Primitive ideals of $\mathrm{C}_q[SL(n)]$

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Abstract

The primitive ideals of the quantum group $\mathbf{C}_q[SL(n)]$ are classified in the case where q is a non-zero complex number which is not a root of unity. It is shown that the orbits in Prim $\mathbf{C}_q[SL(n)]$ under the action of the character group $H \cong (\mathbf{C}^*)^{n-1}$ are parameterized naturally by $W \times W$ where W is the associated Weyl group. It is shown that there is a natural one-to-one correspondence between primitive ideals of $\mathbf{C}_q[SL(n)]$ and symplectic leaves of the associated Poisson algebraic group $SL(n, \mathbf{C})$.

Let q be a non-zero complex number which is not a root of unity. In [3] the authors classified the primitive ideals of the quantum group $\mathbf{C}_q[SL(3)]$, showing that there is a natural bijection between the primitive ideals of $\mathbf{C}_q[SL(3)]$ and the symplectic leaves of $SL(3, \mathbf{C})$ for the associated Poisson group structure. Here we generalize this result to the quantum group $\mathbf{C}_q[SL(n)]$. Denote by W the associated Weyl group and let $l(w)$ be the length of the element $w \in W$. Let H be the usual maximal torus of $SL(n)$. Then $SL(n)$ has a natural H-invariant Poisson structure and H acts by left translation on the set $Symp\, SL(n)$ of symplectic leaves of $SL(n)$. The H-orbits in $Symp\, SL(n)$ are parameterized by the double Weyl group $W \times W$. For more details the reader is referred to the appendix of [3] where a complete description of the symplectic leaves is given using results of Semenov-Tian-Shansky and Lu and Weinstein (see section four for a definition of $s(w)$).

Theorem 1) *Symp* $SL(n) \cong \bigsqcup_{w \in W \times W} Symp_w SL(n)$.

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2) *For each* $w \in W \times W$, $Symp_w SL(n)$ *is a non-empty* H*-orbit. If* $\mathcal{A}_{\dot{w}} \in Symp_w SL(n)$, *then* $H/Stab_H \mathcal{A}_{\dot{w}}$ *is a torus of rank equal to* $n-1-s(w)$.

3) *The dimension of* $\mathcal{A}_{\dot{w}}$ *is* $l(w)+s(w)$.

The group H occurs again in the quantum case as the character group which acts naturally as automorphisms on $\mathbf{C}_q[G]$n. The primitive spectrum $Prim\, \mathbf{C}_q[SL(n)]$ therefore decomposes into a union of H-orbits. Following ideas of Soibelman [7, 8], we define for each $w \in W \times W$ a locally closed H-invariant subset $Prim_w\, \mathbf{C}_q[SL(n)]$ of $Prim\, \mathbf{C}_q[SL(n)]$. The main result of this paper is the following theorem which was conjectured in [3].

Theorem 4.2 1) *Prim* $\mathbf{C}_q[SL(n)] \cong \bigsqcup_{w \in W \times W} Prim_w \mathbf{C}_q[SL(n)]$.

2) *For each* $w \in W \times W$, $Prim_w \mathbf{C}_q[SL(n)]$ *is a non-empty* H*-orbit. If* $P_{\dot{w}} \in Prim_w \mathbf{C}_q[SL(n)]$, *then* $H/Stab_H P_{\dot{w}}$ *is a torus of rank equal to* $n-1-s(w)$.

3) *The Gelfand-Kirillov dimension of* $\mathbf{C}_q[SL(n)]/P_{\dot{w}}$ *is* $l(w)+s(w)$.

Since the proof follows the geometry closely, it is useful for the reader to understand in a little more detail the d escription of the symplectic leaves of $G = SL(n, \mathbf{C})$. Let $D = G \times G$, identify G with the diagonal subgroup of D and let G_r be the dual group. Denote by p the natural projection $G \to D/G_r$. The symplectic leaves of G are precisely the connected components of the inverse images of the left G_r-orbits in D/G_r. Set $\Gamma = \ker p$ and $\bar{G} = p(G)$. Then Γ is a finite subgroup of H and $\bar{G} = G/\Gamma$ is an open subset of D/G_r. For each $w \in W \times W$, let $\mathcal{C}_w$ be the image of the corresponding Bruhat cell of D in D/G_r. Let $\mathcal{B}_w = \mathcal{C}_w \cap \bar{G}$ and let $\mathcal{A}_w = p^{-1}(\mathcal{C}_w)$. Since the G_r-orbits in $\mathcal{C}_w$ form a single H-orbit, it follows that the symplectic leaves in $\mathcal{A}_w$ also form a single H-orbit. The algebras A_w, B_w and C_w defined below may be considered as quantizations of the algebras of functions on $\mathcal{A}_w$, $\mathcal{B}_w$ and $\mathcal{C}_w$ respectively. The role of the hamiltonian vector fields in the quantized situation is played by the adjoint action of $\mathbf{C}_q[SL(n)]$ on itself. For this reason, the key result is the description of the adjoint action given in section three.

Much of the inspiration for this work came from work of Soibelman [7, 8]. When q is real, $q \neq 1$, the quantum group $\mathbf{C}_q[G]$ together wit h a natural involution * can be viewed as a deformation of $\mathbf{C}[K]$, the algebra of functions on a maximal compact subgroup K of G. Soibelman showed that the irreducible unitary representations of $\mathbf{C}_q[K]$ correspond to the symplectic leaves of K. A number of key definitions and results are taken from these papers.

These results were first announced at the Symposium on Noncommutative Rings in Durham in July 1992. More recently Joseph has generalized the main result to the case of an arbitrary simply connected semi-simple group.

1 Preliminaries

Denote by $\mathbf{g}$ and G the Lie algebra $sl(n, \mathbf{C})$ and the Lie group $SL(n, \mathbf{C})$ respectively. Let q be a non-zero complex number which is not a root of unity and let $\mathcal{Q}$ be the subgroup of $\mathbf{C}^*$ generated by q. Denote by $\mathbf{C}_q[G]$ the usual quantization of $\mathbf{C}[G]$ and by $U_q(\mathbf{g})$ the quantized enveloping algebra. The notation used for the weights and roots of $\mathbf{g}$ is as in

Bourbaki [1]. All other undefined notation will be as in [3]. The Weyl group W can be identified in a standard way with the symmetric group S_n and we shall often make this identification.

Let $L(\varpi_i)$ be the finite dimensional simple $U_q(\mathbf{g})$-module corresponding to the fundamental weight ϖ_i. Then $L(\varpi_i)$ has a basis of the form $\{v_{w\varpi_i}\}_{w\in W}$ where each $v_{w\varpi_i}$ has weight $w\varpi_i$. Let $l_{-w\varpi_i}$ be a dual basis of $L(\varpi_i)^*$. For $y,t \in W$, the elements $c^{\varpi_i}_{-y\varpi_i,t\varpi_i}$ of $\mathbf{C}_q[G]$ are defined by:

$$\forall u \in U_q(\mathbf{g}), \quad c^{\varpi_i}_{-y\varpi_i,t\varpi_i}(u) = l_{-y\varpi_i}(uv_{t\varpi_i}).$$

We set $c^+_{i,y} = c^{\varpi_i}_{-y\varpi_i,\varpi_i}$ and $c^-_{i,y} = c^{\varpi_{n-i}}_{-yw_0\varpi_{n-i},w_0\varpi_{n-i}}$ where w_0 is the longest element of W.

For each $i \in \{1,\dots,n-1\}$, we define a relation on W by $y \leq_i w$ if and only if $y\varpi_i \geq w\varpi_i$. We say $y =_i w$ if and only if $y^{-1}w\varpi_i = \varpi_i$. If we identify W with S_n then this partial order has the following interpretation. If $I = \{1,\dots,i\}$, then $y \leq_i w$ if and only if $yI \leq wI$ (where $yI \leq wI$ means that if $yI = \{y_1 < \dots < y_i\}$ and $wI = \{w_1 < \dots < w_i\}$, then $y_i \leq w_i$ for all i). Set $W_i = Stab_W(\varpi_i)$.

The following proposition is proved as in [8, Proposition 3.2]. It is a consequence of the isomorphism $L(\varpi_i) \otimes L(\varpi_j) \cong L(\varpi_j) \otimes L(\varpi_i)$ given by the universal R-matrix.

Proposition 1.1 *Let ϖ_i, ϖ_j be fundamental weights and $x, y, t \in W$. Then*

$$c^{\varpi_i}_{-y\varpi_i,t\varpi_i} c^{\varpi_j}_{-x\varpi_j,\varpi_j} = (q^2)^{(t\varpi_i,\varpi_j)-(x\varpi_j,y\varpi_i)} c^{\varpi_j}_{-x\varpi_j,\varpi_j} c^{\varpi_i}_{-y\varpi_i,t\varpi_i} + \sum_{\substack{u\in W \\ x<_j ux, uy<_i y}} g_u(q) c^{\varpi_j}_{-ux\varpi_j,\varpi_j} c^{\varpi_i}_{-uy\varpi_i,t\varpi_i}$$

for some $g_u(q) \in \mathbf{C}$.

Definition Let $w = (w_+, w_-) \in W \times W$. Set $I_w = \langle c^\varepsilon_{i,y} \mid y \not\leq_i w_\varepsilon \rangle$ and let $\mathcal{E}_w = \{c^+_{i,w_+}, c^-_{i,w_-} \mid i = 1,\dots,n-1\}$.

Theorem 1.2 *Let $P \in Spec\ \mathbf{C}_q[G]$. Then there exists a unique $w \in W \times W$ such that $P \supseteq I_w$ and $P \cap \mathcal{E}_w = \emptyset$.*

Proof. First observe that for all $i = 1,\dots,n-1$, there exists a $w_i \in W$ such that $c^+_{i,w_i} \notin P$ but $c^+_{i,y} \in P$ for all $y \not\leq_i w_i$. To see this let w and w' be two distinct elements of $\{y \in W \mid c^+_{i,y} \notin P\}$ which are maximal for $\leq_i$. Suppose that $w \neq_i w'$. It follows from Proposition 1.1 that $c^+_{i,w}$ and $c^+_{i,w'}$ are normal and regular modulo P. Apply Proposition 1.1 to these elements and notice that the exponent of q^2 is symmetric in w and w'. This implies that $(\varpi_i, \varpi_i) = (w\varpi_i, w'\varpi_i)$. Hence $w =_i w'$. Thus we may take w_i to be any maximal element of $\{y \in W \mid c^+_{i,y} \notin P\}$.

The same remark as above concerning the symmetry in w_i and w_j in Proposition 1.1 shows that

$$(\varpi_i, \varpi_j) = (w_i\varpi_i, w_j\varpi_j), \text{ for all } 1 \leq i,j \leq n-1.$$

It is easy to see that this condition is equivalent to the existence of a unique element $w_+ \in W$ such that $w_+ =_i w_i$ for all i.

A similar argument produces an analogous element w_-. The element $w = (w_+, w_-)$ is then the unique element such that $P \supseteq I_w$ and $P \cap \mathcal{E}_w = \emptyset$. $\square$

Corollary 1.3 *Let* $Spec_w\mathbf{C}_q[G] = \{P \in Spec\ \mathbf{C}_q[G] \mid P \supseteq I_w \text{ and } P \cap \mathcal{E}_w = \emptyset\}$. *Then* $Spec\ \mathbf{C}_q[G] = \bigsqcup_{w \in W\times W} Spec_w\mathbf{C}_q[G]$.

Let E_w be the multiplicatively closed set generated by the images of the elements of $\mathcal{E}_w$ in $\mathbf{C}_q[G]/I_w$. Since the elements of E_w are normal we may localise with respect to these elements. Denote by A_w the localised algebra $(\mathbf{C}_q[G]/I_w)_{E_w}$. The analysis of Spec $\mathbf{C}_q[G]$ reduces to the analysis of Spec A_w. Recall that $A_w \neq 0$ for all $w \in W \times W$ [3, Theorem 2.2.2].

Another important relation between the $c^+_{i,y}$'s is the following.

Lemma 1.4 *Let* $y, y' \in W$ *and* $i, j \in \{1, \ldots, n-1\}$ *such that* $yW_i \cap y'W_j = \emptyset$. *Then there exist scalars* $a_{u,u'}$ *such that*

$$c^+_{i,y}c^+_{j,y'} = \sum_{u,u' \in W} a_{u,u'}c^+_{i,u}c^+_{j,u'}$$

and $a_{u,u'} \neq 0$ *only if* $u\varpi_i + u'\varpi_j = y\varpi_i + y'\varpi_j$ *and either* $u >_i y$ *or* $u' >_j y'$.

Proof. This is a consequence of the Plucker relations [5, 1.2]. Suppose that $i \leq j$. Then the hypothesis $yW_i \cap y'W_j = \emptyset$ is equivalent to $yI \not\subseteq y'J$ where $I = \{1, \ldots, i\}$ and $J = \{1, \ldots, j\}$. Let $t \in yI \backslash y'J$. Apply [5, 1.2] in the following situation:

$$I_1 = I,\ I_2 = J,\ J_1 = yI\backslash\{t\},\ J_2 = \emptyset,\ K = y'J \cup \{t\}.$$

In the notation of that paper, we conclude that:

$$\sum_{K' \cup K'' = K} sgn_q(J_1, K')sgn_q(K', K'')sgn_q(K'', J_2)\xi_I^{J_1 \cup K'}\xi_J^{K''} = 0$$

where $\#(K') = 1$ and $\#(K'') = j$. We may pick $u, u' \in W$ such that $J_1 \cup K' = uI$ and $K'' = u'J$. In this case $\xi_I^{J_1\cup K'} = c^+_{i,u}$ and $\xi_J^{K''} = c^+_{j,u'}$. It is easily verified that if $K' = \{t'\}$, then $t' > t \Rightarrow u >_i y$, $t = t' \Rightarrow u =_i y,\ u' =_i y'$ and $t' < t \Rightarrow u' >_j y'$. $\square$

2 The Structure of C_w^H

Recall some of the notation and basic results from section 2 of [3]. Denote by H the group of one dimensional representations of $\mathbf{C}_q[G]$. Notice that H identifies naturally with the usual maximal torus of $SL(n, \mathbf{C})$ via $\chi \mapsto (\chi(X_{ij}))$. The group H acts naturally on $\mathbf{C}_q[G]$ by $a^\chi = \sum a_{(1)}\chi(a_{(2)})$. The elements $c^\varepsilon_{i,y}$ are eigenvectors for this action and hence there is an induced action of H on A_w for all w. Denote by Γ the subgroup of H consisting of representations χ such that $\chi(X_{ii}) = \pm 1$. Set $B_w = A_w^\Gamma$. Then B_w contains the elements:

$$z^\varepsilon_{i,y} = c^\varepsilon_{i,y}(c^\varepsilon_{i,w_\varepsilon})^{-1}, \quad t_i = c^-_{i,w_-}(c^+_{i,w_+})^{-1}.$$

We define C_w to be the algebra

$$C_w = \mathbf{C}[t_i^{\pm 1}, z_{i,y}^{\varepsilon} \mid \varepsilon = \pm,\ y <_i w_\varepsilon,\ i = 1, \ldots, n-1].$$

It is shown in [3, 2.6] that there exists a normal element $d \in C_w$ such that $B_w = C_w[d^{-1}]$. The action of H restricts to an action on C_w and the fixed ring is the subalgebra generated by the z's. That is

$$C_w^H = \mathbf{C}[z_{i,y}^{\varepsilon} \mid \varepsilon = \pm,\ y <_i w_\varepsilon,\ i = 1, \ldots, n-1].$$

The monomials in the $t_i^{\pm 1}$ form a basis for C_w over C_w^H. Hence C_w is an iterated skew Laurent extension of C_w^H.

Definition Set $R^+(w) = \{\alpha \in R \mid \alpha > 0,\ w_+(\alpha) < 0\}$, $R^-(w) = \{\alpha \in R \mid \alpha < 0,\ w_-(\alpha) > 0\}$ and set $R(w) = R^+(w) \cup R^-(w)$. For $\alpha \in R$ define $j_\alpha = \min\{i \mid (\alpha, \varpi_i) \neq 0\}$ and $l_\alpha = 1 + \max\{i \mid (\alpha, \varpi_i) \neq 0\}$. For $\alpha \in R^\varepsilon(w)$ set $z_\alpha = z^{\varepsilon}_{j_\alpha, w_\varepsilon s_\alpha}$. Notice that α is completely determined by ε, j_α and l_α and that $\#R(w) = l(w)$.

Theorem 2.1 *Assume that $1 \leq j \leq n-1$, $y \leq_j w_\varepsilon$. Then for $\varepsilon = \pm$,*

$$z_{j,y}^{\varepsilon} \in \mathbf{C}[z_\alpha \mid \alpha \in R^\varepsilon(w) \text{ and } j_\alpha < j \text{ or } j_\alpha = j \text{ and } w_\varepsilon s_\alpha \geq_j y].$$

Hence $C_w^H = \mathbf{C}[z_\alpha \mid \alpha \in R(w)]$.

Proof. We prove the assertion for $z_{j,y}^+$, the proof in the other case being similar. The result is trivially true in the case $z_{1,w_+}^+ = 1$. By induction we may assume that the result is true for all $z_{i,y'}^+$ where $i < j$ or $i = j$ and $y <_j y' \leq_j w_+$. First suppose that there exists an $i < j$ such that $w_+ W_i \cap y W_j = \emptyset$. Then Lemma 1.4 implies that

$$z_{j,y}^+ = \sum_{u,u'} a_{u,u'} z_{i,u}^+ z_{j,u'}^+$$

where $y <_j u' \leq_j w_+$ and $a_{u,u'} \in \mathbf{C}$. The result then follows by induction. On the other hand suppose that $w_+ W_i \cap y W_j \neq \emptyset$ for all $i < j$. In this case we may assume that $y =_i w_+$ for all $i < j$. But then it is easily verified that

$$w_+ \varpi_j - y \varpi_j = \varepsilon_{w_+(j)} - \varepsilon_{y(j)} = w_+(\gamma)$$

where $\gamma = \varepsilon_j - \varepsilon_{w_+^{-1} y(j)} \in R^+(w)$. Thus $y =_j w_+ s_\gamma$ and $j_\gamma = j$. Hence $z_{j,y}^+ = z_\gamma$. $\square$

Definition Define a total ordering $\prec$ on $R(w)$ by the following rule: if $\alpha \in R^\varepsilon(w)$ and $\beta \in R^\eta(w)$ then

$$\beta \prec \alpha \text{ if } \begin{cases} j_\beta < j_\alpha, & \text{or} \\ j_\beta = j_\alpha,\ \eta = +, \varepsilon = -, & \text{or} \\ j_\beta = j_\alpha,\ \varepsilon = \eta,\ w_\varepsilon s_\beta >_{j_\alpha} w_\varepsilon s_\alpha. & \end{cases}$$

Notice that when $j_\beta = j_\alpha$, the condition $w_\varepsilon s_\beta >_{j_\alpha} w_\varepsilon s_\alpha$ is equivalent to $w_\varepsilon(l_\beta) > w_\varepsilon(l_\alpha)$. (This is not quite the same order as that defined in [3, Definition 3.3].)

Theorem 2.2 *Let $\alpha \succ \beta \in R(w)$. Then there exists an $a \in \mathcal{Q}$ such that $z_\alpha z_\beta - a z_\beta z_\alpha \in \mathbf{C}[z_\gamma \mid \gamma \prec \alpha]$.*

Proof. Let $j = j_\alpha$, $i = j_\beta$. First consider the case when $\alpha, \beta \in R^+(w)$. If $i = j$, then it follows from the Plucker relations that z_α and z_β quasi-commute. Suppose on the other hand that $i < j$. Set $x = w_+ s_\alpha$ and $y = w_+ s_\beta$. It follows from Proposition 1.1 that there exist $a \in \mathcal{Q}$, and $b_u \in \mathbf{C}$ such that

$$z_\alpha z_\beta - a z_\beta z_\alpha = \sum_{\substack{x <_j ux \leq_j w \\ uy <_i y}} b_u z^+_{j,ux} z^+_{i,uy}.$$

From the previous theorem we have that

$$z^+_{j,ux} \in \mathbf{C}[z_\gamma \mid \gamma \in R^+(w) \text{ and } j_\gamma < j \text{ or } j_\gamma = j \text{ and } ws_\gamma \geq_j ux],$$

and $z^+_{i,uy} \in \mathbf{C}[z_\gamma \mid j_\gamma \leq i]$. Since $j_\gamma = j$ and $ws_\gamma \geq_j ux >_j x = ws_\alpha$ implies that $\gamma \prec \alpha$, the result follows in this case. The proof in the case when α and β belong to $R^-(w)$ is similar. Now consider the case $\alpha \in R^-(w)$ and $\beta \in R^+(w)$. Set $x = w_- s_\beta$ and $y = w_+ s_\alpha$. From Proposition 1.1, we have that :

$$z_\alpha z_\beta - a z_\beta z_\alpha = \sum_{x <_i ux, uyw_0 <_{n-j} yw_0} b_u z^+_{i,ux} z^-_{j,uy}.$$

From the previous theorem, we deduce that

$$z^+_{i,ux} \in \mathbf{C}[z_\gamma \mid \gamma \in R^+(w),\ \gamma < \beta], \quad z^-_{j,ux} \in \mathbf{C}[z_\gamma \mid \gamma \in R^-(w),\ \gamma < \alpha].$$

Hence the result follows easily when α and β belong to distinct $R^\varepsilon(w)$. □

3 The Adjoint Action

Denote by X_{ij} the usual generators for $\mathbf{C}_q[G]$; that is, $X_{ij} = c^{\varpi_1}_{-(1,i)\varpi_1,(1,j)\varpi_1}$. For $J \subset \mathbf{Z}$ and $k \in \mathbf{Z}$, define $\delta_{k,J} = \begin{cases} 1 & \text{if } k \in J, \\ 0 & \text{if } k \notin J; \end{cases}$ and set $\hat{q} = q^2 - q^{-2}$.

Lemma 3.1 *Let $k, l \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$ and $y \in W$. Set $J = \{1, \ldots, j\}$.*

$$X_{kl} c^+_{j,y} = (q^2)^{\delta_{l,J}} [(q^2)^{-\delta_{k,yJ}} c^+_{j,y} X_{kl} + (\delta_{k,yJ} - 1) \sum_{a<k, a \in yJ} \hat{q}(-q^2)^{d(a,y,k)} c^+_{j,(a,k)y} X_{al}]$$

where $d(a, y, k) = \#\{x \in yJ \mid a < x < k\}$.

Proof. This follows from the identities [6, 4.5.1,5.1.2]. Let $\mathbf{C}_q[M(m)]$ be the usual quantized algebra of functions on $m \times m$ matrices and let D_{ij} be the quantum minor obtained by deleting the i-th row and the j-th column. Then the formulae in [6, 4.5.1,5.1.2] may be combined to give:

$$X_{kl}D_{rm} = (q^2)^{1-\delta_{l,m}}[(q^2)^{\delta_{k,r}-1}D_{rm}X_{kl} - \delta_{k,r}\sum_{a=1}^{k-1}\hat{q}(-q^2)^{k-a-1}D_{al}X_{al}].$$

Interpreting this formula for the $c^+_{j,y}$ yields the desired result. $\square$

The adjoint action of an element $a \in \mathbf{C}_q[G]$ on $b \in A_w$ is given by $(\mathrm{ad}\, a)b = \sum a_{(1)}bS(a_{(2)})$ where S is the antipode. We denote $\mathrm{ad}\, X_{kl}$ by $\mathrm{ad}_{k,l}$. It follows from the description of the action of Γ that B_w is invariant under the adjoint action. The following results imply in particular that C_w and C_w^H are also invariant and that the adjoint action on C_w is locally finite.

Theorem 3.2 a) *Let $\alpha \in R^+(w)$, $j = j_\alpha$, $l = l_\alpha$, and $y = w_+s_\alpha$. Then*

$$ad_{k,m}z_\alpha = \begin{cases} (q^2)^{\delta_{k,w_+J}-\delta_{k,yJ}}z_\alpha & \text{if } k = m \\ a_{k,m,y}z^+_{j,(m,k)y} & \text{if } w_+(l) \le m < k \le w_+(j),\ k \notin yJ,\ m \in yJ \\ 0 & \text{otherwise} \end{cases}$$

where $a_{k,m,y} \in \mathbf{C}^*$. In particular $ad_{w_+(j),w_+(l)}\, z_\alpha \in \mathbf{C}^*$.

b) *Let $\beta \in R^-(w)$, $j = j_\beta$, $l = l_\beta$, and $y = w_-s_\beta$. Then*

$$ad_{k,m}z_\beta = \begin{cases} (q^2)^{\delta_{m,w_-J}-\delta_{m,yJ}}z_\beta & \text{if } k = m \\ b_{k,m,y}z^-_{j,(m,k)y} & \text{if } w_-(l) \le k < m \le w_-(j),\ k \in yJ,\ m \notin yJ \\ 0 & \text{otherwise} \end{cases}$$

where $b_{k,m,y} \in \mathbf{C}^*$. In particular $ad_{w_-(l),w_-(j)}\, z_\beta \in \mathbf{C}^*$.

Proof. a) Lemma 3.1 implies that

$$c^+_{j,w_+}X_{kl} = (q^2)^{-\delta_{l,J}+\delta_{k,w_+J}}X_{kl}c^+_{j,w_+}$$

modulo I_w. Therefore we deduce from the lemma that

$$X_{kl}z^+_{j,y} = (q^2)^{\delta_{k,w_+J}}[(q^2)^{-\delta_{k,yJ}}z^+_{j,y}X_{kl} + (\delta_{k,yJ}-1)\sum_{a<k,a\in yJ}\hat{q}(-q^2)^{d(a,y,k)}z^+_{j,(a,k)y}X_{al}]$$

Since $\mathrm{ad}_{k,m}z^+_{j,y} = \sum_l X_{kl}z^+_{j,y}S(X_{lm})$ and $\sum_b X_{ab}S(X_{bd}) = \delta_{a,d}$ we deduce that

$$\mathrm{ad}_{k,m}z^+_{j,y} = \begin{cases} (q^2)^{\delta_{k,w_+J}-\delta_{k,yJ}}z^+_{j,y} & \text{if } k = m \\ -\hat{q}(q^2)^{\delta_{k,w_+J}}(-q^2)^{d(m,y,k)}z^+_{j,(m,k)y} & \text{if } m < k, m \in yJ, k \notin yJ, (m,k)y \le_j w_+ \\ 0 & \text{otherwise.} \end{cases}$$

In the case where $y = ws_\alpha$ and $j = j_\alpha$, it is easily seen that the condition $w_+(l) \le m < k \le w_+(j),\ k \notin yJ,\ m \in yJ$ is equivalent to $m < k, m \in yJ, k \notin yJ, (m,k)y \le_j w_+$. A similar argument proves part (b). $\square$

Corollary 3.3 *Let $r \in \mathbf{N}$.*
1. *If $\alpha \in R^+(w)$ and $y = w_+ s_\alpha$, then*

$$ad_{k,m} z_\alpha^r = \begin{cases} a z_\alpha^r & \text{if } k = m \\ a' z_\alpha^{r-1} & \text{if } k = w_+(j_\alpha) \text{ and } m = w_+(l_\alpha) \\ 0 & \text{unless } w_+(l_\alpha) \le m < k \le w_+(j_\alpha),\ k \notin yJ,\ m \in yJ \\ & \quad \text{or } k = m \end{cases}$$

where $a \in \mathcal{Q}$ and $a' \in \mathbf{C}^*$.

2. *If $\beta \in R^-(w)$ and $y = w_- s_\beta$, then*

$$ad_{k,m} z_\beta^r = \begin{cases} b z_\beta^r & \text{if } k = m \\ b' z_\beta^{r-1} & \text{if } k = w_-(l_\beta) \text{ and } m = w_-(j_\beta) \\ 0 & \text{unless } w_-(l_\beta) \le k < m \le w_-(j_\beta),\ k \in yJ,\ m \notin yJ \\ & \quad \text{or } k = m \end{cases}$$

where $b \in \mathcal{Q}$ and $b' \in \mathbf{C}^*$.

Proof. The proof is analogous to that of [3, 3.9.2]. □

It follows from Theorem 2.2 that the algebra C_w^H is generated by monomials of the form

$$\mathbf{z} = \prod_{\gamma \in R^+(w)} z_\gamma^{m_\gamma} \prod_{\beta \in R^-(w)} z_\beta^{m_\beta}$$

where the product is taken over the γ's according to the inverse ordering given by $\prec$ on $R^+(w)$ and the second product is taken over the β's according to the ordering $\prec$ on $R^-(w)$. Such monomials will be called *standard monomials.* In order to define an ordering on such expressions we define a new ordering on $R(w)$.

Definition. Define a new total ordering $\prec'$ in the following way: if $\alpha \in R^\varepsilon(w)$ and $\beta \in R^\eta(w)$, then

$$\alpha \succ' \beta \text{ if } \begin{cases} \varepsilon = - \quad \eta = + & \text{or} \\ \varepsilon = \eta, \quad w_\varepsilon(l_\alpha) < w_\varepsilon(l_\beta), & \text{or} \\ \varepsilon = \eta, \quad w_\varepsilon(l_\alpha) = w_\varepsilon(l_\beta), & j_\alpha < j_\beta. \end{cases}$$

The following proposition follows easily from the definition.

Proposition 3.4 *Suppose $\alpha, \beta \in R^\varepsilon(w)$.*
1. *If $\alpha \succ' \beta$ and $\alpha \succ \beta$, then $j_\alpha \ge j_\beta$ and $w_\varepsilon(l_\alpha) < w_\varepsilon(l_\beta)$.*
2. *If $\alpha \succ' \beta$ and $\alpha \prec \beta$, then $j_\alpha < j_\beta$ and $w_\varepsilon(l_\alpha) \le w_\varepsilon(l_\beta)$.*

For a standard monomial $\mathbf{z} = \prod_{\alpha \in R(w)} z_\alpha^{m_\alpha}$ of the form given above, we define

$$Supp(\mathbf{z}) = \{\alpha \in R(w) \mid m_\alpha \neq 0\}$$

and $MSupp(\mathbf{z})$ to be the largest element of $Supp(\mathbf{z})$ under the ordering $\prec'$. Define the degree of a non-zero standard monomial, $\deg(\mathbf{z})$ to be the element $(m_\alpha)_\alpha$ of $\mathbf{N}^{R(w)}$ and define e_α by $(e_\alpha)_\beta = \delta_{\alpha,\beta}$.

Proposition 3.5 *Let $\alpha \in R(w)$. Set $(k,m) = (w_+(j_\alpha), w_+(l_\alpha))$ if $\alpha \in R^+(w)$, $(k,m) = (w_-(l_\alpha), w_-(j_\alpha))$ if $\alpha \in R^-(w)$.*

1. *If $MSupp(\mathbf{z}) = \alpha$, then $ad_{k,m}\mathbf{z}$ is a scalar multiple of a standard monomial and* $\deg(ad_{k,m}\mathbf{z}) = \deg(\mathbf{z}) - e_\alpha$.
2. *If $MSupp(\mathbf{z}) \prec' \alpha$, then $ad_{k,m}\mathbf{z} = 0$.*

Proof. Let $\mathbf{z}$ be a monomial such that $MSupp(\mathbf{z}) \preceq' \alpha$. Then we may write

$$\mathbf{z} = \left(\prod_\gamma z_\gamma^{m_\gamma}\right) z_\alpha^{m_\alpha} \left(\prod_\beta z_\beta^{m_\beta}\right)$$

where the products are ordered as described above. We prove the result by induction on $e(\mathbf{z}) = \#(Supp(\mathbf{z})\backslash\{\alpha\})$. We first consider the case where $\alpha \in R^-(w)$. We prove that if $k' \leq k$ then

$$\mathrm{ad}_{k',m}\mathbf{z} = \delta_{k',k}\left(\prod_\gamma \mathrm{ad}_{k,k} z_\gamma^{m_\gamma}\right) \mathrm{ad}_{k,m} z_\alpha^{m_\alpha} \left(\prod_\beta \mathrm{ad}_{m,m} z_\beta^{m_\beta}\right)$$

Suppose that $e(\mathbf{z}) = 0$. Then $\mathbf{z} = z_\alpha^{m_\alpha}$ and the result is true by Corollary 3.3. Now assume that the proposition is true for all monomials $\mathbf{y}$ such that $MSupp(\mathbf{y}) \preceq' \alpha$ and $e(\mathbf{y}) < e(\mathbf{z})$. We consider three cases.

(1) Suppose that $\mathbf{z} = \xi\mathbf{y}$, where $\xi = z_\gamma^{m_\gamma}$ and $\gamma \in R^+(w)$. Write $\mathrm{ad}_{k',m}\mathbf{z} = \sum_s \mathrm{ad}_{k',s}\xi\mathrm{ad}_{s,m}\mathbf{y}$. Suppose that $\mathrm{ad}_{k',s}\xi$ and $\mathrm{ad}_{s,m}\mathbf{y}$ are non-zero for some s. From $\mathrm{ad}_{k',s}\xi \neq 0$ it follows that $s \leq k' \leq k$. By induction we conclude from $\mathrm{ad}_{s,m}\mathbf{y} \neq 0$ that $s = k = k'$.

(2) Suppose that $\mathbf{z} = \xi\mathbf{y}$, $\xi = z_\gamma^{m_\gamma}$ where $\gamma \in R^-(w)$ and $\gamma \prec \alpha$. Proposition 3.4 implies that $j_\gamma \leq j_\alpha$ and $k = w_-(l_\alpha) < w_-(l_\gamma)$. Write $\mathrm{ad}_{k',m}\mathbf{z} = \sum_s \mathrm{ad}_{k',s}\xi\mathrm{ad}_{s,m}\mathbf{y}$. Suppose that $\mathrm{ad}_{k',s}\xi$ and $\mathrm{ad}_{s,m}\mathbf{y}$ are nonzero for some s. Then $k' \leq s$. If $s \neq k'$, we have $w_-(l_\gamma) \leq k' < s \leq w_-(j_\gamma)$. Hence $w_-(l_\gamma) \leq k' \leq k$, a contradiction. Thus $s = k' \leq k$ and by induction $k \leq k' = s \leq m$. Hence $k' = s = k$.

(3) Suppose that $\mathbf{z} = \mathbf{y}\xi$ where $\xi = z_\beta^{m_\beta}$ where $\beta \in R^-(w)$ and $\beta \succ \alpha$. Then Proposition 3.4 implies that $j_\beta > j_\alpha$ and $k = w_-(l_\alpha) \leq w_-(l_\beta)$. Write $\mathrm{ad}_{k',m}\mathbf{z} = \mathrm{ad}_{k',s}\mathbf{y}\mathrm{ad}_{s,m}\xi$ and suppose that $\mathrm{ad}_{k',s}\mathbf{y} \neq 0$ and $\mathrm{ad}_{s,m}\xi \neq 0$. Then $s \leq m$. If $s < m$, we must have

$$w_-(j_\alpha) = m \notin yJ = \{w_-(1), \ldots, w_-(j_\beta - 1), w_-(l_\beta\}.$$

But this contradicts $j_\alpha < j_\beta$. Hence $s = m$. Then $\mathrm{ad}_{k',m}\mathbf{y} \neq 0$ and induction yields the result. The case when $\alpha \in R^+(w)$ is similar. □

Using induction on the degree of a standard monomial we then obtain:

Theorem 3.6 *Let $\mathbf{z}$ be a standard monomial. Then there exists a monomial f in the X_{ij} such that*

1. $(adf)\mathbf{z} \in \mathbf{C}^*$;
2. $(adf)\mathbf{z}' = 0$ *if $\mathbf{z}'$ is a standard monomial with* $\deg \mathbf{z}' \prec' \deg \mathbf{z}$.

The theorem implies in particular that the standard monomials are linearly independent. This fact together with the commutation relations of Theorem 2.2 implies the following.

Corollary 3.7 *The algebra C_w^H is an iterated skew polynomial ring in $l(w)$ variables.*

As we noted above, C_w is a skew Laurent extension of C_w^H in $rk\, G$ variables. Thus C_w is a deformation of the algebra of functions on the variety $\mathcal{C}_w \cong \mathbf{C}^{l(w)} \times (\mathbf{C}^*)^{rk\, G}$ described in [3, A.2]. Similarly the algebra $B_w = C_w[d^{-1}]$ is a deformation of the algebra of functions on the open subset $\mathcal{B}_w$ of $\mathcal{C}_w$.

Lemma 3.8 *Let $k, l \in \{1, \dots, n\}$ and let $j \in \{1, \dots, n-1\}$. Then*

$$ad_{k,l} t_j = \delta_{kl} q^{2\psi_{kj}} t_j$$

where $\psi_{kj} = \delta_{k,w_+(J)} - \delta_{k,w_-(J)}$.

Proof. It follows from Lemma 3.1 that $X_{kl} c^+_{j,w_+} = (q^2)^{\delta_{l,J} - \delta_{k,w_+(J)}} c^+_{j,w_+} X_{kl}$. Similarly $X_{kl} c^-_{j,w_-} = (q^2)^{\delta_{l,J} - \delta_{k,w_-(J)}} c^-_{j,w_-} X_{kl}$. The result is then clear. □

Denote by Ψ the $n \times (n-1)$ matrix (ψ_{kj}). It induces a map $\Psi : \mathbf{Z}^{n-1} \to \mathbf{Z}^n$.

Theorem 3.9 *The subalgebra $\mathbf{C}[t_i^{\pm 1} \mid i = 1, \dots n-1]$ is invariant and diagonalisable under the adjoint action. The subalgebra of ad-invariant elements is*

$$\mathbf{C}[t_i^{\pm 1} \mid i = 1, \dots n-1]^{ad} = \mathbf{C}[t^{\mathbf{m}} \mid \Psi \mathbf{m} = 0]$$

which is the algebra of functions on a (commutative) torus with rank $\dim \ker_{\mathbf{h}^*}(w_+ w_-^{-1} - Id)$.

Proof. It is clear from the lemma that $\mathrm{ad}_{i,i} t^{\mathbf{m}} = q^{2(\Psi \mathbf{m})_i} t^{\mathbf{m}}$. Thus the subalgebra of ad-invariants is the linear span of the monomials $t^{\mathbf{m}}$ for $\mathbf{m} \in \ker \Psi$. Identify $\mathbf{h}^*$ with the usual subspace of $V = \langle \varepsilon_1, \dots, \varepsilon_n \rangle_{\mathbf{C}}$. Let Φ be the linear map on V given by the matrix $[\Psi : 0]$ with respect to this basis. Then $\Phi(\varepsilon_i) = (w_+ - w_-)(\varepsilon_1 + \cdots + \varepsilon_i)$. Hence $\mathrm{rank}(\ker \Psi) = \dim(\ker \Phi) - 1 = \dim(\ker_{\mathbf{h}^*}(w_+ - w_-)) = \dim(\ker_{\mathbf{h}^*}(w_+ w_-^{-1} - Id))$. □

4 Primitive Spectrum of $\mathbf{C}_q[G]$

We are now able to deduce the main result.

Definition. Let K be a subgroup of $\mathbf{Z}^{n-1}$ such that $\mathbf{Z}^{n-1} = \ker \Psi \oplus K$. Set

$$C_{\dot{w}} = C_w^H[t^{\mathbf{m}} \mid \mathbf{m} \in K]$$

and let $B_{\dot{w}} = C_{\dot{w}}[d^{-1}]$.

It follows from [3, 2.7] and Corollary 3.7 that $C_{\dot{w}}$ is a localization of an iterated skew polynomial ring. Hence $C_{\dot{w}}$ and $B_{\dot{w}}$ are integral domains. Furthermore $C_w \cong C_{\dot{w}} \otimes C_w^{ad}$, $B_w \cong B_{\dot{w}} \otimes C_w^{ad}$ as $\mathbf{C}_q[G]$-module algebras and $C_w^{ad} = \mathbf{C}[t_i^{\pm 1} \mid i = 1, \dots n-1]^{ad}$ as described in Theorem 3.9.

Set $s(w) = \operatorname{codim}(\ker_{\mathbf{h}^*}(w_+ w_-^{-1} - Id))$, (c.f. [3, A.2]).

Theorem 4.1 *$B_w \cong B_{\dot{w}} \otimes C_w^{ad}$ and $B_{\dot{w}}$ is a simple algebra. The center of B_w is C_w^{ad} and the ideals of B_w are generated by their intersection with the center. Thus Spec $B_w \cong$ Spec C_w^{ad} and Prim $B_w \cong$ Prim C_w^{ad}. The primitive ideals of B_w are maximal and form an H-orbit. If $P \in$ Prim B_w, then $H/Stab_H P$ is a torus of rank equal to rank $G - s(w)$ and the Gelfand-Kirillov dimension of B_w/P is $l(w) + s(w)$.*

Proof. Let P_e be the ideal of B_w generated by $\{t^{\mathbf{m}} - 1 \mid \mathbf{m} \in \ker \Psi\}$. Then P_e is an ad-invariant ideal and $B_{\dot{w}} \cong B_w/P_e$ as $\mathbf{C}_q[G]$-module algebras. Suppose that $B_{\dot{w}}$ is not simple. Then by 'going up' [4, 10.5.15] $B_{\dot{w}}$ contains a proper non-zero ad-invariant ideal, I. In this case $I \cap C_{\dot{w}}$ is a proper non-zero ad-invariant ideal of $C_{\dot{w}}$. Let $\mathbf{C}K = \mathbf{C}[t^{\mathbf{m}} \mid \mathbf{m} \in K]$. Since $\mathbf{C}K$ is diagonalizable under the adjoint action and $C_{\dot{w}} \cong C_w^H \otimes \mathbf{C}K$ as $\mathbf{C}_q[G]$-modules, it follows that the socle of $C_{\dot{w}}$ under the adjoint action is given by

$$Soc\, C_{\dot{w}} \cong Soc\, C_w^H \otimes \mathbf{C}K \cong \mathbf{C} \otimes \mathbf{C}K.$$

Thus $Soc\, C_{\dot{w}}$ is a direct sum of distinct one-dimensional submodules each generated by a unit. Since $I \cap C_{\dot{w}}$ must intersect nontrivially at least one summand of the socle, it must contain a unit, a contradiction. Thus $B_{\dot{w}}$ is simple.

Let $\sigma : H \to H/\Gamma$ be the map $\sigma(h) = h^2\Gamma$. Then the action of H on $\mathbf{C}[t_i^{\pm 1} \mid i = 1, \dots n-1]$ factors through σ and the induced action of H/Γ is the natural action of a torus on its algebra of functions. This, together with Theorem 3.9 implies the assertions concerning the H-action. The assertion concerning the Gelfand-Kirillov dimension follows from a slight generalization of [4, 8.2.10]. $\square$

Notice that Theorem 4.1 implies that all prime ideals of $\mathbf{C}_q[G]^\Gamma$ are completely prime. This result may also be deduced from [2] where it is proved that all prime ideals of $\mathbf{C}_q[G]$ are completely prime.

Recall that a Noetherian $\mathbf{C}$-algebra A is said to satisfy the Dixmier-Moeglin condition if the following are equivalent for $P \in Spec\, A$: a) P is primitive; b) P is rational (the center of the ring of fractions of A/P is $\mathbf{C}$); c) P is locally closed in $Spec\, A$.

Theorem 4.2 1) *Prim $\mathbf{C}_q[G] \cong \bigsqcup_{w \in W \times W}$ Prim$_w \mathbf{C}_q[G]$.*

2) *For each $w \in W \times W$, Prim$_w \mathbf{C}_q[G]$ is a non-empty H-orbit. If $P_{\dot{w}} \in$ Prim$_w \mathbf{C}_q[G]$, then $H/Stab_H P_{\dot{w}}$ is a torus of rank equal to rank $G - s(w)$.*

3) *The Gelfand-Kirillov dimension of $\mathbf{C}_q[G]/P_{\dot{w}}$ is $l(w) + s(w)$.*

4) *$\mathbf{C}_q[G]$ satisfies the Dixmier-Moeglin condition.*

Proof. Notice first that $\mathbf{C}_q[G]$ satisies the nullstellensatz, [4, 9.1.8]. Therefore a) $\Rightarrow$ b) and c) $\Rightarrow$ a) for any $P \in Spec\, \mathbf{C}_q[G]$. Let $P \in Spec_w \mathbf{C}_q[G]$ be rational and let $Q = P \cap \mathbf{C}_q[G]^\Gamma$.

Since C_w^{ad} is central in A_w, $PA_w \cap C_w^{ad}$ must be a maximal ideal of C_w^{ad}. Thus QB_w is maximal by Theorem 4.1. It then follows from 'going up' [4, 10.5.15] that any prime ideal strictly containing P must contain the product of the elements of $\mathcal{E}_w$. Hence P is locally closed and primitive. Thus $\mathbf{C}_q[G]$ satisfies the Dixmier-Moeglin condition. Since Γ acts transitively on the primitive ideals of $\mathbf{C}_q[G]$ lying over Q and H acts transitively on the maximal ideals of B_w, it follows easily that H acts transitively on $Prim_w\,\mathbf{C}_q[G]$. Since the action of H is algebraic $H/Stab_H P$ must be a torus whose rank is $rk\,G - s(w)$ by Theorem 4.1. The assertion concerning the Gelfand-Kirillov dimension follows from [4, 8.2.9]. $\square$

Comparing this result with Theorem A.3.2 of [3], we see that there exists an H-equivariant bijection $\beta : \text{Prim } \mathbf{C}_q[G] \longrightarrow \text{Symp } G$ such that $\beta(\text{Prim}_w\mathbf{C}_q[G]) = \text{Symp}_w G$ for all $w \in W \times W$ and such that $\dim \beta(P) = \text{GKdim } \mathbf{C}_q[G]/P$ for all $P \in \text{Prim } \mathbf{C}_q[G]$. Thus Conjecture 1 of [3] is true.

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