak{m}$, with G acting on $\mathfrak{g}$ through the adjoint action. The (reduced) quadratic cone $Z=\{\mathtt{F}=0\}\subset\mathbb{C}^n$ can be identified with the closure of the M-orbit of a highest weight vector $x_{\tilde{\alpha}}\in\mathfrak{r}^+\equiv\mathbb{C}^n$ and this then identifies $R_1=A/\mathtt{F}A$ with the algebra of regular functions on Z. Recall that the orbit $\mathbf{O}_{\min}=G.x_{\tilde{\alpha}}$ is the nonzero nilpotent orbit of minimal dimension and that $\frac{1}{2}\dim\mathbf{O}_{\min}=\dim M.x_{\tilde{\alpha}}=n-1$.

Recall that the Verma module $M(\mu)$ with highest weight $\mu\in\mathfrak{h}^*$ has a unique simple quotient $L(\mu)$ and set $I(\mu)=\mathrm{ann}_{U(\mathfrak{g})}L(\mu)$. The associated variety, $\mathcal{V}(J)\subset\mathfrak{g}$, of $J=I(\mu)$ is the closure of a nilpotent orbit, cf. [Jos3]. Let W be the associated Weyl group of Φ and denote by $w\cdot\lambda=w(\lambda+\rho)-\rho$ the "dot" action of $w\in W$ on $\lambda\in\mathfrak{h}^*$, cf. [Jan, 2.3].

The starting point for this paper is the following result from [LSS], see also [Lev], which gives a construction of the Joseph ideal. (See [Jos4] for another proof of this theorem.)

Theorem 4.6. *The ideal* $J_1=\mathrm{Ker}\,\psi_1$ *is a completely prime maximal ideal such that* $\mathcal{V}(J_1)=\overline{\mathbf{O}}_{\min}$*. Moreover,* $U(\mathfrak{g})/J_1\cong\mathcal{D}(R_1)$*.*

Recall [Jos1] that if $\mathfrak{s}$ is a complex simple Lie algebra not of type A_ℓ, there exits a unique completely prime ideal $J\subset U(\mathfrak{s})$ such that $\mathcal{V}(J)=\overline{\mathbf{O}}_{\min}$. The primitive ideal J is called the *Joseph ideal.* Thus, when $n\neq 4$, the ideal J_1 is the Joseph ideal. When $n=4$, i.e. $\mathfrak{g}$ is of type $\mathsf{D}_3=\mathsf{A}_3$, by a slight abuse of notation, we will still call J_1 the Joseph ideal.

Remark 4.7. The ideal K_1 from Theorem 1.1 is the intersection of J_1 with $U(\mathfrak{so}(p+1,q+1))$. Eastwood also notes this fact, but his argument relies on an explicit set of generators for the Joseph ideal (see [ESS, Str]). In contrast, in Theorem 4.6 this is almost automatic; it follows almost immediately from the fact that R_1 has Krull dimension $(n-1)$; see for example [LSS, Proposition 3.5].

In the general case, $r\ge 1$, we begin with an easy lemma.

Lemma 4.8. *The* $\mathfrak{g}$*-module* $R=R_r$ *is a highest weight module with highest weight* $\lambda=-d\varpi_1=-d\varepsilon_1$*.*

Remark 4.9. We typically write $R=N(\lambda)$ when thinking of R as a $\mathfrak{g}$-module.

Proof. Since $R \subset \operatorname{Im}(\psi_r) \subset \mathcal{D}(R)$, one has $R = U(\mathfrak{g}).1$. By construction, $D_{s,t}.1 = P_j.1 = 0$ and so the choice of Chevalley system in Lemma 4.3 implies that $\mathfrak{n}^+.1 = 0$. Moreover, $h_j.1 = 0$ for $2 \leq j \leq \ell$ whereas $h_1.1 = -I(d).1 = -d.1 = -d\varepsilon_1(h_1).1$. The choice of $\mathsf{B} \subset \Phi^+$ yields $\varpi_1 = \varepsilon_1$, see [Bou2, Planche II, pp. 252-253, and Planche IV, pp. 256-257]. Hence R is a highest weight module with highest weight $\lambda = -d\varpi_1$. $\square$

We need to analyze the structure of the $\mathfrak{g}$-module $N(\lambda)$ in more detail for which we need some notation. Grade $A = \bigoplus_{m\geq 0} A(m)$ by total polynomial degree (denoted by deg). Let $H = \{p \in A : \Delta_1(p) = 0\}$ be the space of harmonic polynomials. It is graded and we set $H(m) = A(m) \cap H$. The algebraic group $K = \mathrm{SO}(\mathbb{C}^n, \mathtt{F}) \cong \mathrm{SO}(n, \mathbb{C})$ acts naturally on A, through the identification of A with the ring of regular functions of the standard representation $\mathbb{C}^n$ of K. The differential of this K-action is given by the natural representation of

$$\mathfrak{k} \cong \bigoplus_{1\leq s<t\leq n} \mathbb{C}D_{s,t} \cong \mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(2\ell', \mathbb{C}).$$

It is clear that that $\mathbb{C}\mathtt{F}^k$ is the trivial $\mathfrak{k}$-module for all k and that each $H(m)$ is a $\mathfrak{k}$-module. Notice that $\mathfrak{h}' = \bigoplus_{j=2}^{\ell} \mathbb{C}h_j$ is a Cartan subalgebra of $\mathfrak{k}$ and set $\mathfrak{n}' = \bigoplus_{1\leq a<b\leq \ell'} \mathfrak{g}^{\varepsilon_{a+1}\pm\varepsilon_{b+1}} \subset \mathfrak{k}$. Then, $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$ is a Borel subalgebra of $\mathfrak{k}$. The following result is classical; see, for example, [GW, §5.2.3].

Theorem 4.10. *Each space $H(k)$ is an irreducible $\mathfrak{k}$-module, of highest weight $k\varpi_1$ (in Bourbaki's notation for type $\mathsf{B}_{\frac{n-1}{2}}$ or $\mathsf{D}_{\frac{n}{2}}$). The $\mathfrak{k}$-module $A(m)$ decomposes as the direct sum of irreducible modules:*

$$A(m) = \sum_{k=0}^{[m/2]} H(m-2k)\mathtt{F}^k \cong \bigoplus_{k=0}^{[m/2]} H(m-2k) \otimes \mathbb{C}\mathtt{F}^k.$$

Let B be the polar form of $\mathtt{F}$ and $\xi \in \mathbb{C}^n$ be a non zero isotropic vector with respect to $\mathtt{F}$. If $\xi^k \in A(k)$ is the function given by $v \mapsto B(v, \xi)^k$, then $H(k) = U(\mathfrak{k}).\xi^k$. $\square$

If $(v_1, \dots, v_n)$ is the canonical basis of $\mathbb{C}^n$, the vector $\xi = v_1 - i\,v_\ell$ is isotropic and the associated polynomial function ξ^k is simply $U_2^k = (X_1 - i\,X_\ell)^k$. As observed in Lemma 4.3, ξ considered as an element of $\mathfrak{g}$ has weight $-\varepsilon_1 + \varepsilon_2$ under the adjoint action of $\mathfrak{g}$. It follows from the choice of the Borel subalgebra $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$ of $\mathfrak{k}$ that $\xi^k \in H(k)$ is a highest weight vector. We will choose this generator for the simple finite dimensional $\mathfrak{k}$-module $H(k) \cong L(k\varpi_1')$ (where $\varpi_1' = \varepsilon_{2|\mathfrak{h}'}$ is the first fundamental weight for $\mathfrak{k}$). Observe the following consequence of Theorem 4.10.

Corollary 4.11. *Let $0 \leq v \leq t$ and $p = \sum_{j=v}^{t} p_j\mathtt{F}^j \in A(m)$, where $p_j \in H(m-2j)$ for each j. Then, for each $k \in \{v, \dots, t\}$, there exists $u \in U(\mathfrak{k})$ such that $p_k\mathtt{F}^k = u.p$.*

Proof. Since $\bigoplus_{j=v}^{t} H(m-2j)\mathtt{F}^j$ is a multiplicity free, semi-simple $U(\mathfrak{k})$-module, the claim follows from the Jacobson's Density Theorem, see [Bou1, § 4, n° 2, Corollaire 2]. $\square$

Let $0 \neq p \in A(m)$. By Theorem 4.10 we may uniquely write $p = \sum_{k=v}^{t} p_k\mathtt{F}^k$ with $p_k \in H(m-2k)$, $v \leq t$ and $p_t \neq 0 \neq p_v$. The integer $v = v(p)$ will be called the *harmonic valuation* of p. Let $P_j(d')$ be as in (3.11) (with $d' \in \mathbb{C}$ arbitrary). For $a \in H(m-2k)$ one has $P_j(d')(a) = -(k+d')\partial_{X_j}(a) \in H(m-2k-1)$ (with the convention that $H(j) = 0$ when $j < 0$). Also, $\Delta_1(X_j a) = [\Delta_1, X_j](a) + X_j\Delta_1(a) = \partial_{X_j}(a)$.

Therefore, by Lemma 3.10 with $d = \frac{n}{2} - r$, we obtain:

$$P_j(d)(a\mathtt{F}^k) = P_j(2k+d)(a)\mathtt{F}^k + k(n-2k-2d)X_j a\mathtt{F}^{k-1}$$

with $P_j(2k+d)(a) \in H(m-2k-1)$ and $\Delta_1(X_j a) = \partial_{X_j}(a)$. In particular, this gives:

Lemma 4.12. (1) *If $\partial_{X_j}(a) = 0$, for $a \in A$, then $P_j(d)(a\mathtt{F}^k) = P_j(2k+d)(a)\mathtt{F}^k + k(n-2k-2d)X_j a\mathtt{F}^{k-1}$. If in addition $a \in H(m-2k)$, then $P_j(2k+d)(a) \in H(m-2k-1)$ and $X_j a \in H(m-2k+1)$.*
(2) *If $a = \xi^{m-2k}$ with $\xi = v_1 - iv_\ell$, and $j \neq 1, \ell$, then the condition $\partial_{X_j}(a) = 0$ is satisfied.* $\square$

The structure of the $\mathfrak{g}$-module $N(\lambda) = R_r$ is given by the next result. Recall that $\lambda = -d\varpi_1$.

Theorem 4.13. (1) *Assume that either n is even with $r < \frac{n}{2}$ or n is odd. Then, $R_r = N(\lambda) \cong L(\lambda)$ is a simple $\mathfrak{g}$-module.*
(2) *Assume that n is even with $r \geq \frac{n}{2}$; thus $d = \frac{n}{2} - r \leq 0$. Then, as a $\mathfrak{g}$-module, $R_r = N(\lambda)$ has a simple socle $Z_r \cong L(\mu)$, where*

$$\mu = \lambda + (d-1)\alpha_1 = \begin{cases} (d-2)\varpi_1 + (1-d)\varpi_2 & \text{when } \ell \geq 4, \\ (d-2)\varpi_1 + (1-d)(\varpi_2 + \varpi_3) & \text{when } \ell = 3. \end{cases} \tag{4.14}$$

The quotient $N(\lambda)/Z_r \cong L(\lambda)$ is an irreducible finite dimensional $\mathfrak{g}$-module, isomorphic to the module of harmonic polynomials of degree $(-d)$ in $n+2$ variables. Also, Z_r is the ideal of R_r generated by $H(1-d)$.

Proof. Let $0 \neq \overline{M} \subseteq N(\lambda)$ be a $\mathfrak{g}$-submodule. Since $R_r \subset \mathrm{Im}(\psi_r)$, the module $\overline{M}$ is also an ideal of R_r. Therefore $\overline{M} = M/(\mathtt{F}^r)$, for some $\widetilde{\mathfrak{g}}_d$-stable ideal M of A. In particular, by Proposition 3.8, $(\mathtt{E}_1 + d)(M) \subseteq M$, whence $\mathtt{E}_1(M) \subseteq M$, and so $M = \bigoplus_m M \cap A(m)$ is homogeneous.

Let $p \in M \cap A(m)$, with $p \notin (\mathtt{F}^r)$, and write $p = \sum_{k=v}^{t} p_k\mathtt{F}^k$ for $p_k \in H(m-2k)$ with $p_v \neq 0 \neq p_t$. Note that $t < r$. We want to simplify our choice of p. Recall that $\mathfrak{g} \supset \mathfrak{k} \cong \mathfrak{so}(n, \mathbb{C})$. Thus, by Corollary 4.11, $p_k\mathtt{F}^k \in M \cap H(m-2k)\mathtt{F}^k$ for all k and so, as $H(m-2k)\mathtt{F}^k$ is an irreducible $\mathfrak{k}$-module, $H(m-2k)\mathtt{F}^k \subset M$ for all k such that $p_k \neq 0$. In particular, $H(m-2v)\mathtt{F}^v \subset M$ and we can replace p by $p_v\mathtt{F}^v$. There exists $u \in U(\mathfrak{k})$ such that $u.p_v = \xi^{m-2v} = (X_1 - iX_\ell)^{m-2v}$. From $D_{s,t}.\mathtt{F}^k = 0$ we then deduce $u.p = (u.p_v)\mathtt{F}^v \in M$ and, replacing p by $u.p$, we may therefore assume that $p_v = \xi^{m-2v}$. Finally, we assume that the harmonic valuation $v = v(p)$ is as small as possible among such choices of p.

Now apply the operator $P_{\ell+1} = P_{\ell+1}(d)$ to p; from Lemma 4.12 and $\partial_{X_{\ell+1}}(p_v) = 0$ we obtain

$$P_{\ell+1}.p \;=\; (P_{\ell+1}(2v+d).p_v)\mathtt{F}^v + v(n-2v-2d)X_{\ell+1}p_v\mathtt{F}^{v-1} \tag{4.15}$$

where $P_{\ell+1}(2v+d).p_v \in H(m-2v-1)$ and $X_{\ell+1}p_v \in H(m-2v+1)$. Since $P_{\ell+1}.p \in M \smallsetminus (\mathtt{F}^r)$ and $X_{\ell+1}p_v \neq 0$, the minimality of v, combined with (4.15), forces $v(n-2v-2d) = 0$. In other words, either $v = 0$ or $n - 2v - 2d = 0$. The latter is equivalent to $v = r$, which is excluded. Thus we have proved that M contains a non zero element $p = p_0 \in H(m)$. In particular, $H(m)A \subseteq M$.

Now pick $0 \neq p = \xi^m \in M \cap H(m)$ with m minimal. As p is homogeneous, if $m = 0$ then $1 \in M$ and $\overline{M} = N(\lambda)$. Otherwise $m \geq 1$. In this case, applying $P_1 = X_1\Delta_1 - (\mathtt{E}_1 + d)\partial_{X_1}$ gives

$$P_1.\xi^m = -m(\mathtt{E}_1 + d)(\xi^{m-1}) = -m(m-1+d)\xi^{m-1} \;\in\; M \cap H(m-1).$$

The minimality of m then implies that $m(m-1+d)=0$, whence $m-1+d=0$. Thus, we have shown that $\overline{M}=N(\lambda)$ unless d satisfies $d=-m+1$ for some $m\geq 1$. In the latter case $0\geq d=\frac{n}{2}-r$. Since $r,m\in\mathbb{N}$, this is equivalent to r being even with $r\geq\frac{n}{2}$. This proves (1).

It remains to prove (2), where $r=\frac{n}{2}+m-1$ for $m=1-d\geq 1$. Note that, in this case, we have shown that every non-zero $\mathfrak{g}$-submodule $\overline{M}$ of $R_r=N(\lambda)$ contains the image S of the ideal $\mathfrak{J}=H(m)A$.

Since $\lambda=-d\varpi_1=(m-1)\varpi_1\in\mathsf{P}^{++}$ is a dominant integral weight, $L(\lambda)$ is finite dimensional. Moreover, it is isomorphic to the $\mathfrak{g}$-module of harmonic polynomials of degree $m-1$ in $n+2$ variables (apply Theorem 4.10 to $\mathfrak{so}(n+2,\mathbb{C})$).

Since $D_{s,t}(H(m))\subseteq H(m)$ and $\mathfrak{J}$ is homogeneous, certainly $\Theta.\mathfrak{J}\subseteq\mathfrak{J}$ for $\Theta=D_{s,t},X_j$ and $(\mathtt{E}_1+d)$. Thus, in order to show that $\mathfrak{J}$ is a $\mathfrak{g}$-module, it remains to prove that $P_j.\mathfrak{J}\subseteq\mathfrak{J}$ or, equivalently, that $P_j.X^\alpha f\in\mathfrak{J}$ for any $f\in H(m)$ and monomial $X^\alpha=X_1^{\alpha_1}\cdots X_n^{\alpha_n}$. We argue by induction on $|\alpha|=\sum_i\alpha_i$. If $|\alpha|=0$, we have

$$P_j.f=-(m-1+d)\partial_{X_j}(f)=0 \tag{4.16}$$

by the choice of m. If $|\alpha|\geq 1$ write $X^\alpha=X_kX^\beta$, for $|\beta|=|\alpha|-1$. Then

$$P_j.X^\alpha f=[P_j,X_k].X^\beta f+X_kP_j.X^\beta f.$$

By induction, $P_j.X^\beta f\in\mathfrak{J}$, while $[P_j,X_k]=D_{j,k}-\delta_{j,k}(\mathtt{E}_1+d)$ from Proposition 3.8(b), which implies that $[P_j,X_k].X^\beta f\in\mathfrak{J}$. Hence $P_j.X^\alpha f\in\mathfrak{J}$ and $\mathfrak{J}$ is indeed a $\mathfrak{g}$-module. Since $H(m)\neq A$ and $H(m)\not\subseteq(\mathtt{F}^r)$, the image S of $\mathfrak{J}$ in R_r is a nontrivial $\mathfrak{g}$-submodule of $R_r=N(\lambda)$. However, we have already noted that S is contained in *every* non-zero $\mathfrak{g}$-submodule $\overline{M}\subseteq N(\lambda)$; in other words $S=\operatorname{Soc}(N(\lambda))=Z_r$.

Recall that $H(m)$ contains all the functions ζ^m associated to the isotropic vectors $\zeta\in\mathbb{C}^n$. Since $\mathbb{C}^n$ has a basis $(\zeta_j)_j$ of such vectors, the ideal $\sum_{j=1}^n\zeta^mA$ has finite codimension in A. Therefore, $\dim R_r/Z_r\leq\dim A/H(m)A<\infty$. Thus, on the one hand, $N(\lambda)/Z_r$ is finite dimensional and hence completely reducible, but on the other hand, as it is a factor of the Verma module $M(\lambda)$, it has a unique simple quotient. This forces $N(\lambda)/Z_r\cong L(\lambda)$.

Finally, we need to compute the highest weight of Z_r. Recall that the $\mathfrak{k}$-module $H(m)$ is generated by the function ξ^m associated to the isotropic vector $\xi=v_1-iv_\ell\in\mathbb{C}^n$. We have shown in (4.16) that $P_j.\xi^m=0$ for all j and hence $\mathfrak{r}^+.\xi^m=0$. Since $\xi^m=U_2^m$, Lemma 4.3 shows that $x_{\varepsilon_{a+1}\pm\varepsilon_{b+1}}.\xi^m=0$ for $1\leq a<b\leq\ell'$. Hence, $\mathfrak{n}'.\xi^m=0$ (recall that n is even) and so that $\mathfrak{n}^+.\xi^m=0$. Moreover, from our choice of $h_j=iD_{j,j+\ell'}$ one obtains

$$h_1.\xi^m=-\xi^m,\quad h_2.\xi^m=m\xi^m\quad\text{and}\quad h_j.\xi^m=0\ \text{ for } j\geq 3.$$

Thus $\xi^m\in Z_r$ is a highest weight vector, with weight $\mu=-\varepsilon_1+m\varepsilon_2$. Hence $Z_r\cong L(\mu)$. The fact that μ equals $\lambda-m\alpha_1$ and satisfies (4.14) are easy exercises using [Bou2, Planche IV]. □

Recall that $s_{\alpha_1}\in W$ is the transposition $(1,2)$. In the notation of Theorem 4.13, an easy calculation shows that $\mu=\lambda+(d-1)\alpha_1=s_{\alpha_1}\cdot\lambda$. We can therefore unify the cases in that result by defining

$$\omega=\begin{cases}s_{\alpha_1}\cdot\lambda=\lambda-(r-\frac{n}{2}+1)\alpha_1 & \text{if } n \text{ is even with } r\geq\frac{n}{2}\\ \lambda & \text{otherwise.}\end{cases}$$

Write $\operatorname{GKdim} M = \operatorname{GKdim}_U M$ for the *Gelfand-Kirillov dimension* of a module M over an algebra U, whenever it is defined, see [MR, §8.1.11] for more details.

Corollary 4.17. (1) *The ideal J_r is equal to $I(\omega) = \operatorname{ann}_{U(\mathfrak{g})} L(\omega)$.*
(2) *The associated variety of J_r is $\mathcal{V}(J_r) = \overline{\mathbf{O}}_{\min}$.*

Proof. (1) By Theorem 4.13, $Z_r = \operatorname{Soc} N(\lambda) \cong L(\omega)$ while $N(\lambda)/Z_r$ is finite dimensional (possibly zero). Thus, by [Jan, Lemma 8.14], $\operatorname{ann}_{U(\mathfrak{g})} Z_r = \operatorname{ann}_{U(\mathfrak{g})} N(\lambda)$; equivalently $J_r = I(\lambda) = I(\omega)$.

(2) Since $\operatorname{GKdim} L(\omega) = \operatorname{GKdim} R_r = n-1$, [Jan, Satz 10.9] implies that $\operatorname{GKdim} U(\mathfrak{g})/J(\omega) = 2(n-1)$. Thus $\mathcal{V}(J_r) = \overline{\mathbf{O}}$ for a nilpotent orbit $\mathbf{O}$ with $\dim \mathbf{O} = 2(n-1)$. This forces $\mathbf{O} = \mathbf{O}_{\min}$. $\square$

The ideal J_r contains the kernel of the character χ_λ in the centre of $U(\mathfrak{g})$. Since $\lambda = -d\varpi_1 \in \mathbb{Q}\varpi_1$, there exists a unique weight ν such that $\nu \in W(\lambda+\rho)$, and $\langle \nu\,, \alpha^\vee\rangle \geq 0$ for all simple roots α; equivalently ν is in the dominant chamber $\overline{\mathsf{C}}$ for the given choice of positive roots Φ^+. Moreover, $\chi_\lambda = \chi_{\nu-\rho}$, since $w\cdot\lambda = \nu - \rho$. For these and related Lie-theoretic facts, see [Bou2, Bou3], especially [Bou2, Chap. V.3.3, VI.1.10] and [Bou3, Chap. VIII.8.5].

Proposition 4.18. *Let $\lambda = -d\varpi_1$, so that $\lambda + \rho = (1-d)\varpi_1 + \varpi_2 + \cdots + \varpi_\ell$.*
(1) *If $d \leq 1$, then the weight ν in the dominant chamber is given by $\nu = \lambda + \rho \in \overline{\mathsf{C}}$.*
(2) *Assume that n is odd ($\mathfrak{g}$ of type B_ℓ) and $d > 1$ (i.e. $1 \leq r \leq \ell - 2 = \frac{n-3}{2}$). Then*

$$\nu \;=\; \varpi_1 + \varpi_2 + \cdots + \varpi_{\ell-r-2} + \frac{1}{2}\varpi_{\ell-r-1} + \frac{1}{2}\varpi_{\ell-r} + \varpi_{\ell-r+1} + \cdots + \varpi_\ell \;\in\; \overline{\mathsf{C}} \cap W(\lambda+\rho).$$

(3) *Assume that n is even ($\mathfrak{g}$ of type D_ℓ) and $d > 1$ (i.e. $1 \leq r \leq \ell - 3 = \frac{n-4}{2}$). Then*

$$\nu \;=\; \varpi_1 + \varpi_2 + \cdots + \varpi_{\ell-r-2} + \varpi_{\ell-r} + \varpi_{\ell-r+1} + \cdots + \varpi_\ell \;\in\; \overline{\mathsf{C}} \cap W(\lambda+\rho).$$

Proof. (1) This is obvious.

(2) Observe that $1 - d < 0 \iff r < \frac{n}{2} - 1 \iff r \leq \ell - 2 = \frac{n-3}{2}$. Let w^{-1} be the cycle $(\ell - r, \ell - r - 1, \ldots, 2, 1) \in W$. Then, $\nu = w^{-1}(\lambda + \rho)$ has the desired form.

(3) Here, $1 - d < 0 \iff r < \ell - 2$. If w^{-1} is the cycle $(\ell - r - 1, \ell - r - 2, , \ldots, 2, 1) \in W$, the weight $\nu = w^{-1}(\lambda + \rho)$ has the desired form. $\square$

Corollary 4.19. (i) *If n is even with $r \geq \frac{n}{2}$ then J_r is not maximal. The unique primitive ideal containing J_r is the finite codimensional (maximal) ideal $I((r - \frac{n}{2})\varpi_1)$.*
(ii) *In all other cases, J_r is maximal.*

Proof. Since $\mathcal{V}(J_r) = \overline{\mathbf{O}}_{\min}$, any proper factor of $U(\mathfrak{g})/J_r$ must be finite dimensional and that factor then has to be unique. On the other hand, if $U(\mathfrak{g})/J_r$ is a simple ring, then it cannot have a nonzero, finite dimensional module. Thus part (i) already follows from Theorem 4.13.

Conversely, suppose that $J_r = I(\lambda)$ is not maximal; thus it has a finite dimensional quotient. This implies that there exists $\nu \in \mathsf{P}_{++} \subset \overline{\mathsf{C}}$ such that $\nu \in W(\lambda + \rho)$ (see [Bou3, Chap. VIII.7.2, Corollaire 1 and VIII.8.5, Corollaire 1]). Proposition 4.18 gives the unique weight ν in $W(\lambda+\rho)\cap\overline{\mathsf{C}}$. By inspection, $\nu \in \mathsf{P}_{++}$ if and only if $1 - d \in \mathbb{N}^*$. As usual this forces n to be even and $r - \frac{n}{2} \geq 0$. $\square$

5. Rings in category $\mathcal{O}$

In this section we study rings R on which a reductive Lie algebra $\mathfrak{g}$ acts as differential operators, abstracting the situation from the introduction. Although we need a considerable number of hypotheses, they do hold occur frequently and the consequences are surprisingly strong. In particular, they force a factor ring of $U(\mathfrak{g})$ to be a ring of $\mathfrak{g}$-finite vectors (see Theorem 5.7 and Definition 5.3).

We are concerned with the situation described in the following hypotheses, for which we need a definition. A module M over a ring U of finite Gelfand-Kirillov dimension is called *quasi-simple* if its socle $\operatorname{Soc}(M)$ is simple, with $\operatorname{GKdim}_U(M/\operatorname{Soc}(M)) < \operatorname{GKdim}_U M$.

Hypotheses 5.1. *Let $\mathfrak{g}$ be a finite dimensional, complex reductive Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and assume that R is a commutative, finitely generated $\mathbb{C}$-algebra that is a $\mathfrak{g}$-module via a morphism $\chi : U(\mathfrak{g}) \to \mathcal{D}(R)$. Set $I = \operatorname{Ker}(\chi)$ and identify $\overline{U} = U(\mathfrak{g})/I$ with its image $\chi(U(\mathfrak{g}))$ in $\mathcal{D}(R)$. Assume moreover that:*

1. *under the above action, R is a quasi-simple, highest weight $\mathfrak{g}$-module;*
2. *R is generated by $\chi(\mathfrak{r})$ for some Lie subalgebra $\mathfrak{r} \subseteq \mathfrak{n}^-$;*
3. *if $N = \operatorname{Soc}_{U(\mathfrak{g})}(R)$, then $\operatorname{End}_R(N) = R$.*

Although these hypotheses are clearly strong, they do occur for the rings relevant to this paper (see Lemma 5.2 for (3)). It is important however, that we do not require that $\mathfrak{g}$ act by derivations and so, in that sense at least, our definition is weaker than Joseph's concept of $\mathcal{O}$-rings from [Jos5].

We remark that, by (2), the $\mathfrak{g}$-socle N of R is also an R-module and so (3) makes sense. Moreover, R has finite length as a $U(\mathfrak{g})$-module. A routine argument (see, for example, [Jan, Lemma 8.14]) shows that $I = \operatorname{ann}_{U(\mathfrak{g})}(R) = \operatorname{ann}_{U(\mathfrak{g})}(N)$. Hence $\overline{U}$ is primitive.

First we clarify when Hypothesis 5.1(3) holds.

Lemma 5.2. *Let C be a finitely generated commutative $\mathbb{C}$-algebra and $f \in C$. Suppose that C is a Cohen-Macaulay domain and set $R = C/(f)$. Let M be an ideal of R such that $\operatorname{GKdim}(R/M) \le \operatorname{GKdim}(R) - 2$. Then Hypothesis 5.1(3) holds for M in the sense that $\operatorname{End}_R(M) = R$.*

Proof. The hypothesis on C implies that R is Cohen-Macaulay and that the grade of the R-module R/M is at least 2, see [Mat, (16.A), (16.B)]; that is, $\operatorname{Hom}_R(R/M, R) = \operatorname{Ext}^1_R(R/M, R) = 0$. Applying $\operatorname{Hom}_R(-, R)$ to the short exact sequence $0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0$ then gives $R = \operatorname{End}_R(R) = \operatorname{Hom}_R(M, R)$. It follows that $\operatorname{Hom}_R(M, R) = \operatorname{End}_R(M)$ and so $\operatorname{End}_R(M) = R$. $\square$

Definition 5.3. Let $R, \mathfrak{g}$ satisfy Hypotheses 5.1 and let M, N be (left) $\mathfrak{g}$-modules. Define the adjoint action of $x \in \mathfrak{g}$ on $\theta \in \operatorname{Hom}_{\mathbb{C}}(M, N)$ by $\operatorname{ad}(x)(\theta) = x\theta - \theta x$. Set

$$\mathcal{A}(M, N) = \{\theta \in \operatorname{Hom}_{\mathbb{C}}(M, N) : \mathfrak{r} \text{ acts ad-nilpotently on } \theta\}.$$

Analogously to the order filtration for $\mathcal{D}(R)$ in (3.1), we filter the spaces $\mathcal{A}(M, N)$ by $\mathcal{A}(M, N) = \bigcup_{k \ge 0} \mathcal{A}_k(M, N)$, where $\mathcal{A}_{-1}(M, N) = 0$ and, for $k \ge 0$,

$$\mathcal{A}_k(M, N) \;=\; \big\{\theta \in \operatorname{Hom}_{\mathbb{C}}(M, N) : [x, \theta] = x\theta - \theta x \in \mathcal{A}_{k-1}(M, N) \text{ for all } x \in \mathfrak{r}\big\}. \tag{5.4}$$

Recall, for example from [Jan, §6.8], that one also has the set of $\mathfrak{g}$-*finite elements* $\mathcal{L}(M,N) \subseteq \mathrm{Hom}_{\mathbb{C}}(M,N)$, on which $\mathfrak{g}$ acts locally finitely under the adjoint action. If $M=N$, then [Jan, §6.8(5)] implies that $\mathcal{L}(M,M)$ contains $U(\mathfrak{g})/\operatorname{ann}(M)$ as a subalgebra.

Proposition 5.5. *Assume that $(R,\mathfrak{g})$ satisfies Hypotheses 5.1 and let M,N be non-zero left $\mathfrak{g}$-modules.*

(i) *Under the natural R-module action induced from (5.1)(2), each $\mathcal{A}_k(M,N)$ is an R-bimodule.*

(ii) *$\mathcal{D}(M,N)=\mathcal{A}(M,N)\supseteq\mathcal{L}(M,N)$. Indeed $\mathcal{D}_k(M,N)=\mathcal{A}_k(M,N)$ for all $k\geq 0$.*

(iii) *Assume that $M\subseteq R$ with $\mathrm{GKdim}(R/M)<\mathrm{GKdim}(R)$ and $\mathrm{End}_R(M)=R$. Then $\mathcal{L}(M,M)=\overline{U}$.*

Proof. (i) By induction, assume that $\mathcal{A}_{k-1}(M,N)$ is an R-bimodule. Let $\theta\in\mathcal{A}_k(M,N)$, $y\in R$ and $x\in\mathfrak{r}$. Then, by induction, and the fact that R is commutative,

$$[x,\theta y]=x\theta y-\theta xy=(x\theta-\theta x)y\in\mathcal{A}_{k-1}(M,N)R=\mathcal{A}_{k-1}(M,N).$$

Thus, $\theta y\in\mathcal{A}_k(M,N)$ and, by symmetry, the result follows.

(ii) As $\mathfrak{n}^-$ acts ad-nilpotently on any finite dimensional $\mathfrak{g}$-module and hence on $\mathcal{L}(M,N)$, Hypotheses 5.1(2) implies that $\mathcal{A}(M,N)\supseteq\mathcal{L}(M,N)$. Since the inclusion $\mathcal{A}_k(M,N)\supseteq\mathcal{D}_k(M,N)$ is obvious, we are reduced to proving that $\mathcal{A}_k(M,N)\subseteq\mathcal{D}_k(M,N)$ for each k. By induction and for some m, assume that the result holds for all $k<m$ and fix $\theta\in\mathcal{A}_m(M,N)$.

Filter $R=\bigcup\Lambda_pR$, by taking $\Lambda_0=\mathbb{C}$ and $\Lambda_pR=\chi(\mathfrak{r})^p+\Lambda_{p-1}R$ for $p>0$. For some q, assume that $[\theta,z]\in\mathcal{D}_{m-1}(M,N)$ for all $z\in\Lambda_sR$ and $s<q$. (The case $s=1$ holds automatically by the definition of $\mathcal{A}(M,N)$, so the induction does start.) Let $y\in\Lambda_qR$ and suppose that $y=x_1x_2$ for $x_1\in\chi(\mathfrak{r})$ and $x_2\in\Lambda_{q-1}R$. Then, by part (i) and the induction on q,

$$\begin{aligned}\theta y-y\theta &= \theta x_1x_2-x_1x_2\theta &&= \theta x_1x_2-x_1\theta x_2+x_1\theta x_2-x_1x_2\theta\\ &&&= (\theta x_1-x_1\theta)x_2+x_1(\theta x_2-x_2\theta)\\ &&&\in \mathcal{A}_{m-1}(M,N)x_2+x_1\mathcal{A}_{m-1}(M,N)\ \subseteq\ \mathcal{A}_{m-1}(M,N).\end{aligned}$$

Hence $[\theta,y]\in\mathcal{A}_{m-1}(M,N)$. By linearity and our inductive hypotheses, $[\theta,z]\in\mathcal{A}_{m-1}(M,N)=\mathcal{D}_{m-1}(M,N)$ for all $z\in R$. Hence $\mathcal{A}_m(M,N)\subseteq\mathcal{D}_m(M,N)$.

(iii) Since R is a quasi-simple $U(\mathfrak{g})$-module, [Jan, Lemma 8.14] implies that $\operatorname{ann}_{U(\mathfrak{g})}(M)=\operatorname{ann}_{U(\mathfrak{g})}(R)$ and so $\overline{U}\hookrightarrow\mathcal{L}(M,M)$ by [Jan, §6.8(5)]. By definition, $\mathcal{M}=\mathcal{L}(M,M)$ is a locally finite $\mathfrak{g}$-module under the adjoint action and hence is semi-simple. Thus, we may write $\mathcal{M}=\overline{U}\oplus P$ for some complementary $(\operatorname{ad}\mathfrak{g})$-module P. Assume that $P\neq 0$.

Pick $p\in P\smallsetminus\{0\}$ with $p\in\mathcal{A}_k(M,M)$ for k as small as possible. Then, for $x\in\mathfrak{r}$, we have both $[x,p]\in\mathcal{A}_{k-1}(M,M)$ by the definition of the order filtration (5.4), and $[x,p]\in P$ as P is an $(\operatorname{ad}\mathfrak{r})$-module. Thus by the choice of k we have $[x,p]=0$ for all $x\in\mathfrak{r}$. Therefore, by hypothesis and Part (ii),

$$p\in\mathcal{A}_0(M,M)=\mathcal{D}_0(M)=\mathrm{End}_R(M)=R.$$

However, by Hypothesis 5.1(2), $R\subseteq\overline{U}$ as subrings of $\mathcal{D}(R)$. Hence $p\in\overline{U}$, giving the required contradiction. $\square$

Definition 5.6. If A, B are prime Goldie rings with the same simple artinian ring of fractions Q, then A and B are *equivalent orders* if $a_1 B a_2 \subseteq A$ and $b_1 A b_2 \subseteq B$ for some regular elements $a_i \in A$ and $b_j \in B$. The ring A is called a *maximal order* if it is not equivalent to any $B \supsetneq A$.

The property of being a maximal order is the appropriate noncommutative analogue of being an integrally closed commutative ring and has a number of useful consequences; see for example [JS] or [MaR].

Theorem 5.7. *Let R and $\mathfrak{g}$ satisfy Hypotheses 5.1. Then*

$$\mathcal{L}(R,R) = \mathcal{L}(N,N) = \overline{U}.$$

In particular, $\mathcal{L}(R,R)$ is a maximal order.

Remark 5.8. In the sense of, say, [Jos2] this solves the Kostant problem for the ring $\overline{U}$.

Proof. Both $M = R$ and $M = N$ satisfy the hypotheses of Proposition 5.5 and so the displayed equation follows. The final assertion then follows from [JS, Theorem 2.9]. □

Here is a simple example that illustrates the restrictions of Hypothesis 5.1(3), even if one assumes that $R/N(R)$ has pleasant properties.

Example. Take $A = \mathbb{C}[x, y, xz, yz, z^2, z^3] \subset \mathbb{C}[x, y, z] = C$; thus $C = A + \mathbb{C}z$. Set $I = A \cap z^2 C$ and $R = A/I$. Clearly the nil radical $N(R)$ equals $(zC \cap A)/I = ((xz, yz) + I)/I$ and so $R/N(R) \cong \mathbb{C}[x, y]$. However, $N(R)/I = (xz, yz) \cong (x, y)$ as modules over $k[x, y] \cong R/N(R)$.

Now let $M = (x, y, zx, zy, z^2, z^3)/I$ be the augmentation ideal of R. The maximal ring of fractions $\operatorname{Quot}(R)$ clearly contains $z = (zx)x^{-1}$. However $z \notin R$ yet, by the computations of the last paragraph, $zM \subseteq R$. Indeed $zM \subseteq M$ and so $\operatorname{End}_R(M) \supsetneq R$.

6. Modules of $\mathfrak{so}(n+2,\mathbb{C})$-finite vectors

As we noted in Theorem 4.6, $\mathcal{D}(R_1) = U(\mathfrak{so}(n+2))/J$ for the Joseph ideal J. We would like to pass from $\mathcal{D}(R_1)$ to $\mathcal{D}(R_r)$ for any r to obtain the analogous result for $\mathcal{D}(R_r)$. There are two potential ways of creating such a passage; through differential operators $\mathcal{D}(R_r, R_1)$ and through the $\mathfrak{g}$-finite vectors $\mathcal{L}(R_r, R_1)$. The first step, however, is to prove that the latter is non-zero. This we accomplish in this section and then we use it in the next section to prove the major part of Theorem 1.3 from the introduction.

There are a number of different rings involved in this discussion and so we need to refine the earlier notation. As in Section 3, set $A = \mathbb{C}[X_1, \dots, X_n] \ni \mathtt{F} = \sum_{i=1}^n X_i^2$ with $R_r = A/(\mathtt{F}^r)$ for any $r \geq 1$. Recall the elements $\Delta_1, \mathtt{E}_1, D_{k\ell} \in \mathcal{D}(A)$ from (3.5) as well as the elements

$$P_{kj} = P_j(\tfrac{n}{2} - k) = X_j \Delta_1 - \big(\mathtt{E}_1 + (\tfrac{n}{2} - k)\big)\partial_j \in \mathcal{D}(A) \quad \text{for } 1 \leq j \leq n \text{ and } k \geq 1,$$

defined by (3.7). By Corollary 3.14, the images of $\{P_{rj}, D_{ij}, (\mathtt{E}_1 + \frac{n}{2} - r), X_i : 1 \leq i, j \leq n\}$ span a copy $\mathfrak{g}_r$ of $\mathfrak{g} = \mathfrak{so}(n+2,\mathbb{C})$ inside $\mathcal{D}(R_r)$. Moreover, for $r, s \geq 1$, Proposition 3.8 proves that the map $P_{rj} \mapsto P_{sj}$ and $\mathtt{E}_1 + \frac{n}{2} - r \mapsto \mathtt{E}_1 + \frac{n}{2} - s$, with $D_{ij} \mapsto D_{ij}$ and $X_i \mapsto X_i$ induces a Lie algebra isomorphism $\mathfrak{g}_r \to \mathfrak{g}_s$.

We may therefore write $\{\widetilde{P}_j, D_{ij}, \widetilde{\mathtt{E}}_1, X_i : 1 \leq i,j \leq n\}$ for the corresponding generators of the abstract copy $\mathfrak{g}$ of $\mathfrak{so}(n+2,\mathbb{C})$ and rephrase Corollary 3.14 as saying that there is a Lie algebra isomorphism $\psi_r : \mathfrak{g} \to \mathfrak{g}_r$ given by $\widetilde{P}_j \mapsto P_{rj}$ and $\widetilde{\mathtt{E}}_1 \mapsto \mathtt{E}_1 + \frac{n}{2} - r$, etc.

Lemma 6.1. *Fix $r \geq 2$ and recall the notation $\mathcal{D}(M,N)$ from (3.1). Then, regarded as differential operators acting on A, the elements Δ_1, ∂_j and 1 induce differential operators in $\mathcal{D}(R_r,\, R_{r-1})$.*

Proof. Suppose first that $\phi \in \mathcal{D}(A)$ satisfies $\phi(\mathtt{F}^r A) \subseteq \mathtt{F}^{r-1}A$ and so induces a function $\widetilde{\phi} : R_r \to R_{r-1}$. Then we claim that $\widetilde{\phi} \in \mathcal{D}(R_r,\, R_{r-1})$. To see this, assume that $\phi \in \mathcal{D}_t(A)$ and that the analogous statement is true for all $\phi' \in \mathcal{D}_{t-1}(A)$. Then, for $a \in A$, the function $\phi' = [\phi, a]$ also satisfies $\phi'(\mathtt{F}^r A) \subseteq \mathtt{F}^{r-1}A$, but now $\phi' \in \mathcal{D}_{t-1}(A)$. Hence $\phi' \in \mathcal{D}_{t-1}(R_r,\, R_{r-1})$. Thus, $\phi \in \mathcal{D}_t(R_r,\, R_{r-1})$, proving the claim.

In order to prove the lemma it therefore suffices to show that each of our elements ϕ satisfies $\phi(\mathtt{F}^r A) \subseteq \mathtt{F}^{r-1}A$. This is obvious for 1 and ∂_i, while for Δ_1 it follows from (3.13). □

Notation 6.2. For a fixed value of r, the images of Δ_1, ∂_j and 1 in $\mathcal{D}(R_r,\, R_{r-1})$ will be written, respectively, $\overline{\Delta}_1$, $\overline{\partial}_j$ and $\overline{1}$. For concreteness, we note that the *adjoint action* $\widetilde{\psi}$ of $\mathfrak{g} = \mathfrak{so}(n+2,\mathbb{C})$ on $\mathcal{D} = \mathcal{D}(R_r,\, R_{r-1})$ is defined by $\widetilde{\psi}(x)(\theta) = \psi_{r-1}(x) \circ \theta - \theta \circ \psi_r(x)$ for $x \in \mathfrak{g}$ and $\theta \in \mathcal{D}$.

Proposition 6.3. *Fix $r \geq 2$ and let $V = \mathbb{C}\overline{1} + \sum_i \mathbb{C}\overline{\partial}_i + \mathbb{C}\overline{\Delta}_1 \subset \mathcal{D}(R_r,\, R_{r-1})$. Then, under the adjoint action $\widetilde{\psi}$, V is a $\mathfrak{g}$-submodule of $\mathcal{D}(R_r,\, R_{r-1})$. In particular, $V \subseteq \mathcal{L}(R_r,\, R_{r-1})$.*

Proof. By the definition of $\mathcal{L}(R_r,\, R_{r-1})$ it suffices to prove that V is a $\mathfrak{g}$-module. This requires some explicit computations, for which the following three formulæ in $\mathfrak{g}$ will prove useful and will be used frequently without comment: for all $1 \leq i,j \leq n$ we have $[\Delta_1,\, \partial_j] = [\Delta_1,\, D_{ij}] = 0$, while

$$[\Delta_1,\, X_j] \;=\; [\tfrac{1}{2}\partial_j^2,\, X_j] \;=\; \partial_j \quad\text{and}\quad [\mathtt{E}_1,\, \Delta_1] \;=\; \tfrac{1}{2}\textstyle\sum_j [\mathtt{E}_1,\, \partial_j^2] \;=\; -2\Delta_1.$$

First, computing in $\mathcal{D}(R_r,\, R_{r-1})$ we have

$$\begin{aligned}
\widetilde{\psi}(\widetilde{P}_j)(\overline{\Delta}_1) &= P_j(\tfrac{n}{2} - r + 1)\overline{\Delta}_1 - \overline{\Delta}_1 P_j(\tfrac{n}{2} - r) \\
&= \Big(X_j\Delta_1\overline{\Delta}_1 - \overline{\Delta}_1 X_j \Delta_1\Big) + \Big(-\mathtt{E}_1\partial_j\overline{\Delta}_1 + \overline{\Delta}_1\mathtt{E}_1\partial_j\Big) \\
&\qquad\qquad + \Big(-(\tfrac{n}{2} - r + 1)\partial_j\overline{\Delta}_1 + (\tfrac{n}{2} - r)\overline{\Delta}_1\partial_j\Big) \\
&= -\partial_j\overline{\Delta}_1 + 2\partial_j\overline{\Delta}_1 - \partial_j\overline{\Delta}_1 \;=\; 0.
\end{aligned}$$

Next,

$$\widetilde{\psi}(\widetilde{\mathtt{E}}_1)(\overline{\Delta}_1) \;=\; (\mathtt{E}_1 + \tfrac{n}{2} - r + 1)\overline{\Delta}_1 - \overline{\Delta}_1(\mathtt{E}_1 + \tfrac{n}{2} - r) \;=\; -\overline{\Delta}_1 + 2\overline{\Delta}_1 \;=\; -\overline{\Delta}_1.$$

Similarly, $\widetilde{\psi}(D_{ij})(\overline{\Delta}_1) = [D_{ij}, \overline{\Delta}_1] = 0$ and $\widetilde{\psi}(X_j)(\overline{\Delta}_1) = [D_{ij}, \overline{\Delta}_1] = 0$. Therefore, $\widetilde{\psi}(\mathfrak{g})(\overline{\Delta}_1) \subseteq V$. Next, we compute that

$$\begin{aligned}
\widetilde{\psi}(\widetilde{P}_i)(\overline{\partial}_i) &= P_i(\tfrac{n}{2} - r + 1)\overline{\partial}_i - \overline{\partial}_i P_i(\tfrac{n}{2} - r) \\
&= \Big(X_i\Delta_1\overline{\partial}_i - \overline{\partial}_i X_i\Delta_1\Big) + \Big(-\mathtt{E}_1\partial_i\overline{\partial}_i + \overline{\partial}_i\mathtt{E}_1\partial_i\Big) + \Big(-(\tfrac{n}{2} - r + 1)\partial_i\overline{\partial}_i + (\tfrac{n}{2} - r)\overline{\partial}_i\partial_i\Big) \\
&= -\overline{\Delta}_1 - \partial_i\overline{\partial}_i + \overline{\partial}_i\partial_i \;=\; -\overline{\Delta}_1.
\end{aligned}$$

If $i \neq j$ we obtain

$$\widetilde{\psi}(\widetilde{P}_j)(\overline{\partial}_i) = \Big(X_j\Delta_1\overline{\partial}_i - \overline{\partial}_i X_j\Delta_1\Big) + \Big(-\mathtt{E}_1\partial_j\overline{\partial}_i + \overline{\partial}_i\mathtt{E}_1\partial_j\Big) + \Big(-(\tfrac{n}{2}-r+1)\partial_j\overline{\partial}_i + (\tfrac{n}{2}-r)\overline{\partial}_i\partial_j\Big)$$
$$= -\partial_j\overline{\partial}_i + \overline{\partial}_i\partial_j \;\; = 0.$$

On the other hand, $\widetilde{\psi}(\widetilde{\mathtt{E}}_1)(\overline{\partial}_i) \;=\; (\mathtt{E}_1 + \frac{n}{2} - r + 1)\overline{\partial}_i - \overline{\partial}_i(\mathtt{E}_1 + \frac{n}{2} - r) \;=\; \overline{\partial}_i - \overline{\partial}_i \;= 0$ while

$$\widetilde{\psi}(D_{ab})\overline{\partial}_i = (X_a\partial_b - X_b\partial_a)\overline{\partial}_i - \overline{\partial}_i(X_a\partial_b - X_b\partial_a) \;=\; \sum_u \alpha_u\overline{\partial}_u,$$

for any $1 \leq a, b, i \leq n$, and the appropriate scalars $\alpha_u \in \mathbb{C}$. Since $\widetilde{\psi}(X_k)\overline{\partial}_i = [X_k, \overline{\partial}_i] \;= -\delta_{k,i}\overline{1}$, it follows that $\widetilde{\psi}(\mathfrak{g})(\overline{\partial}_i) \subseteq V$. The easy fact that $\widetilde{\psi}(\mathfrak{g})(\overline{1}) \subseteq V$ is left to the reader and completes the proof. □

Corollary 6.4. *Fix $r > s \geq 1$. Then the following hold.*

(1) *Both the projection map $\pi_{rs} : R_r \to R_s$ and the operator Δ_1^{r-s} belong to $\mathcal{L}(R_r,\, R_s)$.*

(2) *Set $Z_s = \mathrm{Soc}_{U(\mathfrak{g})} R_s$. Then $\mathcal{L}(Z_r, Z_s) \neq 0 \neq \mathcal{L}(Z_s, Z_r)$.*

Proof. (1) If $\theta \in \mathcal{L}(A, B)$ and $\phi \in \mathcal{L}(B, C)$ for $\mathfrak{g}$-modules A, B, C then $\phi\theta \in \mathcal{L}(A, C)$ (see[Jan, 6.8(3)]). Thus the result follows from Proposition 6.3 and induction on $r - s$.

(2) By Theorem 4.13 each R_s/Z_s is finite dimensional (or zero), say with $\mathfrak{a}_s = \mathrm{ann}_{U(\mathfrak{g})}(R_s/Z_s)$. Since Z_r is infinite dimensional, the projection map $\pi_{rs} \in \mathcal{L}(R_r,\, R_s)$ has an infinite dimensional image, as does its restriction $\pi'_{rs} : Z_r \to R_s$. Hence $\mathfrak{a}_s\pi'_{rs} \neq 0$ and so, for some $a \in \mathfrak{a}_s$, one has $0 \neq a \circ \pi'_{rs} \in \mathcal{L}(Z_r, Z_s)$. Thus $\mathcal{L}(Z_r, Z_s) \neq 0$. By [Jan, §6.9(5)], this also implies that $\mathcal{L}(Z_s,\, Z_r) \neq 0$. □

7. Differential operators and primitive factor rings of $U(\mathfrak{so}(n+2, \mathbb{C}))$

As usual, we write $R_r = A/(\mathtt{F}^r)$ for $A = \mathbb{C}[X_1, \ldots, X_n]$ and some $r \geq 1$. The aim of this section is to combine the earlier results to prove that $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r) = U(\mathfrak{g})/J_r$, where $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$ and $J_r = \mathrm{ann}_{U(\mathfrak{g})}(R_r)$. In particular, this proves the major part of Theorem 1.3 from the introduction.

We begin with some elementary results, the first of which is a minor generalisation of [JS, Corollary 2.10].

Lemma 7.1. *If M is a simple $U(\mathfrak{g})$-module with $I = \mathrm{ann}_{U(\mathfrak{g})}(M)$, then $\mathcal{L}(M, M)$ is the unique maximal object among orders containing and equivalent to $U = U(\mathfrak{g})/I$.*

Proof. By [JS, Corollary 2.10], $\mathcal{L}(M, M)$ is a maximal order and it is noetherian since it is finitely generated as both a left and a right U-module. So it remains to prove uniqueness. Suppose that $U \subseteq T$ for some other order T equivalent to U. Thus, $aTb \subseteq U$ for some regular elements $a, b \in U$. Let $T' = UaT + U$; this is an overring of U with $T'b \subseteq U$. So, since b is regular, T' is certainly finitely generated as a left U-module. Therefore, by [JS, Theorem 2.9], $T' \subseteq \mathcal{L}(M, M)$. In particular, T' is also finitely generated as a right U-module. Since $aT \subseteq T'$ it follows that T is also a finitely generated right U-module. Hence [JS, Theorem 2.9] implies that $T \subseteq \mathcal{L}(M, M)$. □

Lemma 7.2. *Let $a, b, c \in \mathbb{N}^*$ and set $Z_a = \mathrm{Soc}_{U(\mathfrak{g})}(R_a)$.*

(i) *$\mathcal{L}(R_a, R_a) = \mathcal{L}(Z_a, Z_a) = U(\mathfrak{g})/J_a$ is a primitive ring.*

(ii) *$\mathcal{L}(R_a, R_b)$ and $\mathcal{L}(Z_a, Z_b)$ are non-zero and torsion-free both as left $U(\mathfrak{g})/J_b$-modules and as right $U(\mathfrak{g})/J_a$-modules.*

(iii) *Under composition of operators, both $\mathcal{L}(R_b, R_a)\mathcal{L}(R_a, R_b)$ and $\mathcal{L}(Z_b, Z_a)\mathcal{L}(Z_a, Z_b)$ are non-zero ideals of $\mathcal{L}(R_a)$.*

Proof. (i) By Theorem 4.13, Hypotheses 5.1 is satisfied and so the result follows from Theorem 5.7.

(ii) By [Jan, Lemma 8.14], $\mathrm{ann}_{U(\mathfrak{g})}(R_b) = \mathrm{ann}_{U(\mathfrak{g})}(Z_b) = J_b$ and so R_b is *ann-homogeneous* in the sense of [JS, §2.7]. Thus, by [JS, Lemma 2.8], $\mathcal{L}(R_a, R_b)$ is *GK-homogeneous* in the sense that any non-zero, left or right $\mathfrak{g}$-submodule N of $\mathcal{L}(R_a, R_b)$ has $\mathrm{GKdim}(N) = \mathrm{GKdim}(\mathcal{L}(R_a, R_b))$. Since each $U(\mathfrak{g})/J_c$ is a prime ring by (i), this is equivalent to $\mathcal{L}(R_a, R_b)$ being torsion-free on both sides. Exactly the same argument works for $\mathcal{L}(Z_a, Z_b)$.

It therefore remains to prove that these modules are non-zero. For $\mathcal{L}(Z_a, Z_b)$ this is Corollary 6.4. In particular, $\mathcal{L}(Z_a, Z_b)$ is infinite dimensional, and hence $\mathcal{L}(Z_a, Z_b)\mathfrak{a}_a \neq 0$, where $\mathfrak{a}_a$ is the annihilator of the finite dimensional $\overline{U}_a$-module R_a/Z_a. Thus $0 \neq \mathcal{L}(Z_a, Z_b)\mathfrak{a}_a \subseteq \mathcal{L}(R_a, Z_b) \subseteq \mathcal{L}(R_a, R_b)$.

(iii) This is automatic from (ii). $\square$

We also have analogous elementary results about differential operators, although as we do not (yet) know that these rings are noetherian the proof is a little more involved.

Lemma 7.3. *Set $\{1, r\} = \{a, b\}$. Then:*

(i) *$\mathcal{D}(R_a)$ is a prime Goldie ring;*

(ii) *$\mathcal{D}(R_a)$ has* Goldie rank *a; indeed the simple artinian ring of fractions* $\mathrm{Quot}(\mathcal{D}(R_a))$ *is even isomorphic to the $a \times a$ matrix ring $M_a(\mathrm{Quot}(\mathcal{D}(R_1)))$;*

(iii) *$\mathcal{D}(R_a, R_b)$ is non-zero and torsion-free both as a left $\mathcal{D}(R_b)$-module and as a right $\mathcal{D}(R_a)$-module.*

Proof. (i) Clearly R_a has a local artinian ring of fractions $(R_a)_{(\mathtt{F})}$. Now apply [Mus, Corollary 2.6].

(ii) This follows from [Mus, Lemma 2.4 and (2.5)].

(iii) The projection map $\pi : R_r \to R_1$ is an A-module map and so belongs to $\mathcal{D}(R_r, R_1)$. Similarly, the A-module isomorphism $\phi : R_1 \to \mathtt{F}^{r-1}R_r$ belongs to $\mathcal{D}(R_1, R_r)$. Thus, $\mathcal{D}(R_r, R_1) \neq 0 \neq \mathcal{D}(R_1, R_r)$.

As right $\mathcal{D}(R_r)$-modules, $\mathcal{D}(R_r, R_1) \cong \mathcal{D}(R_r, \mathtt{F}^{r-1}R_r) \hookrightarrow \mathcal{D}(R_r, R_r)$. Thus by (i), it is a torsion-free right $\mathcal{D}(R_r)$-module. For the left action, consider $0 \neq \phi \in \mathcal{D}(R_r, R_1)$ and pick s maximal such that $\phi(\mathtt{F}^s R_r) \neq 0$. If $\bar{r} \in R_1$, write $\bar{r} = \pi(r)$ for some $r \in R_r$ and define $\widetilde{\phi}(\bar{r}) = \phi(\mathtt{F}^s r)$. Since $\phi(\mathtt{F}^{s+1}R_r) = 0$, this is a well defined morphism and hence a differential operator; thus $\widetilde{\phi} \in \mathcal{D}(R_1, R_1)$. Moreover, $\widetilde{\phi} \neq 0$ by the choice of s, and so $\widetilde{\phi}$ is not torsion as an element of the left $\mathcal{D}(R_1)$-module $\mathcal{D}(R_1)$. Consequently, ϕ is not torsion under the left $\mathcal{D}(R_1)$ action.

Conversely, if $0 \neq \theta \in \mathcal{D}(R_1, R_r)$, then $0 \neq \theta \circ \pi \in \mathcal{D}(R_r, R_r)$. Thus neither $\theta \circ \pi$ nor θ is torsion under the left $\mathcal{D}(R_r)$ action. Hence $\mathcal{D}(R_1, R_r)$ is torsion-free as a left $\mathcal{D}(R_r)$-module. Finally, let $0 \neq \phi \in \mathcal{D}(R_1, R_r)$ and choose s minimal such that $\phi(R_1) \subseteq \mathtt{F}^s R_r$. Thus, $\phi \in \mathcal{D}(R_1, \mathtt{F}^s R_r)$. Moreover, if $\rho : \mathtt{F}^s R_r \to \mathtt{F}^s R_r / \mathtt{F}^{s+1} R_r \cong R_1$, then $0 \neq \rho \circ \phi \in \mathcal{D}(R_1, R_1)$. As in the last paragraph, this ensures that ϕ is not torsion under the right $\mathcal{D}(R_1)$ action. Hence $\mathcal{D}(R_1, R_r)$ is torsion-free on the right. $\square$

By Lemma 4.3, we know that $A = \mathbb{C}[X_1, \dots, X_n] \hookrightarrow U(\mathfrak{n}^-) \subset U(\mathfrak{g})$. As before, set $Z_a = \mathrm{Soc}_{U(\mathfrak{g})}(R_a)$ for $a \in \mathbb{N}^*$. Then, by Proposition 5.5(ii) and Theorem 4.13 we have

$$\mathcal{L}(R_a, R_b) \subseteq \mathcal{D}(R_a, R_b) \qquad \text{and} \qquad \mathcal{L}(Z_a, Z_b) \subseteq \mathcal{D}(Z_a, Z_b) \qquad \text{for } \{a, b\} = \{1, r\}. \tag{7.4}$$

Proposition 7.5. *The rings $\mathcal{D}(R_r)$ and $\mathcal{L}(R_r, R_r)$ are equivalent orders, with $\mathcal{D}(R_r) \supseteq \mathcal{L}(R_r, R_r)$.*

Proof. Consider

$$F = \mathcal{D}(R_1, R_r)\mathcal{L}(R_r, R_1) \quad \text{and} \quad G = \mathcal{L}(R_1, R_r)\mathcal{D}(R_r, R_1),$$

where multiplication is composition of functions. We claim that $F \neq 0 \neq G$. To see this it suffices, by (7.4), to prove that $\mathcal{L}(R_a, R_b) \cdot \mathcal{L}(R_a, R_b) \neq 0$ (where $\{a, b\} = \{1, r\}$). This follows from Lemma 7.2(ii).

Recall from Theorem 4.6 that $\mathcal{D}(R_1) = \mathcal{L}(R_1, R_1)$. Thus, repeating the argument of the last paragraph shows that

$$\begin{aligned} 0 \neq GF = \;& \mathcal{L}(R_1, R_r)\mathcal{D}(R_r, R_1) \cdot \mathcal{D}(R_1, R_r)\mathcal{L}(R_r, R_1) \\ \subseteq \;& \mathcal{L}(R_1, R_r)\mathcal{D}(R_1)\mathcal{L}(R_r, R_1) \;=\; \mathcal{L}(R_1, R_r)\mathcal{L}(R_1, R_1)\mathcal{L}(R_r, R_1) \\ = \;& \mathcal{L}(R_1, R_r)\mathcal{L}(R_r, R_1) \;\subseteq\; \mathcal{L}(R_r, R_r). \end{aligned}$$

Both F and G contain $\mathcal{L}(R_1, R_r)\mathcal{L}(R_r, R_1)$ which, by Lemma 7.2(i,iii), contains a regular element $a \in \mathcal{L}(R_r, R_r)$. Thus from the last display $a\mathcal{D}(R_r)a \subseteq G\mathcal{D}(R_r)F = GF \subseteq \mathcal{L}(R_r, R_r)$, as required. ☐

Finally, putting everything together we get:

Theorem 7.6. *One has $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r) = \mathcal{L}(Z_r, Z_r) = U(\mathfrak{so}(n+2, \mathbb{C}))/J_r$. Moreover this ring is a maximal order in its simple artinian ring of fractions and has Goldie rank r.*

Proof. By Theorem 4.13, Hypotheses 5.1 are satisfied and so $U(\mathfrak{g})/J_r = \mathcal{L}(R_r, R_r) = \mathcal{L}(Z_r, Z_r)$ is a maximal order by Theorem 5.7. Thus, by Proposition 7.5, $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r)$. The fact that $\mathcal{D}_r$ has Goldie rank r follows from Lemma 7.3. ☐

Remark 7.7. There are other ways to prove aspects of this theorem. For example, recall that the associated variety of the primitive ideal J_r is equal to $\overline{\mathbf{O}}_{\min}$, cf. Corollary 4.17. Assume that $n \neq 4$ and that $R_r \cong L(\lambda)$ is an irreducible $\mathfrak{g}$-module (so we are in Case (1) of Theorem 4.13). Then the minimal orbit $\mathbf{O}_{\min}$ is rigid, in the sense that it is not induced from any proper parabolic subalgebra; this implies that the $\mathfrak{g}$-module R_r is rigid in the sense of [Jos4, 1.2]. Then, [Jos4, 5.8] can be applied in this situation to show that $\mathcal{L}(R_r, R_r) = \mathcal{D}(R_r)$ and one deduces from Lemma 7.2 that $U(\mathfrak{g})/J_r = \mathcal{D}(R_r)$.

8. Higher symmetries of powers of the Laplacian

Recall that the starting point of this paper was to examine the symmetries of powers of the Laplacian $\Delta_1 = \frac{1}{2}\sum \partial_{X_i}^2$, as defined below, and thereby to extend the work of Eastwood and others [Eas, EL, Mic] on this concept. In this section we translate Theorem 7.6 into a result about those symmetries. In particular, this proves the second half of Theorem 1.3 from the introduction, by identifying the ring of symmetries of Δ_1^r with a factor ring of $U(\mathfrak{so}(n+2, \mathbb{C}))$.

We briefly retain the notation of Section 3 for a general field $\mathbb{K}$ of characteristic zero; in particular setting $A = \mathbb{K}[X_1, \dots, X_n] \ni \mathtt{F} = \sum X_i^2$. We begin with the definition of higher symmetries as described, for example, in [BS, (1.2)], [ShS, (4.3)] or [Eas, Definition 1].

Definition 8.1. Fix an operator $P \in \mathcal{D}(A)$. An operator $Q \in \mathcal{D}(A)$ is a *symmetry* of P if there exists $Q' \in \mathcal{D}(A)$ such that $PQ = Q'P$. Equivalently, in the notation of (3.2), the symmetries of P equals the idealizer $\mathbb{I}(\mathcal{D}(A)P)$. The elements $Q \in \mathcal{D}(A)P$ are trivially symmetries of P and so one usually factors them out and defines *the algebra of symmetries of* P to be the factor algebra

$$\mathscr{S}(P) = \mathbb{I}(\mathcal{D}(A)P)/\mathcal{D}(A)P.$$

The definition of $\mathscr{S}(\Delta_1^r)$ is of course similar in style to that of the ring of differential operators $\mathcal{D}(A/\mathtt{F}^r) \cong \mathbb{I}(\mathtt{F}^r\mathcal{D}(A))/\mathtt{F}^r\mathcal{D}(A)$ from (3.3); except that one works with constant coefficient differential operators rather than polynomials and left rather than right ideals. However, as we show next, one can easily pass from the one to the other by taking a Fourier transform.

We now return to the field $\mathbb{K} = \mathbb{C}$. In our applications it will be convenient to use the Chevalley system from Lemma 4.3 and so we will use the Fourier transform $\mathcal{F}$ in $\mathcal{D}(A)$ with respect to the variables $U_{\pm j}$ from (4.1). Thus we write $A = \mathbb{C}[U_{\pm 2}, \dots, U_{\pm \ell}, U_0]$ and define

$$\mathcal{F}(U_j) = \partial_{U_j}, \quad \mathcal{F}(\partial_{U_j}) = U_j, \quad j \in \{0, \pm 2, \dots, \pm \ell\}. \tag{8.2}$$

In terms of the variables X_j it is routine to check that

$$\mathcal{F}(X_j) = \begin{cases} \frac{1}{2}\partial_{X_j} & \text{if } 1 \leq j \leq \ell' \text{ or if } j = n \text{ when } n \text{ is odd;} \\ -\frac{1}{2}\partial_{X_j} & \text{if } \ell' + 1 \leq j \leq 2\ell'. \end{cases} \tag{8.3}$$

Further routine properties of $\mathcal{F}$ are given by the following lemma. For this we recall that $d = \frac{n}{2} - r$ for some $r \in \mathbb{N}^*$ and that $P_j = P_j(d) = X_j\Delta_1 - (\mathtt{E}_1 + d)\partial_{X_j}$.

Lemma 8.4. *The map $\mathcal{F}$ is an involutive anti-automorphism of $\mathcal{D}(A)$. Moreover,*

$$\mathcal{F}(\mathtt{F}) = \frac{1}{2}\Delta_1, \; \mathcal{F}(\Delta_1) = 2\mathtt{F}, \; \mathcal{F}(\mathtt{E}_1) = \mathtt{E}_1, \; \mathcal{F}(D_{jk}) = \pm D_{jk}, \; \mathcal{F}(P_j) = \pm\big(\mathtt{F}\partial_{X_j} - 2X_j(\mathtt{E}_1 + d)\big).$$

Proof. Use the formulæ $\partial_{X_j} = \frac{1}{2}\big(\partial_{U_{j+1}} + \partial_{U_{-(j+1)}}\big)$ and $\partial_{X_{j+\ell'}} = \frac{1}{2i}\big(\partial_{U_{j+1}} - \partial_{U_{-(j+1)}}\big)$, for $1 \leq j \leq \ell'$, while $\partial_{X_n} = \sqrt{2}\partial_{U_0}$. $\square$

Proposition 8.5. *Let $r \geq 1$ and $R_r = A/\mathtt{F}^r A$. Then the Fourier transform $\mathcal{F}$ induces an anti-isomorphism $\mathcal{F} = \mathcal{F}_r : \mathcal{D}(R_r) \to \mathscr{S}(\Delta_1^r)$.*

Proof. Set $\mathcal{D} = \mathcal{D}(A)$. By Lemma 8.4, $\mathcal{F}(\mathtt{F}^r) = 2^{-r}\Delta_1^r$. Thus, in the notation of Definition 8.1, $\Delta_1^r Q = Q'\Delta_1^r$ for some $Q, Q' \in \mathcal{D} \iff \mathcal{F}(Q)\mathtt{F}^r = \mathtt{F}^r\mathcal{F}(Q')$. Equivalently, $Q \in \mathbb{I}_{\mathcal{D}}(\mathcal{D}\Delta_1^r) \iff \mathcal{F}(Q) \in \mathbb{I}_{\mathcal{D}}(\mathtt{F}^r\mathcal{D})$. Hence $\mathcal{F}$ induces an anti-isomorphism $\mathbb{I}_{\mathcal{D}}(\mathcal{D}\Delta_1^r)/\mathcal{D}\Delta_1^r \to \mathbb{I}_{\mathcal{D}}(\mathtt{F}^r\mathcal{D})/\mathtt{F}^r\mathcal{D}$. $\square$

Recall from Theorem 7.6 that we have an isomorphism $\psi_r : U(\mathfrak{g})/J_r \to \mathcal{D}(R_r)$ and hence an anti-isomorphism from $U(\mathfrak{g})/J_r$ to $\mathscr{S}(\Delta_1^r)$. We want to convert this into an automorphism. By [Jan, 5.2 (2)] $\vartheta(J_r) = J_r$ for any Chevalley anti-involution ϑ of $\mathfrak{g}$, and hence for the anti-involution ϑ defined by the Chevalley system $\{y_\alpha\}_\alpha \subset \mathfrak{g}$ described in Lemma 4.3. Notice that $\widetilde{\psi}_d \circ \vartheta = \vartheta_d \circ \widetilde{\psi}_d$ in the notation of Remark 4.5. In particular, using Theorem 7.6, ϑ_d induces an anti-automorphism ϑ_r on $\mathcal{D}(R_r)$ and combining the earlier results of the paper we obtain the main result of this section.

Theorem 8.6. (1) *Fix $r \geq 1$. Then there are algebra isomorphisms:*

$$\mathcal{F} \circ \vartheta_r \circ \psi_r : U(\mathfrak{so}(n+2, \mathbb{C}))/J_r \xrightarrow{\sim} \mathcal{D}(R_r) \xrightarrow{\sim} \mathscr{S}(\Delta_1^r).$$

In particular, the algebra of symmetries $\mathscr{S}(\Delta_1^r)$ is a noetherian maximal order in its simple artinian ring of fractions and has Goldie rank r. Moreover, it is generated by the elements:

$$\mathtt{E}_1 + d; \quad \partial_{X_k},\ 1 \leq k \leq n; \quad D_{j,k},\ 1 \leq j, k \leq n; \quad \mathtt{F}\partial_{X_j} - 2X_j(\mathtt{E}_1 + d),\ 1 \leq j \leq n.$$

(2) *Consider $\mathcal{A} = \mathbb{C}[\partial_{X_1}, \dots, \partial_{X_n}]/(\Delta_1^r)$. If n is even with $r < \frac{n}{2}$ or n is odd, then $\mathcal{A}$ is a simple $\mathscr{S}(\Delta_1^r)$-module. If n is even with $r \geq \frac{n}{2}$, then $\mathcal{A}$ has a unique proper factor module, which is finite dimensional.*

Proof. (1) Since $\mathscr{S}(\Delta_1^r)$ is generated by the Fourier transforms of the generators of $\mathcal{D}(R_r)$, the assertion on generators is simply the translation of Proposition 3.8 and Lemma 8.4. For the remaining assertions, combine Proposition 8.5 and Theorem 7.6.

(2) Under the Fourier transform, this is a direct consequence of (1) and Theorem 4.13. Up to a change of Borel, the weights of these modules are also given by that theorem. □

9. The real case

In this section we show that Theorems 7.6 and 8.6 have natural analogues for differential operators with real coefficients, thereby proving Theorem 1.1 from the introduction. There are in fact a number of different real forms of $\mathfrak{so}(n+2, \mathbb{C})$, each with their own Laplacian, and these are all covered by our results.

To make this precise, we will always write $A_{\mathbb{R}} = \mathbb{R}[X_1, \dots, X_n] \subset A = \mathbb{C}[X_1, \dots, X_n]$ and set $\mathcal{D}(A_{\mathbb{R}}) = \mathbb{R}[X_1, \dots, X_n, \partial_{X_1}, \dots, \partial_{X_n}]$. Throughout the section we fix $p, q \in \mathbb{N}$ with $p + q = n \geq 3$. Then the elements $\mathtt{F} = \sum X_j^2$ and $\Delta_1 = \sum \frac{1}{2}\partial_{X_j}^2$ can be replaced by, respectively,

$$\widetilde{\mathtt{F}} = \widetilde{\mathtt{F}}_p = \sum_{j=1}^{p} X_j^2 - \sum_{j=p+1}^{q} X_j^2 \in A_{\mathbb{R}}$$

and

$$\square = \square_p = \frac{1}{2}\left(\sum_{j=1}^{p} \partial_{X_j}^2 - \sum_{j=p+1}^{n} \partial_{X_j}^2\right) \in \mathcal{D}(A_{\mathbb{R}}).$$

(Of course, $\mathtt{F} = \widetilde{\mathtt{F}}_n$ and $\Delta_1 = \square_n$.) The signature of the quadratic form $\widetilde{\mathtt{F}}$ will be denoted by (p, q) and the operator $\square$ is called the d'Alembertian (or Laplacian) on the quadratic space $\mathbb{R}^{p,q} = (\mathbb{R}^n, \widetilde{\mathtt{F}})$.

For fixed $r \geq 1$, set $S = S_{p,r} = A_{\mathbb{R}}/\widetilde{\mathtt{F}}^r A_{\mathbb{R}}$ and $\widetilde{S} = A/\widetilde{\mathtt{F}}^r A = S \otimes_{\mathbb{R}} \mathbb{C}$. Define an automorphism $\phi = \phi_{p,q}$ of $\mathcal{D}(A)$, with $\phi(A) = A$, by setting:

$$\phi(X_j) = \begin{cases} X_j & \text{if } 1 \leq j \leq p, \\ iX_j & \text{if } p+1 \leq j \leq n; \end{cases} \quad \text{and} \quad \phi(\partial_{X_j}) = \begin{cases} \partial_{X_j} & \text{if } 1 \leq j \leq p, \\ -i\partial_{X_j} & \text{if } p+1 \leq j \leq n. \end{cases} \tag{9.1}$$

Note that $\phi(\mathtt{F}) = \widetilde{\mathtt{F}}$ and so $\mathcal{D}(A/\widetilde{\mathtt{F}}^r A) \cong \mathcal{D}(A/\mathtt{F}^r A)$.

Fix $d = d_n = \frac{n}{2} - r$ and recall from Notation 3.9 that we have a copy $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$ of $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$. We now want to describe a real analogue $\mathfrak{s} \subset \mathcal{D}(A_{\mathbb{R}})$ of $\widetilde{\mathfrak{g}}$. By analogy with (3.5) and (3.7), set:

(a) $\widetilde{D}_{kl} = \widetilde{D}_{lk} = X_k\partial_{X_l} + X_l\partial_{X_k}$ if $1 \leq k \leq p < l \leq n$, while $\widetilde{D}_{kk} = 0$;

(b) $\widetilde{P}_j = \widetilde{P}_j(d) = \begin{cases} X_j\,\Box - (\mathtt{E}_1 + d)\partial_{X_j} & \text{if } 1 \leq j \leq p \\ X_j\,\Box + (\mathtt{E}_1 + d)\partial_{X_j} & \text{if } p+1 \leq j \leq n; \end{cases}$.

Observe that:

$$\begin{aligned} &X_j = \phi(X_j), \ \text{ for } j \leq p; && X_j = -i\phi(X_j), \ \text{ for } j \geq p+1; \\ &\widetilde{P}_j = \phi(P_j), \ \text{ for } j \leq p; && \widetilde{P}_j = -i\phi(P_j), \ \text{ for } j \geq p+1; \\ &\widetilde{D}_{kl} = i\phi(D_{kl}) = -i\phi(D_{lk}), && \text{for } 1 \leq k \leq p < l \leq n. \end{aligned} \tag{9.2}$$

The next result is standard.

Lemma 9.3. *Let $\mathfrak{s} = \mathfrak{s}(p,r) \subset \mathcal{D}(A_{\mathbb{R}})$ be the real subspace spanned by the elements*

$$\{\mathtt{E}_1 + d;\ X_k;\ \widetilde{P}_j,\ 1 \leq j,k \leq n;\ D_{k,l},\ 1 \leq k,l \leq p,\ \text{ or } p+1 \leq k,l \leq n;\ \widetilde{D}_{kl},\ 1 \leq k \leq p < l \leq n\}.$$

(1) *Then $\mathfrak{s}$ is a Lie subalgebra of $(\mathcal{D}(A_{\mathbb{R}}), [\ ,\ ])$ with $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C} \cong \widetilde{\mathfrak{g}} \cong \mathfrak{so}(n+2,\mathbb{C})$.*

(2) *Moreover, $\mathfrak{s} \subset \mathbb{I}_{\mathcal{D}(A_{\mathbb{R}})}\big(\widetilde{\mathtt{F}}_p^r\mathcal{D}(A_{\mathbb{R}})\big)$ and hence induces a ring homomorphism $\widetilde{\varphi}_{\mathbb{R}} : U(\mathfrak{s}) \longrightarrow \mathcal{D}(S_{p,r})$.*

Proof. (1) It is a routine exercise to see that $\mathfrak{s}$ is a Lie algebra; indeed, one can use (9.2) to reduce this claim to the formulæ in Lemma 3.6 and Proposition 3.8. By construction, the generators of $\phi(\mathfrak{s})$ also span $\widetilde{\mathfrak{g}}$; whence $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C} \cong \widetilde{\mathfrak{g}}$.

(2) Mimic the proof of Lemma 3.10 and Corollary 3.14. □

Proposition 9.4. *Let $d = d_n = \frac{n}{2} - r$ and $\mathfrak{s} = \mathfrak{s}(p,r)$. Then $\widetilde{\varphi}_{\mathbb{R}}$ yields an isomorphism*

$$\widetilde{\psi}_{\mathbb{R}} : U(\mathfrak{s})/(J_r \cap U(\mathfrak{s})) \xrightarrow{\ \sim\ } \mathcal{D}(S_{p,r}).$$

Proof. Recall that $U(\mathfrak{s}) \otimes_{\mathbb{R}} \mathbb{C} \cong U(\widetilde{\mathfrak{g}})$, by Lemma 9.3. Also, by construction, $\phi(\mathtt{F}^r) = \widetilde{\mathtt{F}}^r$ and hence induces an isomorphism $\phi : \mathcal{D}(R_r) \to \mathcal{D}(\widetilde{S})$. Thus one has a natural map $\widetilde{\varphi} = \phi \circ \psi_r$ from $U(\mathfrak{g})$ to $\mathcal{D}(\widetilde{S})$ which, by Theorem 7.6, is surjective.

But, if $\widetilde{\varphi}_{\mathbb{R}}$ is the map from Lemma 9.3, then $\widetilde{\varphi}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \widetilde{\varphi}$. By faithful flatness of the functor $- \otimes_{\mathbb{R}} \mathbb{C}$ this implies that $\widetilde{\varphi}_{\mathbb{R}}$ is surjective and $\mathrm{Ker}(\widetilde{\varphi}_{\mathbb{R}}) = \mathrm{Ker}(\widetilde{\varphi}) \cap U(\mathfrak{s})$. (Notice that this also proves that $\mathcal{D}(R_r) \cong \mathcal{D}(\widetilde{S}) = \mathcal{D}(S_{p,r}) \otimes_{\mathbb{R}} \mathbb{C}$.) Finally, by Theorem 7.6, again, $\mathrm{Ker}(\widetilde{\varphi}) = \mathrm{Ker}(\psi) = J_r$, as required. □

The real forms of $\mathfrak{so}(n+2,\mathbb{C})$ are classified, see [Hel, Chap. X], and in Helgason's notation are the real Lie algebras $\mathfrak{so}(p'+1, q'+1)$ for $p'+q' = n$. It is therefore not surprising that we have the following result.

Proposition 9.5. *The Lie algebra $\mathfrak{s} = \mathfrak{s}(p,r)$ is isomorphic to $\mathfrak{so}(p+1, q+1)$ for $p+q = n$.*

Proof. The proof is omitted since it follows from straightforward, but not particularly illuminating examination of the relations in $\widetilde{\mathfrak{g}}$. □

Combined with Proposition 9.4 this gives:

Theorem 9.6. *Let $p+q=n\geq 3$, $r\geq 1$ and $\widetilde{\mathtt{F}}_p=\sum_{j=1}^p X_j^2-\sum_{j=p+1}^n X_j^2\in A_{\mathbb{R}}=\mathbb{R}[X_1,\dots,X_n]$. Then,*

$$\mathcal{D}\big(A_{\mathbb{R}}/\widetilde{\mathtt{F}}_p^r A_{\mathbb{R}}\big)\;\cong\;\frac{U(\mathfrak{so}(p+1,q+1))}{J_r\cap U(\mathfrak{so}(p+1,q+1))}$$

where J_r is the primitive ideal of $U(\mathfrak{so}(n+2,\mathbb{C}))$ described in Corollary 4.17. □

In Section 8 we showed that the higher symmetries of the Laplacian were equal to a factor of the enveloping algebra of $\mathfrak{so}(n+2,\mathbb{C})$. However, in applications (see, for example, [BS, Eas]), one is interested in the real case. To end this section we show that those results from Section 8 also descend readily to the real case.

We are therefore interested in symmetries of powers of the d'Alembertian $\Box=\Box_p$ for a fixed integer p. Now the basic results from Section 8 do restrict to the real field. In particular, the Fourier transform $\mathcal{F}$ from (8.3) is well-defined on $\mathcal{D}(A_{\mathbb{R}})$ and, by Lemma 8.4, satisfies $\mathcal{F}(\Box)=2\widetilde{\mathtt{F}}_p$. Therefore, the proof of Proposition 8.5 can be used mutatis mutandis to show that $\mathcal{F}$ induces an anti-isomorphism

$$\mathcal{D}(S)\longrightarrow \mathscr{S}(\Box^r)\qquad \text{for } S=S_{p,r}=A_{\mathbb{R}}/\widetilde{\mathtt{F}}_p^r A_{\mathbb{R}}. \tag{9.7}$$

It is known, see for example [DFG, 2.2 and 2.3], that for each real form $\mathfrak{so}(p+1,q+1)$ of $\mathfrak{g}$ there exists a Chevalley anti-involution $\varkappa$ on $\mathfrak{g}$ which stabilizes this real form. Moreover $\varkappa$ fixes J_r by [Jan, 5.2 (2)]. It then follows from Theorem 9.6 that $\varkappa$ induces an anti-automorphism $\varkappa$ on $\mathcal{D}(S)$. We therefore obtain the following analogue of Theorem 8.6:

Theorem 9.8. *Let $n=p+q\geq 3$, $r\geq 1$ and $\Box=\Box_p$ be the d'Alembertian on $\mathbb{R}^{p,q}$. There exists a primitive ideal J_r of $U(\mathfrak{so}(n+2,\mathbb{C}))$ and algebra isomorphisms:*

$$\mathcal{F}\circ\varkappa\circ\widetilde{\psi}_{\mathbb{R}}\;:\;\frac{U(\mathfrak{so}(p+1,q+1))}{(J_r\cap U(\mathfrak{so}(p+1,q+1)))}\xrightarrow{\;\sim\;}\mathcal{D}(S_{p,r})\xrightarrow{\;\sim\;}\mathscr{S}(\Box^r).$$

Consequently, $\mathscr{S}(\Box^r)$ is a primitive noetherian ring that is a maximal order in its simple artinian quotient ring. Moreover it is generated as an algebra by the elements

$$\{\mathtt{E}_1+d;\ \partial_{X_k};\quad \widetilde{Q}_j, 1\leq j,k\leq n;\ D_{k,l}, 1\leq k,l\leq p \text{ or } p+1\leq k,l\leq n;\ \widetilde{D}_{kl}, 1\leq k\leq p<l\leq n\},$$

thought of as differential operators on $S_{p,r}$. Here

$$\widetilde{Q}_j=\mathcal{F}(\widetilde{P}_j)=\begin{cases}\widetilde{\mathtt{F}}\,\partial_{X_j}-2X_j(\mathtt{E}_1+d) & \text{if } 1\leq j\leq p\\ \widetilde{\mathtt{F}}\,\partial_{X_j}+2X_j(\mathtt{E}_1+d) & \text{if } p+1\leq j\leq n.\end{cases}$$

Proof. The isomorphisms follow by combining Theorem 9.6 with (9.7). It follows immediately that $\mathscr{S}=\mathscr{S}(\Box^r)$ is noetherian. The fact that $\mathscr{S}$ has the specified generators then follows from the Fourier transform applied to Lemma 9.3.

It remains to prove that $\mathscr{S}$ (or, equivalently $\mathcal{D}(S)$) is a primitive maximal order. Recall from the proof of Proposition 9.4 that $\mathcal{D}(S)\otimes_{\mathbb{R}}\mathbb{C}=\mathcal{D}(\widetilde{S})\cong\mathcal{D}(R_r)$. Thus $\mathcal{D}(\widetilde{S})$ is primitive and it follows, for example from [MR, 10.1.9], that $\mathcal{D}(S)$ is primitive. If $\mathcal{D}(S)$ is not a maximal order, there exists an overring $\mathcal{D}(S)\subset T$ with $aTb\subseteq\mathcal{D}(S)$, for some regular elements $a,b\in\mathcal{D}(S)$. Tensoring with $\mathbb{C}$ shows that $a(T\otimes\mathbb{C})b\subseteq\mathcal{D}(\widetilde{S})$ and hence, by Theorem 7.6, that $T\otimes\mathbb{C}\subseteq\mathcal{D}(\widetilde{S})$. By the faithful flatness of $-\otimes_{\mathbb{R}}\mathbb{C}$, this forces $T=\mathcal{D}(S)$. Thus, $\mathcal{D}(S)$ is a maximal order. □

Remark 9.9. The theorem recovers the generators for $\mathscr{S}(\Box^r)$ given in [ShS, (4.8)].

Corollary 9.10. *Keep the notation of the theorem and consider* $\mathcal{A} = \mathbb{R}[\partial_{X_1}, \dots, \partial_{X_n}]/(\Box_p^r)$ *under its natural* $\mathscr{S}(\Box_p^r)$*-module structure. If* n *is even with* $r \geq \frac{n}{2}$*, then* $\mathcal{A}$ *has a unique proper factor module, which is finite dimensional. Otherwise* $\mathcal{A}$ *is an irreducible module.*

Proof. Passing from the $\mathscr{S}(\Box_p^r)$-module $\mathcal{A}$ to the $\mathscr{S}(\Delta_1^r)$-module $\mathcal{A}_{\mathbb{C}} = \mathbb{C}[\partial_{X_1}, \dots, \partial_{X_n}]/(\Delta_1^r)$ is given by $\phi^{-1} \circ (- \otimes_{\mathbb{R}} \mathbb{C})$, where the automorphism ϕ of $\mathscr{S}(\Delta_1^r)$ is induced by (9.1). We claim that the corollary follows by the faithful flatness of $\phi^{-1} \circ (- \otimes_{\mathbb{R}} \mathbb{C})$ and Theorem 8.6(2).

In order to prove the claim, we first note that, as a $\mathscr{S}(\Delta_1^r)$-module, $\mathcal{A}_{\mathbb{C}}$ satisfies the claimed results by Theorem 8.6(2). If $\mathcal{A}$ is not a simple $\mathscr{S}(\Delta_1^r)$-module, then faithful flatness implies that the same is true of $\mathcal{A}_{\mathbb{C}}$ as an $\mathscr{S}(\Box_p^r)$-module and hence n must be even with $r \geq \frac{n}{2}$. Moreover, faithful flatness and Theorem 8.6(2) further imply that $\mathcal{A}$ will then have a simple infinite dimensional submodule with a simple finite dimensional factor module.

Conversely, if $\mathcal{A}_{\mathbb{C}}$ is not simple as an $\mathscr{S}(\Delta_1^r)$-module, then it has a finite dimensional factor module, as an $\mathscr{S}(\Delta_1^r)$-module and hence as an $\mathscr{S}(\Box_p^r)$-module. Since $\mathcal{A}_{\mathbb{C}} \cong \mathcal{A}^{(2)}$ as $\mathscr{S}(\Box_p^r)$-modules, this implies that $\mathcal{A}^{(2)}$ and hence $\mathcal{A}$ also has a finite dimensional factor module and so is certainly not simple. $\Box$

10. Harmonic polynomials and $\mathcal{O}$-duality

The set of harmonic polynomials $H = \{p \in A : \Delta_1(p) = 0\}$ and its real analogues are fundamental objects with many applications, notably in Lie theory (see, for example, [GW, HSS, KO]) and physics (see, for example, [Bek1, Bek3]). In the latter subject, Dirac [Dir1, Dir2] constructed a remarkable unitary irreducible representation $D\big(\frac{n}{2} - 1, 0\big)$ of the Lie algebra $\mathfrak{so}(n, 2)$, known as the scalar singleton module. This module can be realized in different ways, see [Bek1, Bek3]: as a highest weight module; as harmonic scalar fields ϕ (in the sense that $\Box_{n-1} . \phi = 0$) on the space $\mathbb{R}^{n-1,1}$ which are preserved by $\mathfrak{so}(n, 2)$; or as harmonic distributions φ of weight $1 - \frac{n}{2}$ on the "ambient space" $\mathbb{R}^{n,2}$ (hence $\Box_{n,2} . \varphi = 0$). The algebra of symmetries of the scalar singleton, as defined in [Bek1, 4.4], acts on $D\big(\frac{n}{2} - 1, 0\big)$. This algebra can be identified, thanks to the results of [Eas] and [Vas], with the algebra $\mathscr{S}(\Box_{n-1,1})$ (see also Remark 11.14 (2)).

Replacing the condition $\Box_{n-1} . \phi = 0$ by $\Box_{n-1}^r . \phi = 0$, generalises the singleton module to give *higher singleton modules*. These have been studied in [BG], with applications to anti-de Sitter gauge fields and related topics.

It is therefore natural to give mathematical description of these higher singletons or, more generally, of the space $\mathcal{M}_{p,r} = \big\{a \in A_{\mathbb{R}} \ : \ \Box_p^r(a) = 0\big\}$. This is the topic of this section where we describe the structure of $\mathcal{M}_{p,r}$ as a representation of the orthogonal Lie algebra $\mathfrak{so}(p+1, q+1)$ and show that its complex analogue is simply the category $\mathcal{O}$ dual of $N(\lambda) = R_r$ (see Corollaries 10.12 and 10.13 for the precise statements).

We now make this precise. We continue with the notation from Sections 2 and 4 and start with the formal definitions. Since we will be able to prove results for both the real and complex cases, we fix $1 \leq p \leq n$ and keep the notation $\widetilde{\mathtt{F}}_p \in A_{\mathbb{R}}$, and $\mathfrak{s} = \mathfrak{s}(p, r) \subset \mathcal{D}(A_{\mathbb{R}})$ and $\Box_p \in \mathcal{D}(A_{\mathbb{R}})$ from the beginning of Section 9. The spaces that interest us are defined as follows.

Definition 10.1. Let $r \in \mathbb{N}^*$. The elements of the subspace

$$\mathcal{M}_{p,r} = \{a \in A_{\mathbb{R}} \ : \ \Box_p^r(a) = 0\}$$

will be called *higher harmonics* or *harmonics of level r in* $A_{\mathbb{R}}$. We clearly also have the analogous complex space

$$M_{p,r} = \{a \in A \ : \ \Box_p^r(a) = 0\}.$$

Observe that $A = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and that $\Box_p^r$ has real coefficients. Hence, if $a = u + iv \in A$ with $u, v \in A_{\mathbb{R}}$, we have: $\Box_p^r(a) = \Box_p^r(u) + i\,\Box_p^r(v) = 0 \iff \Box_p^r(u) = \Box_p^r(v) = 0$. Therefore, $M_{p,r} = \mathcal{M}_{p,r} \otimes_{\mathbb{R}} \mathbb{C}$.

In the notation of Theorem 9.8 set $\tau = \mathcal{F}\circ\varkappa\circ\widetilde{\psi}_{\mathbb{R}}$. Thus $\tau :\ \mathfrak{s} \to \mathscr{S}(\Box_p^r)$ is a Lie algebra homomorphism that induces an isomorphism $\tau : U(\mathfrak{s})/\mathcal{J}_r \cong \mathscr{S}(\Box_p^r)$ for the appropriate ideal $\mathcal{J}_r$. By Proposition 9.4, upon tensoring with the complex numbers, we obtain an isomorphism $\tau \otimes_{\mathbb{R}} \mathbb{C} : U(\mathfrak{g})/J_r \cong \mathscr{S}(\Delta_1^r)$.

Lemma 10.2. (1) *The map τ defines a representation of $\mathfrak{s}$ in the space $\mathcal{M}_{p,r}$ of higher harmonics.*

(2) *Similarly $\tau \otimes_{\mathbb{R}} \mathbb{C}$ defines a representation of $\mathfrak{g}$ in the space $M_{p,r}$.*

Proof. It suffices to prove (1). Let $f \in \mathcal{M}_{p,r}$ and $Y \in \mathfrak{s}$. Then, $\mathcal{Y} = \tau(Y)$ is a symmetry of the operator $\Box_p^r$ whence, $\Box_p^r(\mathcal{Y}(f)) = (\Box_p^r \cdot \mathcal{Y})(f) \in \mathcal{D}(A_{\mathbb{R}}) \cdot \Box_p^r(f) = 0$. Therefore, $\mathcal{Y}(f) \in \mathcal{M}_{p,r}$ and so τ takes values in $\mathrm{End}_{\mathbb{R}}(\mathcal{M}_{p,r})$, as required. □

Remark 10.3. In fact the $\mathfrak{g}$-modules $M_{p,r}$ are twists of each other (see the proof of Corollary 10.13 for the details) and so it will suffice to prove results for just one of them. We will use the module

$$M_r = M_{n,r} = \{a \in A : \Delta_1^r(a) = 0\},$$

as was also defined in the introduction. For most of this section we will study this module.

We are interested in the following pairing.

Definition 10.4. The pairing $\langle\ |\ \rangle$ is defined by:

$$\langle\ |\ \rangle : A \times A \longrightarrow \mathbb{C}, \quad \langle a \mid f\rangle = \mathcal{F}(a)(f)_{|_0} = \mathcal{F}(a)(f)(0).$$

The *orthogonal* of a subspace $W \subseteq A$ is written $W^{\perp} = \{f \in A : \langle W \mid f\rangle = 0\}$.

Recall from Section 2 that the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ generated by the E_{jk}, $2 \le j, k \le \ell$, is isomorphic to $\mathfrak{so}(n, \mathbb{C})$. The next result is an easy variant on classical results, but since we could not find an appropriate reference, we include a proof in the Appendix.

Lemma 10.5. *Set $\mathcal{I}_r = \widetilde{\mathrm{F}}_p^r A$.*

(1) *The bilinear form $\langle\ |\ \rangle$ is symmetric and non-degenerate. It is also $\mathfrak{k}$-invariant in the sense that $\langle Y.a \mid f\rangle + \langle a \mid Y.f\rangle = 0$ for all $a, f \in A$ and $Y \in \mathfrak{k}$.*
(2) *$M_r^{\perp} = \mathcal{I}_r$ and $\mathcal{I}_r^{\perp} = M_r$.*
(3) *The form $\langle\ |\ \rangle$ induces a non-degenerate $\mathfrak{k}$-invariant symmetric pairing $\langle\ |\ \rangle : R_r \times M_r \longrightarrow \mathbb{C}$.* □

By Lemma 10.2, M_r has a $\mathfrak{g}$-module structure given by $a \mapsto \tau(Y).a = \mathcal{F}(\widetilde{\psi}(\vartheta(Y)))(a)$, for all $Y \in \mathfrak{g}$ and $a \in M_r$. On the other hand, $R_r = N(\lambda)$ is a $\mathfrak{g}$-module through the map $\psi_r : U(\mathfrak{g}) \to \mathcal{D}(R)$ induced by $\widetilde{\psi}$. We set $Y.p = \psi_r(Y)(p)$ for all $Y \in \mathfrak{g}$ and $p \in R_r$. These structures are related as follows.

Theorem 10.6. *The pairing* $\langle\ |\ \rangle : R_r \times M_r \longrightarrow \mathbb{C}$ *is* $\mathfrak{g}$*-invariant in the sense that*

$$\langle Y.p \mid g\rangle = \langle p \mid \tau(\vartheta(Y)).g\rangle = \langle p \mid \mathcal{F}(\widetilde{\psi}(Y))(g)\rangle.$$

for all $Y \in \mathfrak{g}$*,* $p \in R_r$ *and* $g \in M_r$*.*

Proof. We first claim that the result is equivalent to proving that

$$\mathcal{F}[\widetilde{\psi}(Y)(p)](g)_{|_0} = \mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)]_{|_0}. \tag{10.7}$$

To see this, note that $\langle Y.p \mid g\rangle = \langle \widetilde{\psi}(Y)(p) \mid g\rangle = \mathcal{F}[\widetilde{\psi}(Y)(p)](g)(0)$ and $\tau(\vartheta(Y)).g = \mathcal{F}(\widetilde{\psi}(Y))(g)$. Hence $\langle p \mid \tau(\vartheta(Y)).g\rangle = \mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)](0)$, as claimed.

Write $V_j = \frac{1}{2}U_j\Delta_1 - I(d)\partial_{U_{-j}}$ as in (4.2). The following formulæ, which are easily checked, will be needed in the proof.

$$\mathcal{F}(E_{ab}) = E_{ba}, \quad \text{and} \quad Q_b = \mathcal{F}(V_b) = \mathtt{F}\partial_{U_b} - U_{-b}I(d), \qquad \text{for } a, b \in \{0, \pm 2, \dots, \pm\ell\}. \tag{10.8}$$

Recall from §2 that $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{m} \oplus \mathfrak{r}^+$ with $\mathfrak{m} = \mathbb{C}E_{11} \oplus \mathfrak{k}$. Observe that if (10.7) is true for $Y, Z \in \mathfrak{g}$, then it is true for $[Y, Z]$. Thus, we can reduce the verification to the case where Y is a (scalar multiple) of a root vector y_α of the Chevalley basis given in Proposition 2.4. We may therefore prove (10.7) by considering the cases $Y \in \mathfrak{k}$, $Y \in \mathfrak{r}^+$ and $Y \in \mathfrak{r}^-$ separately.

Suppose first that $Y = y_\alpha \in \mathfrak{k}$ for $\alpha \in \Phi_1$. From Proposition 2.4 we may assume that $Y = E_{ab}$ with $a, b \in \{0, \pm 2, \dots, \pm\ell\}$. Then $\widetilde{\psi}(Y) = Y = E_{ab}$ and, since Y is a derivation, $Y(p) = [Y, p]$. Hence, $\mathcal{F}(Y(p)) = [\mathcal{F}(p), \mathcal{F}(Y)]$ and we get $\mathcal{F}(Y(p))(g) = \mathcal{F}(p)(\mathcal{F}(Y)(g)) - \mathcal{F}(Y)(\mathcal{F}(p)(g))$. But $\mathcal{F}(Y) = E_{ba} = U_b\partial_{U_a} - U_{-a}\partial_{U_{-b}}$, see (10.8), and so $\mathcal{F}(Y)(q)(0) = 0$ for all $q \in A$ (in particular for $q = \mathcal{F}(p)(g)$). We then get $\mathcal{F}(Y(p))(g)_{|_0} = \mathcal{F}(p)(\mathcal{F}(Y)(g))_{|_0}$, as wanted.

Suppose next that $Y = y_\alpha \in \mathfrak{r}^-$, where either $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$ or $\alpha = -\varepsilon_1$. Here, since $\widetilde{\psi}(y_\alpha) = x_\alpha$, we may assume that $\widetilde{\psi}(Y) = U_b$ for some $b \in \{0, \pm 2, \dots, \pm\ell\}$; see Lemma 4.3. Therefore we have

$$\mathcal{F}[\widetilde{\psi}(Y)(p)](g) = \mathcal{F}(U_b p)(g) = [\mathcal{F}(p)\mathcal{F}(U_b)](g) = \mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)]$$

and, in particular, $\mathcal{F}[\widetilde{\psi}(Y)(p)](g)_{|_0} = \mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)]_{|_0}$.

Finally, suppose that $Y = y_\alpha \in \mathfrak{r}^+$, where either $\alpha = \varepsilon_1 \pm \varepsilon_b$ or $\alpha = \varepsilon_1$. From Lemma 4.3 we may assume that $\widetilde{\psi}(Y) = V_b$ for some $b \in \{0, \pm 2, \dots, \pm\ell\}$. Recall from (10.8) that $\mathcal{F}(V_b) = Q_b = \mathtt{F}\partial_{U_b} - U_{-b}I(d)$. We will need an auxiliary calculation: for $p, f \in A$, we claim that

$$\begin{aligned}\mathcal{F}(\Delta_1 p - \Delta_1(p))(f)_{|_0} &= \left[\mathcal{F}(p\Delta_1)(f) + 2\left\{U_0\mathcal{F}(\partial_{U_0}(p)) + \sum_{\pm j=2}^{\ell} U_{-j}\mathcal{F}(\partial_{U_j}(p))\right\}(f)\right](0) \\ &= \mathcal{F}(p\Delta_1)(f)_{|_0} = 2\mathtt{F}\mathcal{F}(p)(f)_{|_0} = 0.\end{aligned} \tag{10.9}$$

To see this, notice that $\mathcal{F}(\Delta_1 p - \Delta_1(p)) = \mathcal{F}(p\Delta_1) + \mathcal{F}\big(2\{\partial_{U_0}(p)\partial_{U_0} + \sum_{\pm j=2}^{\ell}\partial_{U_j}(p)\partial_{U_{-j}}\}\big)$. Then (10.9) follows by applying this equality to f and then evaluating at 0.

We return now to the proof in the case where $\widetilde{\psi}(Y) = V_b$. We have

$$\mathcal{F}[\widetilde{\psi}(Y)(p)] = \mathcal{F}(V_b(p)) = \mathcal{F}\Big(\frac{1}{2}U_b\Delta_1(p) - I(d)(\partial_{U_{-b}}(p))\Big) = \frac{1}{2}\mathcal{F}(\Delta_1(p))\partial_{U_b} - \mathcal{F}\big[I(d)(\partial_{U_{-b}}(p))\big].$$

On the other hand, using

$$\mathcal{F}\big(I(d)\partial_{U_{-b}}p\big)(g) = [\mathcal{F}(p)U_{-b}\mathcal{F}(I(d))](g) = [\mathcal{F}(p)U_{-b}I(d)](g) = \mathcal{F}(p)\big(U_{-b}I(d)(g)\big)$$

we get:

$$\begin{aligned}\mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)] &= \mathcal{F}(p)(Q_b(g)) = \mathcal{F}(p)\big(\mathtt{F}\partial_{U_b}(g) - U_{-b}I(d)(g)\big)\\ &= \frac{1}{2}\mathcal{F}(\Delta_1 p)(\partial_{U_b}(g)) - \mathcal{F}\big(I(d)\partial_{U_{-b}}p\big)(g).\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)] - \mathcal{F}[\widetilde{\psi}(Y)(p)](g) = \frac{1}{2}\big\{\mathcal{F}(\Delta_1 p) - \mathcal{F}(\Delta_1(p))\big\}\big(\partial_{U_b}(g)\big)\\ - \big\{\mathcal{F}\big(I(d)\partial_{U_{-b}}p\big) - \mathcal{F}\big[I(d)(\partial_{U_{-b}}(p))\big]\big\}(g).\end{aligned} \tag{10.10}$$

It is easily checked that $I(d)\partial_{U_{-b}}p - I(d)(\partial_{U_{-b}}(p)) = I(d)p\partial_{U_{-b}} + \partial_{U_{-b}}(p)\mathtt{E}_1$. Therefore

$$\mathcal{F}\big[I(d)\partial_{U_{-b}}p - I(d)(\partial_{U_{-b}}(p))\big](g) = U_{-b}\mathcal{F}(p)(I(d)(g)) + \mathtt{E}_1\big(\mathcal{F}(\partial_{U_{-b}}(p))(g)\big)$$

and this polynomial vanishes at 0 since $(U_{-b}q)_{|_0} = \mathtt{E}_1(q)_{|_0} = 0$ for all $q \in A$. Equation (10.10) then gives $\big\{\mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)] - \mathcal{F}[\widetilde{\psi}(Y)(p)](g)\big\}(0) = \frac{1}{2}\mathcal{F}\big(\Delta_1 p - \Delta_1(p)\big)\big(\partial_{U_b}(g)\big)(0)$. Now applying (10.9) to $f = \partial_{U_b}(g)$ yields $\big\{\mathcal{F}(p)[\mathcal{F}(\widetilde{\psi}(Y))(g)] - \mathcal{F}[\widetilde{\psi}(Y)(p)](g)\big\}(0) = 2\mathtt{F}\mathcal{F}(p)(\partial_{U_b}(g))_{|_0} = 0$, as required. □

As in [Jan, §2.1], the dual M^* of a $\mathfrak{g}$-module M is endowed with a $\mathfrak{g}$-module structure through

$$(Y.f)(x) = f(\vartheta(Y).x) \quad \text{for all } Y \in \mathfrak{g},\ f \in \mathfrak{g}^* \text{ and } x \in M. \tag{10.11}$$

Let $\mathcal{O}$ denote the category of highest weight modules, as in Section 2. If $M \in \operatorname{Ob}\mathcal{O}$, then $M = \bigoplus_{\mu\in\mathfrak{h}^*} M^\mu$ where M^μ is the μ-weight space of $\mathfrak{h}$ in M. As in [Jan, 4.10], the $\mathcal{O}$-*dual* $M^\vee$ of M is then defined by

$$M^\vee = \bigoplus_{\mu\in\mathfrak{h}^*} (M^\vee)^\mu, \quad (M^\vee)^\mu \cong (M^\mu)^*.$$

By [Jan, 4.10 (2,3)], the contravariant functor $M \to M^\vee$ is exact and satisfies $L(\omega)^\vee \cong L(\omega)$ for all $\omega \in \mathfrak{h}^*$.

For the next result, recall from Theorem 4.13 that, if $R_r = N(\lambda)$ is not a simple $\mathfrak{g}$-module, then it has a simple socle $Z_r \cong L(\mu)$, for the appropriate weight μ, and finite dimensional factor $\overline{R} = R_r/Z_r \cong L(\lambda)$.

Corollary 10.12. *Under the $\mathfrak{g}$-module structure of M_r given by Lemma 10.2, there is a $\mathfrak{g}$-module isomorphism $N(\lambda)^\vee \cong M_r$. Furthermore:*

1. *if n is even with $r < \frac{n}{2}$ or if n is odd, then $M_r \cong L(\lambda)$;*
2. *if n is even with $r \geq \frac{n}{2}$, then M_r has a simple, finite dimensional socle $\overline{R}^\vee \cong L(\lambda)$, with quotient $M_r/\overline{R}^\vee = Z_r^\vee \cong L(\mu)$. The formula of the weight μ is given in (4.14).*

Proof. By Lemma 4.8, $N(\lambda)$ is an object of category $\mathcal{O}$. By Theorem 10.6, the pairing $\langle\ |\ \rangle$ enables us to define a linear injection:

$$\xi\ :\ R_r \hookrightarrow M_r^* \quad \text{given by } \xi(p)(a) = \langle p \mid a\rangle \text{ for all } p \in R_r \text{ and } a \in M_r.$$

Moreover, the $\mathfrak{g}$-invariance of $\langle \mid \rangle$ means that $\xi(Y.p) = Y.\xi(p)$ under the $\mathfrak{g}$-module structure on M_r^* defined by (10.11); thus, ξ is $\mathfrak{g}$-linear. To deduce that ξ restricts to an isomorphism $\xi : R_r \xrightarrow{\sim} M_r^\vee$, it suffices to check that $\xi : R_r^\nu \xrightarrow{\sim} (M_r^\nu)^*$ for all $\nu \in \mathfrak{h}^*$. Let $Y \in \mathfrak{h}$, $p \in R_r^\nu$ and $f \in M_r^{\nu'}$. From $\vartheta(Y) = Y$ one gets that $\nu(Y)\langle p \mid f\rangle = \nu'(Y)\langle p \mid f\rangle$. Therefore $\langle R_r^\nu \mid M_r^{\nu'}\rangle = 0$ for $\nu \neq \nu'$ and it follows that $\langle \mid \rangle$ gives a non-degenerate pairing $R_r^\nu \times M_r^\nu \to \mathbb{C}$, as desired. Statements (1) and (2) are then consequences of the $\mathcal{O}$-duality and the structure of the $\mathfrak{g}$-module $N(\lambda) = R_r$ obtained in Theorem 4.13. □

We now return to the real case, as discussed at the beginning of the section and describe the natural analogues of Corollary 10.12.

Corollary 10.13. *Fix $r \geq 1$ and $1 \leq p \leq n$ and consider the $\mathfrak{s}(p,r)$-module $\mathcal{M}_{p,r}$ from Definition 10.1 and Lemma 10.2 . Then:*

1. *if n is even with $r < \frac{n}{2}$ or if n is odd, then $\mathcal{M}_{p,r}$ is simple;*
2. *if n is even with $r \geq \frac{n}{2}$, then $\mathcal{M}_{p,r}$ has a simple, finite dimensional socle S', with an infinite dimensional simple quotient $\mathcal{M}_{p,r}/S'$.*

Proof. We first want to obtain an analogue of Corollary 10.12 for any $M_{p,r}$. To do this, recall that the automorphism ϕ from (9.1) satisfies $\phi(\Delta_1) = \Box_p$. Hence if we define the *twisted $\mathcal{D}(A)$-module* structure ${}^{\phi}\!A$ on A by $\theta \circ a = \phi(\theta)a$, for $\theta \in \mathcal{D}(A)$ and $a \in A$, then a simple computation shows that ${}^{\phi}M_r = M_{p,r}$. Since twisting preserves submodule lattices and dimensions, it follows from Corollary 10.12 that the statement of the present corollary holds for the $\mathfrak{g}$-modules $M_{p,r}$.

As was observed in (10.1), $\mathcal{M}_{p,r} \otimes_{\mathbb{R}} \mathbb{C} = M_{p,r}$ for each p, r. Thus, just as in the proof of Corollary 9.10, faithful flatness then implies that the desired results for the $\mathcal{M}_{p,r}$ follow from the corresponding results for the $M_{p,r}$. □

11. Conformal densities and the ambient construction

The $\mathfrak{g}$-module $M_1 = H$ of harmonic polynomials from Definition 10.1 is an "incarnation" of the scalar singleton module introduced by Dirac through the ambient method, cf. [Dir1, Dir2, EG, Bek3]. In this section, we will briefly recall this construction, and its generalisation to the higher harmonics, and then show that these objects have a natural algebraic analogue that gives another incarnation of the module $M_r \cong N(\lambda)^\vee$. This will prove Theorem 1.8 from the introduction.

The ambient method allows one to work with the standard representation of $\mathfrak{so}(p+1, q+1)$ by linear vector fields on $\mathbb{R}^{n+2}$ and leads to a conformal model of the representation of $\mathfrak{g}$ on M_r. To see this, one first constructs the following conformal compactification of $\mathbb{R}^{p,q}$. Let $\mathtt{Q} = U_1U_{-1} + \mathtt{F}_p$ be the quadratic form of signature $(p+1, q+1)$ on $\mathbb{R}^{n+2}$ with coordinate functions $U_1, U_{-1}, X_1, \dots, X_n$. and write $\mathcal{C} = \{z \in \mathbb{R}^{n+2} : \mathtt{Q}(z) = 0\}$ for the corresponding light cone. Then the associated projective quadric $\mathcal{Q} = \{\mathtt{Q} = 0\} \subset \mathbb{RP}^{n+1}$ is a conformal compactification of $\mathbb{R}^{p,q}$, under the identification of $\mathbb{R}^{p,q}$ with the open cell $\mathcal{Q} \cap \{U_1 = 1\}$. (The space $\mathcal{Q}$ is sometimes known as the boundary of the anti-de Sitter space when $(p,q) = (n-1, 1)$). In this setting a conformal density of weight w on $\mathcal{Q}$ can be viewed as a homogeneous functions of weight w on the light cone $\mathcal{C}$. The conformal Laplacian is the operator induced by the Laplacian $\Delta = \partial_{U_1}\partial_{U_{-1}} + \Box_p$. It acts on densities of weight $-\frac{n}{2} + 1$, as shown for example in [EG, Propositions 4.4 and 4.7] and [GJMS, Proposition 2.1], and the corresponding space of harmonic densities of weight $-\frac{n}{2}+1$ is then an $\mathfrak{so}(p+1,q+1)$-module. For Minkowski space-time, when $(p,q)=(n-1,1)$, this $\mathfrak{so}(n,2)$-module corresponds to the *scalar singleton module* $D(\frac{n}{2}-1,0)$ of [Bek3].

Analogous questions arise for homogeneous functions on the "generalised light cone" $\{\mathtt{Q}^r=0\}$; equivalently for densities φ of weight w for which $\Delta\cdot\varphi=0 \pmod{\mathtt{Q}^r}$. By [EG] and [GJMS], again, the above construction works for densities of weight $-\frac{n}{2}+r$. This produces an $\mathfrak{so}(p+1,q+1)$-module, which in the case of the Minkowski space time corresponds to the higher-order singleton as defined in [BG].

In this section we aim to give an algebraic version of this construction and relate it to the $\mathfrak{g}$-modules R_r and M_r obtained in the previous sections. Roughly speaking, in our setting (and under the notation of Section 2), the generalised light cone $\{\mathtt{Q}^r=0\}$ is replaced by a factor $B/\mathtt{Q}^rB$ of a polynomial ring in $(n+2)$ variables, while the "conformal Laplacian" is replaced by the operator $\Delta\in\mathcal{D}(B)$ and the "densities" by homogeneous polynomials on (a finite extension of) $B/\mathtt{Q}^rB$.

We now formalise this approach. Adopt the notation of Sections 2 and 3; in particular, we use the presentation of $\mathfrak{g}$ provided by Proposition 2.3. Following (4.1), write $B=A[U_{-1},U_1]$ for $A=\mathbb{C}[U_0,U_{\pm2},\dots,U_{\pm\ell}]$ and let $B[U_1^{-1}]=A[U_{-1},U_1^{\pm1}]$ be the localization of B at the powers of U_1. In the notation of (2.1), $U_{-1}=(\mathtt{Q}-\mathtt{F})U_1^{-1}$, for $\mathtt{Q}=\sum_{j=1}^{\ell}U_jU_{-j}+\frac12U_0^2$, and so $B[U_1^{-1}]=A[\mathtt{Q},U_1^{\pm1}]$. We also need to adjoin a square root of U_1 to $B[U_1^{-1}]$; thus, formally, set

$$S=B[U_1^{-1},T^{\pm1}]/\big(T^2-U_1\big)$$

for an indeterminate T. The class of T in S is denoted by t. Notice that

$$S=A[U_{-1},t^{\pm1}]=A[\mathtt{Q},t^{\pm1}]=\mathbb{C}[t^{\pm1},U_{\pm2},\dots,U_{\pm\ell},U_0,\mathtt{Q}]. \tag{11.1}$$

Clearly, $\mathtt{Q}$, t are indeterminates over A and $S=B[U_1^{-1}]\bigoplus B[U_1^{-1}]t$ is a free $B[U_1^{-1}]$-module of rank 2.

Remark 11.2. A derivation $D\in\operatorname{Der}_{\mathbb{C}}(B)$ extends to $B[U_1^{-1},T^{\pm1}]$ by setting $D(T)=\frac{1}{2T}D(U_1)$. Then $D((T^2-U_1)f)=(T^2-U_1)D(f)$ for any $f\in B[U_1^{-1},T^{\pm1}]$ and so D also defines a derivation on S. In particular, the derivations $E_{ab}\in\mathfrak{g}$, as defined in (2.2), act on S and this endows S with a natural $\mathfrak{g}$-module structure. Furthermore, the derivation ∂_{U_1} extends to S by $\partial_{U_1}(t)=\frac{1}{2t}$ and so we can write $\partial_{U_1}=\frac{1}{2t}\partial_t\in\operatorname{Der}_{\mathbb{C}}(S)$. With this notation and that from (2.1, 2.2), we can therefore write

$$\mathtt{E}=\frac{t}{2}\partial_t+\sum_{j\neq1}U_j\partial_{U_j},\qquad E_{j1}=\frac{U_j}{2t}\partial_t-U_{-1}\partial_{U_{-j}}=\frac{U_j}{2t}\partial_t-\frac{1}{t^2}(\mathtt{Q}-\mathtt{F})\partial_{U_{-j}}$$

as elements of $\operatorname{Der}_{\mathbb{C}}(S)$.

We now fix an integer $r\in\mathbb{N}^*$ and set

$$\overline{S}=\overline{S}_r=S/\mathtt{Q}^rS.$$

Let $D\in\operatorname{Der}_{\mathbb{C}}(S)$ be a derivation such that $D(\mathtt{Q})=c\,\mathtt{Q}$ for some $c\in S$; for example $D=\mathtt{E}$ or E_{ab}. Then, D induces a derivation, again denoted D, in $\operatorname{Der}_{\mathbb{C}}(\overline{S})$. In particular, the algebra $\overline{S}$ also has a $\mathfrak{g}$-module structure. As such, S and $\overline{S}$ are graded by the weights of the Euler operator $\mathtt{E}$ and we can make the following definition.

Definition 11.3. Let $p \in \mathbb{Q}$. An element $u \in C = S$, or $C = \overline{S}$ is said to be *homogeneous of weight* p if $\mathtt{E}(u) = pu$. The space of homogeneous elements of weight p is denoted by $C(p)$. Set $\operatorname{wt}(0) = 0$ and $\operatorname{wt}(u) = p$ if $0 \neq u \in C(p)$; thus $\mathtt{E}(u) = \operatorname{wt}(u)u$. If V is a subspace of C, set $V(p) = C \cap C(p)$.

Notice that $U_j \in S(1)$ for all j and that $t \in S(\frac{1}{2})$. Hence, $S = \bigoplus_{\beta\in\frac{1}{2}\mathbb{Z}} S(\beta)$ and, since $\mathtt{Q}^r$ is homogeneous of degree $2r$, we have:

$$\overline{S} = \bigoplus_{\beta\in\frac{1}{2}\mathbb{Z}} \overline{S}(\beta), \qquad \text{where } \overline{S}(\beta) = S(\beta)/\mathtt{Q}^r S(\beta - 2r).$$

We can think of the elements of $\overline{S}(\beta)$ as "half-densities" on the subscheme $\{\mathtt{Q}^r = 0\} \subset \mathbb{C}^{n+2}$.

Lemma 11.4. *Let $\overline{\mathtt{Q}}$ denote the class of $\mathtt{Q}$ in $\overline{S}$ and pick $\beta \in \frac{1}{2}\mathbb{Z}$. Then:*
(1) *$\overline{S}(\beta)$ is a $\mathfrak{g}$-submodule of the $\mathfrak{g}$-module $\overline{S}$;*
(2) *one has $\overline{S} = \bigoplus_{j=0}^{r-1} A[t^{\pm 1}]\overline{\mathtt{Q}}^j$ and $\overline{S}(\beta) = \bigoplus_{j=0}^{r-1} A[t^{\pm 1}](\beta - 2j)\overline{\mathtt{Q}}^j$ as $A[t^{\pm 1}]$-modules.*

Proof. This follows from (11.1) combined with the fact that $[\mathtt{E}, Y] = 0$ for all $Y \in \mathfrak{g}$. $\square$

By Remark 11.2, the Laplacian $\Delta = 2\sum_{j=1}^{\ell} \partial_{U_j}\partial_{U_{-j}} + \partial_{U_0}^2$ extends trivially to $B[U_1^{-1}, T]$ and then restricts to S. This differential operator on S, still denoted by Δ, can also be written:

$$\Delta = \frac{1}{t}\partial_t\partial_{U_{-1}} + \Delta_1, \qquad \text{for } \ \Delta_1 = 2\sum_{j\geq 2} \partial_{U_j}\partial_{U_{-j}} + \partial_{U_0}^2.$$

As usual, the set of *harmonic elements in* S is defined to be $\mathcal{H}(S) = \{f \in S : \Delta(f) = 0\}$. Since $[\mathtt{E}, \Delta] = -2\Delta$, this is a graded subspace of S and $\mathcal{H}(S) = \bigoplus_{\beta\in\frac{1}{2}\mathbb{Z}} \mathcal{H}(\beta)$. Clearly, $\Delta(S(\beta)) \subseteq S(\beta - 2)$ for all $\beta \in \frac{1}{2}\mathbb{Z}$.

We now want to find β such that Δ induces an operator from $\overline{S}(\beta)$ to $\overline{S}(\beta - 2)$. The next result can be compared with [EG, Propositions 4.4 & 4.7], [GJMS, Proposition 2.1] and [BZ, II Proposition].

Proposition 11.5. *Let $\delta = -d = -\frac{n}{2} + r$. Then, the Laplacian Δ induces an operator*

$$\overline{\Delta}_\delta = \overline{\Delta} \ : \ \overline{S}(\delta) = \frac{S(\delta)}{\mathtt{Q}^r S(\delta - 2r)} \ \longrightarrow \ \overline{S}(\delta - 2) = \frac{S(\delta - 2)}{\mathtt{Q}^r S(\delta - 2(r+1))}$$

by the formula $\overline{\Delta}(\bar{f}) = \overline{\Delta(f)}$ for all $f \in S(\delta)$.

Proof. As in (3.13) one shows that, for all $k \in \mathbb{N}^*$ and homogeneous elements $u \in S$ one has

$$\Delta(u\mathtt{Q}^k) = \Delta(u)\mathtt{Q}^k + k[n + 2 + 2\operatorname{wt}(u) + 2(k-1)]u\mathtt{Q}^{k-1}. \tag{11.6}$$

We are looking for $\delta \in \frac{1}{2}\mathbb{Z}$ such that $\Delta(u\mathtt{Q}^r) \in \mathtt{Q}^r S(\delta - 2(r+1))$ for all $u \in S(\delta - 2r)$. However, by (11.6), if $\delta = -\frac{n}{2} + r$ then

$$\Delta(u\mathtt{Q}^r) = \Delta(u)\mathtt{Q}^r + r[n + 2 + 2\delta - 2(r+1)]u\mathtt{Q}^{r-1} = \Delta(u)\mathtt{Q}^r$$

for all $u \in S(\delta - 2r)$. This proves the proposition. $\square$

Thanks to Proposition 11.5 we can define the following subspace of "harmonic elements of degree δ", or "harmonic conformal densities of weight δ". It gives a (complex) version of the "scalar singleton module" defined by Dirac; see for example [Bek3, §3.5].

Definition 11.7. Let $r \in \mathbb{N}^*$ and $\delta = -\frac{n}{2} + r$ and set:

$$N_\lambda = \operatorname{Ker} \overline{\Delta}_\delta = \{\bar{f} \in \overline{S}(\delta) \ : \ \overline{\Delta}_\delta(\bar{f}) = 0\}.$$

Since $[Y, \Delta] = 0$ for all $Y \in \mathfrak{g}$, it is easy to see that N_λ is a $\mathfrak{g}$-submodule of $\overline{S}(\delta)$.

It follows from Lemma 11.4(2) that any element $\bar{f} \in \overline{S}(\beta)$, for $f \in S(\beta)$, can be uniquely written:

$$\bar{f} = \sum_{j=0}^{r-1} f_j \overline{\mathtt{Q}}^j \quad \text{with} \quad f = \sum_{j=0}^{r-1} f_j \mathtt{Q}^j \text{ and } f_j \in A[t^{\pm 1}](\beta - 2j). \tag{11.8}$$

Let $a \in A$ and $\nu \in \mathbb{Z}$. Since $\partial_{U_{-1}}(at^\nu) = 0$ and $\Delta(a) = \Delta_1(a)$ it follows that $\Delta(at^\nu) = \Delta_1(a)t^\nu$. Applying this observation to the f_j shows that $\Delta(f_j) = \Delta_1(f_j)$ for $0 \le j \le r-1$.

Proposition 11.9. (1) *Let $f = \sum_{j=0}^{s} f_j \mathtt{Q}^j \in S(\delta)$ with $0 \le s \le r-1$ and $f_j \in A[t^{\pm 1}](\delta - 2j)$ as in (11.8). Then,*

$$\overline{\Delta}(\bar{f}) = 0 \iff \Delta(f) = 0 \iff \begin{cases} \Delta_1(f_j) = -2(j+1)(r-(j+1))f_{j+1} & \text{if} \quad 0 \le j \le s-1; \\ \Delta_1(f_s) = 0. \end{cases}$$

Consequently,

$$N_\lambda = \{\bar{f} \in \overline{S}(\delta) \ : \ \Delta(f) = 0\} = \overline{\mathcal{H}(S)(\delta)} \subset \overline{S}(\delta).$$

(2) *There exists an isomorphism of $\mathfrak{m}$-modules*

$$\operatorname{ev}_0 : \ N_\lambda \xrightarrow{\ \sim\ } \mathcal{S}_\lambda = \{a \in A[t^{\pm 1}](\delta) \ : \ \Delta_1^r(a) = 0\}$$

given by $\operatorname{ev}_0(\bar{f}) = f_0$, when $f = \sum_{j=0}^{r-1} f_j \mathtt{Q}^j \in S(\delta)$ with $f_j \in A[t^{\pm 1}](\delta - 2j)$.

Proof. (1) By (11.6) we have:

$$\Delta(f) = \sum_{j=0}^{s} \Delta(f_j \mathtt{Q}^j) = \Delta_1(f_j)\mathtt{Q}^s + \sum_{j=0}^{s-1} \{\Delta_1(f_j) + (j+1)[2\operatorname{wt}(f_{j+1}) + N + 2j]f_{j+1}\}\mathtt{Q}^j.$$

It follows from $\Delta_1(f_j) \in A[t^{\pm 1}]$ that $\Delta(f) \in \bigoplus_{j=0}^{s} A[t^{\pm 1}]\mathtt{Q}^j$. Since $\operatorname{wt}(f_{j+1}) = \delta - 2(j+1)$ if $f_{j+1} \ne 0$, one deduces that $\Delta(f) = 0$ is equivalent to the following system of equations in the ring $A[t, t^{-1}]$:

$$\begin{cases} \Delta_1(f_s) = 0 \\ \Delta_1(f_j) = -2(j+1)(r-(j+1))f_{j+1} \ \text{if } 0 \le j \le s-1. \end{cases}$$

By definition, $\overline{\Delta}(\bar{f}) = \overline{\Delta(f)}$ and so $\overline{\Delta}(\bar{f}) = 0$ is equivalent to $\Delta(f) \in S(\delta - 2(r+1))\mathtt{Q}^r$. Recalling that $\Delta(f) \in \bigoplus_{j=0}^{r-1} A[t^{\pm 1}]\mathtt{Q}^j$ and using $S = \bigoplus_{j \in \mathbb{N}} A[t^{\pm 1}]\mathtt{Q}^j$ one deduces that $\Delta(f) = 0$ if and only if $\overline{\Delta}(\bar{f}) = 0$.

Finally, if $\bar{x} \in N_\lambda$ we can find f as in (11.8) such that $\bar{x} = \bar{f}$. By definition, $\overline{\Delta}(\bar{f}) = 0$ and hence $\Delta(f) = 0$. The equality $N_\lambda = \overline{\mathcal{H}(S)(\delta)}$ follows.

(2) By induction, the equalities $\Delta_1(f_j) = -2(j+1)(r-(j+1))f_{j+1}$ for $0 \le j \le r-2$ yield $\Delta_1^p(f_0) = c_p f_p$, where $c_p = (-2)^p p!(r-1)\cdots(r-p)$, for $0 \le p \le r-1$. It follows from (1) that the map $\bar{f} \mapsto f_0$ with $f \in \mathcal{H}(S)(\delta)$ takes values in $\mathcal{S}_\lambda$ and is injective. This also implies that if $a \in \mathcal{S}_\lambda$ one can obtain $f = \sum_{j=0}^{r-1} f_j \mathtt{Q}^j \in \mathcal{H}(S)(\delta)$ by setting $f_0 = a$ and $f_p = \frac{1}{c_p}\Delta_1^p(a)$ for $1 \le p \le r-1$. This yields a vector space isomorphism $N_\lambda \xrightarrow{\sim} \mathcal{S}_\lambda$.

Recall from §2 that $\mathfrak{m} = \mathbb{C}E_{11} \oplus \mathfrak{k}$ where $E_{11} = \frac{t}{2}\partial_t - U_{-1}\partial_{U_{-1}}$ and $\mathfrak{k} = \sum_{i,j\notin\{\pm1\}} \mathbb{C}E_{ij}$. Clearly, therefore, $A[t^{\pm1}]$ is $\mathfrak{m}$-stable. From $[E_{11}, \Delta_1] = 0 = [\mathfrak{k}, \Delta_1]$ it follows that $\mathcal{S}_\lambda$ is an $\mathfrak{m}$-module. Let $Y \in \mathfrak{m}$ and $\bar{f} \in N_\lambda$ with $f = \sum_{j=0}^{r-1} f_j \mathtt{Q}^j$ as above. Since $Y.\mathtt{Q} = Y(\mathtt{Q}) = 0$ one has $Y.\sum_{j=0}^{r-1} f_j\mathtt{Q}^j = \sum_{j=0}^{r-1}(Y.f_j)\mathtt{Q}^j$ with $Y.f_j \in A[t^{\pm1}](\delta-2j)$. Hence $\mathrm{ev}_0(Y.f) = (Y.f)_0 = Y.f_0 = Y.\mathrm{ev}_0(f)$ and so the linear isomorphism $\mathrm{ev}_0 : N_\lambda \xrightarrow{\sim} \mathcal{S}_\lambda$ is indeed an isomorphism of $\mathfrak{m}$-modules. □

As we next show, this theorem implies that, as $\mathfrak{k}$-modules, N_λ is isomorphic to the module $M_r = M_{n,r} = \{a \in A : \Delta_1^r(a) = 0\}$ from Remark 10.3. Before proving this, note that as $\Delta_1^r(A(m)) \subset A(m-2r)$ for all $m \in \mathbb{N}$, the space M_r is a graded subspace of A. Thus any $a \in M_r$ can be written as $a = \sum_{m\in\mathbb{N}} a(m)$ with $a(m) \in M_r(m)$. On the other hand, by Proposition 11.9(2) and the fact that $\mathrm{wt}(t) = \frac{1}{2}$, any $g \in \mathcal{S}_\lambda$ may be written as $g = \sum_{m\in\mathbb{N}} g(m)t^{2(\delta-m)}$ with $g(m) \in A(m)$.

Corollary 11.10. *The $\mathfrak{k}$-modules $M_r \subset A$ and $\mathcal{S}_\lambda \subset A[t^{\pm1}]$ are isomorphic via the map:*

$$\sigma \ : \ \mathcal{S}_\lambda \xrightarrow{\sim} M_r; \qquad \sum_{m\in\mathbb{N}} a(m)t^{2(\delta-m)} \mapsto \sum_{m\in\mathbb{N}} a(m).$$

In particular, the map $\sigma\circ\mathrm{ev}_0 : N_\lambda \xrightarrow{\sim} M_r$ is an isomorphism of $\mathfrak{k}$-modules.

Proof. Let $g = \sum_{m\in\mathbb{N}} a(m)t^{2(\delta-m)} \in \mathcal{S}_\lambda$ and recall that $\Delta_1(at^\beta) = \Delta_1(a)t^\beta$ for all $a \in A, \beta \in \mathbb{Z}$. Then, $\Delta_1^r(g) = \sum_m \Delta_1(a(m))t^{2(\delta-m)} = 0$ forces $\Delta_1^r(a(m)) = 0$ for all m, hence $\sum_m a(m) \in M_r$. Conversely, the same computation shows that if $\sum_m a(m) \in M_r$ then $\sum_{m\in\mathbb{N}} a(m)t^{2(\delta-m)} \in \mathcal{S}_\lambda$. Thus σ is a linear bijection. The $\mathfrak{k}$-linearity of σ is consequence of $Y.t^\beta = 0$ for all β and $Y \in \mathfrak{k}$. The second assertion then follows from Proposition 11.9(2). □

By Lemma 10.2, M_r is a $\mathfrak{g}$-module so it is natural to ask whether $\sigma\circ\mathrm{ev}_0$ is actually $\mathfrak{g}$-linear in this corollary. This is true and forms the main result of the section.

Theorem 11.11. *The map $\sigma\circ\mathrm{ev}_0 : N_\lambda \to M_r$ is an isomorphism of $\mathfrak{g}$-modules. Equivalently,*

$$\sigma(\mathrm{ev}_0(Y.\bar{f})) = \tau(Y).g \quad \textit{for all } Y \in \mathfrak{g} \textit{ and } g = \sigma(\mathrm{ev}_0(\bar{f})) \in M_r, \textit{ for } \bar{f} \in N_\lambda.$$

Proof. Throughout the proof we write $\bar{f} = \sum_{j=0}^{r-1} f_j\bar{\mathtt{Q}}^j$ with $f = \sum_{j=0}^{r-1} f_j\mathtt{Q}^j$ as in (11.8); thus $\mathrm{ev}_0(\bar{f}) = f_0 \in \mathcal{S}_\lambda$ by Proposition 11.9. Let $Y \in \mathfrak{g}$ and recall that the derivation Y satisfies $Y(\mathtt{Q}) = 0$. Thus, $Y.f = \sum_j (Y.f_j)\mathtt{Q}^j$ and it follows from the definition of ev_0 that $\mathrm{ev}_0(Y.\bar{f}) = \mathrm{ev}_0(Y.f_0)$. Using this, the theorem is then obviously equivalent to proving:

$$\sigma(\mathrm{ev}_0(Y.f_0)) = \tau(Y).\sigma(f_0) \quad \text{for all } Y \in \mathfrak{g} \text{ and } f_0 \in \mathcal{S}_\lambda. \tag{11.12}$$

By Corollary 11.10, $f_0 = \sum_{m\in\mathbb{N}} a(m)t^{2(\delta-m)}$, with $a(m) \in A(m)$ for all m, and $\sigma(f_0) = \sum_m a(m)$. For simplicity we set $\gamma(m) = 2(\delta - m)$. Thus

$$Y.f_0 = \sum_m (Y.a(m))t^{\gamma(m)} + \sum_m a(m)(Y.t^{\gamma(m)}) = \sum_m (Y.a(m))t^{\gamma(m)} + \sum_m \gamma(m)a(m)(Y.t)t^{\gamma(m)-1}$$

while $\tau(Y).\sigma(f_0) = \sum_m \tau(Y).a(m)$. As usual, in order to prove that (11.12) holds, it suffices to prove it when Y is a root vector in $\mathfrak{g}$, hence a (scalar multiple) of some y_α from the Chevalley system given in Proposition 2.4. As in the proof of Theorem 10.6, we will consider the three cases $Y \in \mathfrak{k}$, $Y \in \mathfrak{r}^-$ and $Y \in \mathfrak{r}^+$ separately.

Suppose first that $Y = y_\alpha \in \mathfrak{k}$ for $\alpha \in \Phi_1$. We may assume that $Y = E_{ab}$ with $a, b \in \{0, \pm 2, \dots, \pm\ell\}$. Then $E_{ab}.t = 0$ and $\tau(Y) = \mathcal{F}(\widetilde{\psi}(\vartheta(Y))) = \mathcal{F}(\widetilde{\psi}(E_{ba})) = \mathcal{F}(E_{ba}) = E_{ab} = Y$, by(10.8). Thus (11.12) is equivalent to the $\mathfrak{k}$-linearity of σ, which follows from Corollary 11.10.

Suppose next that $Y = y_\alpha \in \mathfrak{r}^-$, where either $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$ or $\alpha = -\varepsilon_1$. Hence, by Proposition 2.4, $Y = y_\alpha$ is either equal to $E_{\mp b,1}$ when $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$ and $b \in \{2, \dots, \ell\}$, or $Y = \sqrt{2}E_{0,1}$ when $\alpha = -\varepsilon_1$. By the definition of ϑ and $\widetilde{\psi}$ we have $\widetilde{\psi}(\vartheta(Y)) = x_{-\alpha} \in \mathfrak{r}^+$. Using Lemma 4.3, this root vector is either V_{-b} when $\alpha = -(\varepsilon_1 - \varepsilon_b)$, or V_b when $\alpha = -(\varepsilon_1 + \varepsilon_b)$ or $\sqrt{2}V_0$ when $\alpha = -\varepsilon_1$. Recall that $\mathcal{F}(V_a) = Q_a = \mathtt{F}\partial_{U_a} - U_{-a}I(d)$. Therefore, up to the scalar $\sqrt{2}$, we may assume that $Y = E_{j1}$ and $\tau(Y) = Q_{-j}$ with $j \in \{0, \pm 2, \dots, \pm\ell\}$. Since $E_{j1} = \dfrac{U_j}{2t}\partial_t - U_{-1}\partial_{U_{-j}}$ and $a(m) \in A = \mathbb{C}[U_0, U_{\pm 2}, \dots, U_{\pm\ell}]$, we obtain

$$E_{j1}.f_0 = -\sum_m U_{-1}\partial_{U_{-j}}(a(m))t^{\gamma(m)} + \sum_m \frac{\gamma(m)}{2}U_j a(m)t^{\gamma(m)-2}.$$

Recalling from (2.1) that $U_{-1} = \dfrac{1}{U_1}(\mathtt{Q} - \mathtt{F}) = t^{-2}(\mathtt{Q} - \mathtt{F})$, it follows that

$$\begin{aligned} E_{j1}.f_0 &= -\sum_m (\mathtt{Q} - \mathtt{F})\partial_{U_{-j}}(a(m))t^{\gamma(m)-2} + \sum_m \frac{\gamma(m)}{2}U_j a(m)t^{\gamma(m)-2} \\ &= \sum_m \Big(\frac{\gamma(m)}{2}U_j a(m) + \mathtt{F}\partial_{U_{-j}}(a(m))\Big)t^{\gamma(m)-2} - \mathtt{Q}\sum_m \partial_{U_{-j}}(a(m))t^{\gamma(m)-2}. \end{aligned}$$

Thus, as $\gamma(m) = 2(\delta - m) = -2(d + m)$, we have

$$\mathrm{ev}_0(E_{j1}.f_0) = \sum_m \big(-(d+m)U_j + \mathtt{F}\partial_{U_{-j}}\big)a(m)t^{\gamma(m)-2}$$

and $\sigma(\mathrm{ev}_0(E_{j1}.f_0)) = \sum_m \big(-(d+m)U_j + \mathtt{F}\partial_{U_{-j}}\big)(a(m))$. Since $I(d)(a(m)) = (m+d)a(m)$, observe that

$$Q_{-j}.\sigma(f_0) = Q_{-j}.\sum_m a(m) = \sum_m Q_{-j}.a(m) = \sum_m \big(\mathtt{F}\partial_{U_{-j}}(a(m)) - (d+m)U_j a(m)\big)$$

and we obtain $\sigma(\mathrm{ev}_0(E_{j1}.f_0)) = Q_{-j}.\sigma(f_0)$, as required.

Finally, suppose that $Y = y_\alpha \in \mathfrak{r}^+$, where either $\alpha = \varepsilon_1 \pm \varepsilon_b$ or $\alpha = \varepsilon_1$. From Lemma 4.3 we see that we may take $Y = E_{1b}$; using Lemma 4.3 and the definition of $\mathcal{F}$, we may assume that $Y = E_{1j}$, whence $\tau(Y) = \partial_{U_j}$ with $j \in \{0, \pm 2, \dots, \pm\ell\}$. Recall that $E_{1j} = U_1\partial_{U_j} - U_{-j}\partial_{U_{-1}} = t^2\partial_{U_j} - U_{-j}\partial_{U_{-1}}$; from $E_{1j}(t) = 0$ and $\partial_{U_{-1}}(a(m)) = 0$, we get that $E_{1j}.f_0 = \sum_m \partial_{U_j}(a(m))t^{\gamma(m)+2}$. On the other hand we have $\partial_{U_j}.\sigma(f_0) = \sum_m \partial_{U_j}(a(m))$ and it follows that $\sigma(\mathrm{ev}_0(E_{1j}.f_0)) = \sum_m \partial_{U_j}(a(m)) = \partial_{U_j}.\sigma(f_0)$, as desired. $\square$

In summary, by combining Corollary 10.12 with Theorem 11.11, we have the following relationship between the three main $\mathfrak{g}$-modules appearing in this paper: the module $N(\lambda) = R_r$ of regular functions on the scheme $\{\mathtt{F}^r = 0\} \subset \mathbb{C}^n$, the module M_r of higher harmonic polynomials on $\mathbb{C}^n$, and the module N_λ of harmonic conformal densities of weight $-\frac{n}{2} + r$ on $\mathbb{C}^{n+2}$.

Corollary 11.13. *There exist isomorphisms of $\mathfrak{g}$-modules:*

$$N(\lambda) \;\cong\; N_\lambda^\vee \quad \textit{or, equivalently,} \quad R_r \cong M_r^\vee.$$

Moreover:

(1) *if n is odd or if n is even with $r < \frac{n}{2}$, then $R_r \cong L(\lambda) \cong M_r$;*

(2) *if n is even with $r \geq \frac{n}{2}$, then R_r has a simple socle Z_r isomorphic to $L(\mu)$, where μ is given by (4.14), and the quotient $Q_r = R_r/Z_r$ is isomorphic to the simple finite dimensional module $L(\lambda)$.*

(3) *if n is even with $r \geq \frac{n}{2}$, then M_r has a simple (finite dimensional) socle $Q_r^\vee \cong L(\lambda)$ and the quotient $Z_r^\vee = M_r/Q_r^\vee \cong L(\mu)$.* $\square$

We conclude this section with remarks about the module N_λ and its annihilator.

Remark 11.14. (1) Recall that $\delta = -d = -\frac{n}{2} + r$. Since $t^{2\delta} \in \mathcal{H}(S)(\delta)$, the class $e_\lambda = [t^{2\delta}] \in \overline{S}(\delta)$ belongs to N_λ. It is easy to see that e_λ is a highest weight vector in N_λ, with weight $\lambda = -d\varepsilon_1 = \delta\varepsilon_1$. In more detail:

(i) $E_{ab}.t^{2\delta} = 0 = E_{c,0}.t^{2\delta}$ for $1 \leq a < b \leq \ell$ and $1 \leq c \leq \ell$. Thus, by Proposition 2.4, $\mathfrak{n}^+.e_\lambda = 0$.

(ii) $E_{11}.t^{2\delta} = \frac{U_1}{2t}\partial_t(t^{2\delta}) = \delta t^{2\delta}$ and $E_{jj}.t^{2\delta} = 0$, for $2 \leq j \leq \ell$; showing that e_λ has weight $\lambda = \delta\varepsilon_1$.

Therefore, if n is even with $r < \frac{n}{2}$ or if n is odd, one gets $N_\lambda = U(\mathfrak{g}).e_\lambda$.

Assume now that n is even with $r \geq \frac{n}{2}$. Then, $F^\vee = U(\mathfrak{g}).e_\lambda \cong L(\lambda)$ is finite dimensional. The quotient $N_\lambda/F^\vee$ is isomorphic to $L(\mu)$ and, by similar computations, it is easily seen that the class $e_\mu \in N_\lambda/F^\vee$ of the element $U_2^{\delta+1}t^{-2} = U_2^{\delta+1}U_1^{-1} \in \mathcal{H}(S)(\delta)$ is a highest weight vector of weight μ. Thus, $N_\lambda/F^\vee = U(\mathfrak{g}).e_\mu \cong L(\mu)$. Through the isomorphism $N_\lambda \xrightarrow{\sim} M_r$, the elements e_λ and e_μ correspond, respectively, to 1 and $U_2^{\delta+1}$ (which, in turn, gives the element ξ^m in the proof of Theorem 4.13).

(2) By Corollaries 11.13 and 4.17, the annihilator of N_λ is equal to the primitive ideal J_r. Let $\mathcal{U}$ denote the subalgebra of $\mathcal{D}(B)$ generated by the elements $E_{ij} \in \mathfrak{g}$. Then, $\mathcal{U}$ is a factor of the enveloping algebra $U(\mathfrak{g})$ and $\overline{\mathcal{U}} = U(\mathfrak{g})/J_r$ is a primitive quotient.

When $r = 1$, $\mathcal{U}$ and $\overline{\mathcal{U}}$ are, respectively, complexifications of the "off-shell higher-spin algebra" and "on-shell higher-spin algebra" as defined in [Vas, §3], [Bek1, §3.1.1] and [Bek2, §4.1]. In particular, on the Minkowski space-time (i.e. in the real case $\mathbb{R}^{p,q} = \mathbb{R}^{n-1,1}$ as in §9) the algebra of symmetries $\mathscr{S}(\Box_{n-1})$ is isomorphic to the "on-shell higher-spin algebra" of [Bek1, §3.1.3, Corollary 3].

Appendix

Here we provide a proof of Lemma 10.5 and repeat (a minor variant of) the statement of the result for the reader's convenience.

Lemma 12.1. (1) *Set $\mathcal{I}_r = \mathtt{F}^r A$. The bilinear form $\langle\,|\,\rangle$ from Definition 10.4 satisfies the following properties.*

(i) $\langle A(p) \mid A(q)\rangle = 0$ *for* $p \neq q$ *in* $\mathbb{N}$ *and* $\langle a \mid \phi\rangle = \mathcal{F}(a)(\phi) = \mathcal{F}(\phi)(a)$ *for all* $a, \phi \in A(p)$.

(ii) $\langle \mid \rangle$ *is symmetric non degenerate and* $\mathfrak{k}$*-invariant.*

(iii) $L^{\perp\perp} = L$ *for any graded subspace* $L \subset A$.

(iv) $M_r^{\perp} = \mathcal{I}_r$ *and* $\mathcal{I}_r^{\perp} = M_r$.

(2) *The pairing* $\langle \mid \rangle$ *induces a non degenerate* $\mathfrak{k}$*-invariant symmetric pairing* $\langle \mid \rangle : R_r \times M_r \longrightarrow \mathbb{C}$.

Proof. (1) (i,ii) With standard notation, let $U^{\mathbf{j}}, U^{\mathbf{k}}$ be monomials in the U_j (where $\mathbf{j}, \mathbf{k}$ are multi-indices). Then, $\mathcal{F}(U^{\mathbf{j}}) = \partial_U^{\mathbf{j}}$ and $\langle U^{\mathbf{j}} \mid U^{\mathbf{k}}\rangle = \partial_U^{\mathbf{j}}(U^{\mathbf{k}}) = c_{\mathbf{j}}\, \delta_{\mathbf{j},\mathbf{k}}$ for some $c_{\mathbf{j}} \in \mathbb{N}^*$. This proves (i) and implies that the pairing is symmetric non degenerate.

Let $K \cong \mathrm{SO}(\mathtt{F}) \subset \mathrm{Aut}(A)$ be the algebraic group such that $\mathrm{Lie}(K) = \mathfrak{k}$. It is well known (and easy to see) that if $g \in K$, then $\mathcal{F}(g.a) = g.\mathcal{F}(a)$ for all $a \in A$. Hence, $\langle g.a \mid g.\phi\rangle = (g.\mathcal{F}(a))(g.\phi)(0) = [g.\mathcal{F}(a)(\phi)](0) = \mathcal{F}(a)(\phi)(0) = \langle a \mid \phi\rangle$. Thus, $\langle \mid \rangle$ is K-invariant and therefore $\mathfrak{k}$-invariant.

(iii) By (i) we have perfect pairings $\langle \mid \rangle_m : A(m) \times A(m) \to \mathbb{C}$ for all $m \in \mathbb{N}$. For a subspace $P \subset A(m)$, denote by P° its orthogonal w.r.t $\langle \mid \rangle_m$; thus $P^\circ = P^\perp \cap A(m)$. As $A(m)$ is finite dimensional, one has $P^{\circ\circ} = P$. It is easily seen that $L^\perp$ is graded and that $L^\perp(m) = L(m)^\circ$ for all m. Let $a \in L^{\perp\perp}$ and write $a = \sum_m a(m)$ with $a(m) \in A(m)$. We have $\langle a \mid p\rangle = \langle a(k) \mid p\rangle = 0$ for all $p \in L(k)^\circ$. Hence $a(k) \in L(k)^{\circ\circ} = L(k)$ and we get $a \in L$.

(iv) Since $\mathcal{F}(\mathtt{F}^r) = 2^{-r}\Delta_1^r$, it is clear that $M_r \subset \mathcal{I}_r^\perp$. Conversely, if $\phi \in \mathcal{I}_r^\perp$ and $U^{\mathbf{j}} \in A$, one has $\partial_U^{\mathbf{j}}[\Delta_1^r(\phi)](0) = 2^r\langle \mathtt{F}^r U^{\mathbf{j}} \mid \phi\rangle = 0$. It then follows from (i) that $\Delta_1^r(\phi) = 0$ and $\phi \in M_r$. Thus $\mathcal{I}_r^\perp = M_r$.
(2) Observe that $\mathcal{I}_r$ and M_r are $\mathfrak{k}$-invariant subspaces of A (see Corollary 11.10). Since $R_r = A/\mathcal{I}_r$, the claim therefore follows from (1)(ii) and (1)(iv). □

References

[BG] V. G. Bagrov and D. M. Gitman, *Exact Solutions of Relativistic Wave Equations,* Mathematics and its Applications (Soviet Series), **39** Kluwer Academic Publishers Group, Dordrecht, 1990.

[BS] V. G. Bagrov, B. F. Samsonov, A. V. Shapovalov, and A. V. Shirokov, *Identities on solutions of the wave equation in the enveloping algebra of the conformal group.* (Russian) Teoret. Mat. Fiz., **83** (1990), n° 1, 14-22; translation in Theoret. and Math. Phys. **83** (1990), n° 1, 347-353.

[Bek1] X. Bekaert, *Comments on higher-spin symmetries*, Int. J. Geom. Methods Mod. Phys. **6** (2009), 285-342.

[Bek2] X. Bekaert, *Singletons and their maximal symmetry algebras*, in "6th Mathematical physics meeting: Summer School and Conference on Modern Mathematical Physics", 2010. Belgrade, Bulgaria, 2012.

[Bek3] X. Bekaert, *The many faces of singletons*, Physics AUC, vol. **21**-Sp Issue (2011), 154-170.

[BG] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach*, Nuclear Physics B, Volume **876**-2 (2013), 667-714.

[BGG] J.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Differential operators on the cubic cone*, Russian Math. Surveys, **27** (1972), 466-488.

[BZ] B. Binegar and R. Zierau, *Unitarization of a singular representation of $SO(p,q)$*, Commun. Math. Phys. **138** (1991), 245-258.

[Bou1] N. Bourbaki, *Algèbre. Chapitre 8*, Hermann, Paris, 1973.

[Bou2] N. Bourbaki, *Groupes et Algèbres de Lie. Chapitres 4, 5, 6*, Masson, Paris, 1981.

[Bou3] N. Bourbaki, *Groupes et Algèbres de Lie. Chapitres 7, 8*, Hermann, Paris, 1990.

[BKM] C. P. Boyer, E. G. Kalnins and W. Miller, Jr., *Symmetry and separation of variables for the Helmholtz and Laplacian equations*, Nagoya Math. J. **60** (1976), 35-80.

[DFG] H. Dietrich, P. Faccin, and W. de Graaf, *Computing with real Lie algebras: Real forms, Cartan decompositions and Cartan subalgebras*, J. Symbolic Computation **56** (2013), 27-45.

[Dir1] P.A.M. Dirac, *The electron wave equation in de-Sitter space*, Ann. of Math. **36** (1935), 657-669.

[Dir2] P.A.M. Dirac, *Wave equations in conformal space*, Ann. of Math. **37** (1936), 429-442.

[Eas] M.G. Eastwood, *Higher symmetries of the Laplacian*, Ann. of Math. **161** (2005), 1645-1665.

[EG] M.G. Eastwood and C.R. Graham, *Invariants of conformal densities*, Duke Math. J. **63** (1991), 633-671.

[ESS] M.G. Eastwood, P. Somberg, and V. Souček, *The uniqueness of the Joseph ideal for the classical groups*, arXiv:math/0512296v1, 2005.

[EL] M.G. Eastwood and T. Leistner, *Higher symmetries of the square of the Laplacian*, in "Symmetries and overdetermined systems of partial differential equations", IMA Vol. Math. Appl., Vol. **144**, Springer, New York, 2008, 319-338.

[GW] R. Goodman and N.R. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge University Press, Cambridge, 1998.

[GS] A.R. Gover and J. Šilhan, *Higher symmetries of the conformal powers of the Laplacian on conformally flat manifolds*, J. Math. Phys. **53**, 032301 (2012), 26 pages, arXiv:0911.5265.

[GJMS] C.R. Graham, R. Jenne, L.J. Mason, and G.A. Sparling, *Conformally invariant powers of the Laplacian, I: Existence*, J. London Math. Soc. **46** (1992), 557-565.

[EGA] A. Grothendieck, *Éléments de Géométrie Algébrique IV*, Inst. Hautes Études Sci., Publ. Math., N° **32**, 1967.

[Hel] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics **34**, AMS, Providence, 2001.

[How1] R. Howe, *Dual pairs in physics: Harmonic oscillators, photons, electrons, and singletons*, in *Lectures in Applied Mathematics*, Vol. **21**, 179-207, Amer. Math. Soc., 1985.

[How2] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539-570.

[HSS] M. Hunziker, M.R. Sepanski and R.J. Stanke, *The minimal representation of the conformal group and classical solutions to the wave equation*, J. Lie Theory **22** (2012), 301-360.

[Jan] J.C. Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*, Springer-Verlag, Berlin, 1983.

[Jos1] A. Joseph, *The minimal orbit in a simple Lie algbera and its associated ideal*, Ann. Sci. École Normale Sup., **9** (1976), 1-29.

[Jos2] A. Joseph, *Kostant's Problem, Goldie Rank and the Gelfand-Kirillov conjecture*, Invent. Math. **56** (1980), 191-213.

[Jos3] A. Joseph, *On the associated variety of a primitive ideal*, J. Algebra, **93** (1985), 509-523.

[Jos4] A. Joseph, *A surjectivity theorem for rigid highest weight modules*, Invent. Math. **92** (1988), 567-596.

[Jos5] A. Joseph, *Rings which are modules in the Bernstein-Gelfand-Gelfand $\mathcal{O}$ category*, J. Algebra, **113** (1988), 110-126.

[JS] A. Joseph and J.T. Stafford, *Modules of $\mathfrak{k}$-finite vectors over semi-simple Lie algebras*, Proc. London Math. Soc., **49** (1984), 361-384.

[KM] T. Kobayashi and G. Mano, *The Schrödinger model for the minimal representation of the indefinite orthogonal group* $\mathrm{O}(p,q)$, Mem. Amer. Math. Soc., **213**, 2011.

[KO] T. Kobayashi and B. Ørsted, *Analysis on the minimal representation of* $\mathrm{O}(p,q)$ *III. Ultrahyperbolic equations on* $\mathbb{R}^{p-1,q-1}$, Advances in Math., **180** (2003), 551-595.

[Lev] T. Levasseur, *La dimension de Krull de* $U(\mathfrak{sl}(3))$, J. Algebra **102** (1986), 39-59.

[LSS] T. Levasseur, S.P. Smith and J.T. Stafford, *The minimal nilpotent orbit, the Joseph ideal, and differential operators*, J. Algebra **116** (1988), 480-501.

[LS] T. Levasseur and J.T. Stafford, *Rings of differential operators on classical rings of invariants*, Mem. Amer. Math. Soc., **412**, 1989.

[MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Grad. Texts in Math. Vol. **30**, Amer. Math. Soc., Providence, RI, 2000.

[MaR] G. Maury and J. Renaud, *Ordres maximaux au sens de K. Asano*, Lecture Notes in Math., **341**, Springer-Verlag, Berlin, 1973.

[Mat] H. Matsumura, *Commutative Algebra (second edition)*, Math. Lecture Notes Series **56**, Benjamin Publishing Company, Reading, Mass., 1980.

[Mic] J.-P. Michel, *Higher symmetries of the Laplacian via quantization*, Ann. Inst. Fourier **64** (2014), 1581-1609.

[Mil] W. Miller, Jr., *Symmetry and Separation of Variables*, Encyclopedia of Mathematics and its Applications, Vol. 4. Addison-Wesley Publishing Co., Reading, Mass., 1977.

[Mus] I. Musson, *Rings of differential operators and zero divisors*, J. Algebra **125** (1989), 489-501.

[ShS] A.V. Shapovalov and I.V. Shirokov, *On the symmetry algebra of a linear differential equation*, Teoret. Mat. Fiz. **92** (1992), no. 1, 3-12; translation in Theoret. and Math. Phys. **92** (1992), no. 1, 697- 703 (1993).

[SmS] S.P. Smith and J.T. Stafford, *Differential operators on an affine curve*, Proc. London Math. Soc., **56** (1988), 229-259.

[Str] P. Somberg, *Deformations of quadratic algebras, the Joseph ideal for classical Lie algebras, and special tensors*, in "Symmetries and overdetermined systems of partial differential equations", IMA Vol. Math. Appl., Vol. **144**, Springer, New York, 2008, 527-536.

[Vas] M.A. Vassiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, M.A. Phys. Lett. **B567** (2003), 139-151.

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