

DIFFERENTIAL OPERATORS COMMUTING WITH INVARIANT FUNCTIONS

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ABSTRACT. Let \mathfrak{g} be a reductive, complex Lie algebra, with adjoint group G , let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})$ for the differential of this action. We prove that the commutant, in $\mathcal{D}(\mathfrak{g})$, of $\mathcal{O}(\mathfrak{g})^G$ is the algebra generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$, thereby answering a question of Barlet.

1. INTRODUCTION

Fix a reductive, complex Lie algebra \mathfrak{g} , with adjoint group G , let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})$ for the differential of this action. We identify $\mathcal{O}(\mathfrak{g})$, the ring of regular functions on \mathfrak{g} , with $S(\mathfrak{g}^*)$ and let $\mathcal{O}(\mathfrak{g})^G$ denote the subalgebra of G -invariant functions. The aim of this note is to prove:

Theorem 1.1. *The commutant $\mathcal{C} = \mathcal{C}_{\mathcal{D}(\mathfrak{g})}(\mathcal{O}(\mathfrak{g})^G)$, in $\mathcal{D}(\mathfrak{g})$, of $\mathcal{O}(\mathfrak{g})^G$ is the algebra generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$.*

At the level of vector fields, this result follows from [5, Theorem 2.1], in the sense that Dixmier's result implies that $\mathcal{C} \cap \text{Der } \mathcal{O}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$. In [2], D. Barlet raised the question of whether Theorem 1.1 is true, since this would form a natural generalization of Dixmier's result. In the same paper, Barlet was able to prove the theorem in the case when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. We would like to thank M. Raïs for bringing Barlet's question to our attention.

In the process of proving Theorem 1.1, we obtain a considerable amount of information about the structure of \mathcal{C} . Some particular properties are given in the next result. The unexplained definitions can be found in Section 3.

Proposition 1.2. *\mathcal{C} is an Auslander-Gorenstein, CM domain and a maximal order in its quotient division ring.*

In fact, the main theorem of this paper is a result about commutative rings. To state this, let A denote the subalgebra of $\mathcal{D}(\mathfrak{g})$ generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$ and set $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \text{Der } \mathcal{O}(\mathfrak{g})$. If one filters $\mathcal{D}(\mathfrak{g})$ and its subalgebras by degree of differential operators, then it is easy to see that the associated graded rings $\text{gr } A$ and $\text{gr } \mathcal{C}$ are domains with the same quotient field as the symmetric algebra $\text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$. Then, Theorem 1.1 and Proposition 1.2 follow easily from the following result.

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Theorem 1.3. (i) Let $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \text{Der } \mathcal{O}(\mathfrak{g})$. Then $\text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$ is a factorial, complete intersection of Krull dimension $2 \dim \mathfrak{g} - \text{rk } \mathfrak{g}$.
(ii) $\text{gr } A = \text{gr } \mathcal{C} = \text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$.

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2. THE SYMMETRIC ALGEBRA OF THE MODULE GENERATED BY $\tau(\mathfrak{g})$

In this section we prove Theorem 1.3 from the introduction. We begin with some preliminary notation and results.

As before, we fix a complex, reductive Lie algebra \mathfrak{g} of dimension n and rank ℓ . Write G for the adjoint group of \mathfrak{g} . Define the categorical quotient of \mathfrak{g} by $\mathcal{O}(\mathfrak{g}/G) = \text{Spec } \mathcal{O}(\mathfrak{g})^G$ and let $u : \mathfrak{g} \rightarrow \mathfrak{g}/G$ denote the quotient morphism. We will write \mathcal{O} for $\mathcal{O}(\mathfrak{g})$. Define

$$\mathcal{X}_i = \{y \in \mathfrak{g} : \text{rk } d_y u \leq i\},$$

where $d_y u : T_y \mathfrak{g} \rightarrow T_{u(y)} \mathfrak{g}/G$ denotes the differential of u . Observe that each \mathcal{X}_i is a closed G -subvariety of \mathfrak{g} . Recall that $y \in \mathfrak{g}$ is called *regular* if its centralizer in \mathfrak{g} is of dimension ℓ . Then [10, Theorem 10.1], $\text{rk } d_y u = \ell$ if and only if y is regular.

We would like to thank D. Panyushev for the proof of the following proposition, which is considerably easier than our original proof.

Proposition 2.1. *One has: $\text{codim } \mathcal{X}_i \geq \ell - i + 2$, for $0 \leq i \leq \ell - 1$.*

Proof. Notice that u induces a surjective morphism $\varpi : \mathcal{X}_i \rightarrow \mathcal{X}_i/G$ and that, for all $x \in \mathcal{X}_i$, the differential $d_x \varpi$ is the restriction of $d_x u$ to $T_x \mathcal{X}_i$. Set $r = \max\{\text{rk } d_x \varpi : x \in \mathcal{X}_i\}$. Then, by [7, Proposition III.10.6] and the definition of \mathcal{X}_i , we obtain that $\dim \mathcal{X}_i/G \leq r \leq i$.

Since \mathcal{X}_i is stable under the \mathbb{C}^* -action $y \mapsto \lambda y$, $\lambda \in \mathbb{C}^*$, the point 0 belongs to each irreducible component of \mathcal{X}_i . Hence, $\dim \mathcal{X}_i \leq \dim \mathcal{X}_i/G + \dim \varpi^{-1}(\varpi(0))$ (see [11, AI.3.3] or [7, Ex. II.3.22]). But $\varpi^{-1}(\varpi(0)) = \mathcal{X}_i \cap \mathbf{N}$, where \mathbf{N} denotes the nilpotent cone of \mathfrak{g} , and, since $i \leq \ell - 1$, $\mathcal{X}_i \cap \mathbf{N}$ is contained in the subvariety of non-regular nilpotent elements. Therefore $\dim \varpi^{-1}(\varpi(0)) \leq n - \ell - 2$ and it follows that $\dim \mathcal{X}_i \leq i + n - \ell - 2$, as required. \square

Remark . Proposition 2.1 generalizes the well-known fact that $\mathcal{X}_{\ell-1}$ has codimension at least three (see, for example, [18, Theorem 4.12]). It is natural to conjecture that Proposition 2.1 can be improved to the statement that $\text{codim } \mathcal{X}_i \geq 3(\ell - i)$ for $0 \leq i \leq \ell - 1$. D. Panyushev informs us that he has been able to prove this by a case by case analysis.

Fix a G -invariant, non-degenerate, symmetric bilinear form κ on \mathfrak{g} and let $\tilde{\kappa} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ be the induced isomorphism. Thus, $\tilde{\kappa}$ induces an isomorphism between differential one-forms on \mathfrak{g} and vector fields on \mathfrak{g} . If $f \in \mathcal{O}^G$, then we define a G -invariant vector field $\text{grad}(f) \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$ to be the image of df under $\tilde{\kappa}$. Equivalently, if we fix an orthonormal basis $\{e_i\}$ of \mathfrak{g} and write $x_i = e_i^* \in \mathfrak{g}^*$, then

$$(2.1) \quad \text{grad}(f) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes e_j = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}.$$

By Chevalley's Theorem, \mathcal{O}^G is a polynomial ring, say $\mathcal{O}^G = \mathbb{C}[u_1, \dots, u_\ell]$ for homogeneous, algebraically independent polynomials $\{u_i\}_i$. Set $\nabla_i = \text{grad}(u_i)$, for

$1 \leq i \leq \ell$. If $\tau : \mathfrak{g} \rightarrow \text{Der } \mathcal{O}$ is the differential of the adjoint action of G on \mathfrak{g} , then write $E = \mathcal{O}\tau(\mathfrak{g})$. We will also write τ for the induced map:

$$\tau : \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow E \subseteq \text{Der } \mathcal{O}.$$

Notice that if $\theta \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$, the vector field $\tau(\theta)$ is given by $\tau(\theta)_y = [y, \theta_y]$ for all $y \in \mathfrak{g}$. It follows easily that if θ is G -invariant, then $\tau(\theta) = 0$. In particular, one has $\tau(\nabla_i) = 0$ for all i . In fact rather more is true:

Lemma 2.2. *There is a short exact sequence*

$$(2.2) \quad 0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}\nabla_i \longrightarrow \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\tau} E \longrightarrow 0.$$

Proof. This is [16, Theorem 2.5.4]. Using the identification of \mathfrak{g} with \mathfrak{g}^* under $\tilde{\kappa}$, it also follows from [14, Theorem 1.9]. \square

Corollary 2.3. *If $\text{Sym}_{\mathcal{O}}(E)$ denotes the symmetric algebra of the \mathcal{O} -module E , then, $\text{Sym}_{\mathcal{O}}(E) \cong \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g})/(\nabla_1, \dots, \nabla_{\ell})$.*

Proof. This follows from the universal property of symmetric algebras. \square

Set $\text{Sym}(E) = \text{Sym}_{\mathcal{O}}(E)$. The main aim of this section is to understand the structure of $\text{Sym}(E)$, for which we use the results from [1] and [8].

Let $I_t(\mathbf{u})$ be the ideal generated by the $t \times t$ minors of the matrix $\mathbf{u} = \left[\frac{\partial u_i}{\partial x_j} \right]$ and consider the following condition for $s \geq 0$:

$$(\mathcal{F}_s) \quad \text{ht } I_t(\mathbf{u}) \geq \ell - t + 1 + s, \quad \text{for } 1 \leq t \leq \ell.$$

Observe that, if we regard the short exact sequence (2.2) as a sequence

$$0 \longrightarrow \mathcal{O}^{\ell} \xrightarrow{\beta} \mathcal{O}^n \longrightarrow E \longrightarrow 0,$$

then (2.1) implies that $I_t(\mathbf{u})$ is the ideal generated by the $t \times t$ minors of the map β . Thus, the ideals $I_{n-t}(\mathbf{u})$ are nothing more than the Fitting ideals of E (see, for example, [17, 1.1]). In particular, they are independent of the presentation of E and our condition (\mathcal{F}_s) coincides with that of [8].

Proposition 2.4. (i) The condition (\mathcal{F}_2) is satisfied by E .

(ii) $\text{Sym}(E)$ is a factorial domain of Krull dimension $2n - \ell$. In particular, $\text{Sym}(E)$ is a complete intersection and is Gorenstein.

(iii) If P is a prime ideal of \mathcal{O} with $\text{ht } P \geq 2$, then $\text{ht } P\text{Sym}(E) \geq 2$.

Proof. Write $\tilde{\mathcal{X}}_{i-1}$ for the zero set of $I_i(\mathbf{u})$; thus

$$\tilde{\mathcal{X}}_{i-1} = \{x \in \mathfrak{g} : \text{rk}(\nabla_1(x), \dots, \nabla_{\ell}(x)) \leq i - 1\}.$$

Since the ∇_j are the images of the du_j under the isomorphism $\tilde{\kappa}$, clearly $\tilde{\mathcal{X}}_{i-1} = \{x \in \mathfrak{g} : \text{rk}(d_x u_1, \dots, d_x u_{\ell}) \leq i - 1\}$. Since u_1, \dots, u_{ℓ} define the quotient map $u : \mathfrak{g} \rightarrow \mathfrak{g}/G$, this implies that $\tilde{\mathcal{X}}_i = \mathcal{X}_i$. Hence, part (i) is a reformulation of Proposition 2.1.

By Lemma 2.2, E has projective dimension at most 1. Thus, part (ii) follows from part (i), combined with [1, Propositions 3 and 6]. By [8, Remarks, pp. 664-5], the condition of part (iii) is equivalent to the condition (\mathcal{F}_2) . \square

We end this section by giving the geometric significance of Proposition 2.4. This should be compared with [9, § 2] which proves weaker results for much more general G -varieties.

The map τ induces a homomorphism of algebras

$$\tilde{\tau} : \mathcal{O}(\mathfrak{g} \times \mathfrak{g}^*) = \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes \mathfrak{g}) \longrightarrow \mathcal{O}(T^*\mathfrak{g}) = \text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O}).$$

Clearly, the image of $\tilde{\tau}$ is the subring $\mathcal{O}[\tau(\mathfrak{g})]$ of $\text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O})$ generated by \mathcal{O} and $\tau(\mathfrak{g})$. After identification of \mathfrak{g}^* with \mathfrak{g} through $\tilde{\kappa}$, the associated morphism to $\tilde{\tau}$ is:

$$\nu : T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, \quad \nu(x, y) = (x, [y, x])$$

Let $\tilde{T}\mathfrak{g}$ denote the closure of the image of ν ; thus, $\tilde{T}\mathfrak{g}$ is an irreducible affine subvariety of $\mathfrak{g} \times \mathfrak{g}$ with coordinate ring $\mathcal{O}(\tilde{T}\mathfrak{g}) \cong \mathcal{O}[\tau(\mathfrak{g})]$.

Corollary 2.5. (i) $\text{Sym}(E) = \mathcal{O}(\tilde{T}\mathfrak{g})$.

(ii) The variety $\tilde{T}\mathfrak{g}$ is a factorial complete intersection in $\mathfrak{g} \times \mathfrak{g}$.

Proof. By universality, $\tilde{\tau}$ induces a surjective morphism $\pi : \text{Sym}(E) \rightarrow \mathcal{O}[\tau(\mathfrak{g})]$. If we prove that $\dim \tilde{T}\mathfrak{g} \geq 2n - \ell$, then the corollary will follow from Proposition 2.4(ii).

Let $\rho : \tilde{T}\mathfrak{g} \rightarrow \mathfrak{g}$ denote the projection onto the first factor. By [11, AI.3.3] there exists a dense open subset $U \subseteq \tilde{T}\mathfrak{g}$ such that $\dim \tilde{T}\mathfrak{g} = \dim \mathfrak{g} + \dim \rho^{-1}(\rho(u))$ for all $u \in U$. Since $\tilde{T}\mathfrak{g}$ is irreducible, we can pick $u = (x, y) \in \rho^{-1}(\mathfrak{g}') \cap U$, where \mathfrak{g}' denotes the set of generic elements in \mathfrak{g} . Now,

$$\rho^{-1}(\rho(u)) \supseteq \rho^{-1}(\rho(u)) \cap \text{Im } \nu = \{(x, [y, x])\}.$$

Since x is generic, $\dim \rho^{-1}(\rho(u)) \geq \dim[\mathfrak{g}, x] = n - \ell$ and the result follows. \square

3. THE COMMUTANT OF $\mathcal{O}(\mathfrak{g})^G$

As usual, we identify $\mathcal{D}(\mathfrak{g})$, as a vector space, with $\mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$, where $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ and the symmetric algebra $S(\mathfrak{g})$ is identified with the constant coefficient differential operators on \mathfrak{g} . We will always filter $\mathcal{D}(\mathfrak{g})$ by degree of differential operators and so, as algebras, $\text{gr } \mathcal{D}(\mathfrak{g}) = \text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O}) \cong \mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$. Write A for the subring of $\mathcal{D}(\mathfrak{g})$ generated by \mathcal{O} and $\tau(\mathfrak{g})$ and let \mathcal{C} denote the commutant of \mathcal{O}^G , as in the introduction. Obviously, A is contained in \mathcal{C} .

Lemma 3.1. *Let $x \in \mathfrak{g}$ be a regular point and set $R = \mathcal{O}_{\mathfrak{g}, x}$ for the local ring of \mathfrak{g} at x . Then, there exists a basis of derivations $\{\partial_i : 1 \leq i \leq n\}$ of $\text{Der } R$ such that $\partial_i(u_j) = \delta_{ij}$ for all $1 \leq i, j \leq \ell$ and $R\tau(\mathfrak{g}) = \bigoplus_{i=\ell+1}^n R\partial_i$.*

Proof. Let \mathfrak{m} denote the maximal ideal of R . By [10, Theorem 0.1], the $\{d_x u_i : 1 \leq i \leq \ell\}$ are linearly independent. The $\{d_x u_i\}$ may also be regarded as elements of $\mathfrak{m}/\mathfrak{m}^2$, under the usual identification of $T_x^*\mathfrak{g}$ with $\mathfrak{m}/\mathfrak{m}^2$. Thus, for some scalars λ_i , the set $\{u_1 - \lambda_1, \dots, u_\ell - \lambda_\ell\}$ is part of a system of parameters, say $\{z_1 = u_1 - \lambda_1, \dots, z_\ell = u_\ell - \lambda_\ell, z_{\ell+1}, \dots, z_n\}$ for \mathfrak{m} . Let $\partial_i \in \text{Der } R$ be defined by $\partial_i(z_j) = \delta_{ij}$.

If $D \in \text{Der } R$, then $\tilde{D} = D - \sum_{i=1}^{\ell} D(u_i)\partial_i$ satisfies $\tilde{D}(u_j) = 0$, for $1 \leq j \leq \ell$. Thus, $\tilde{D}(\mathcal{O}^G) = 0$ and so, by [5, Theorem 2.1] (or directly), $\tilde{D} \in R\tau(\mathfrak{g})$. Hence, $\text{Der } R = R\tau(\mathfrak{g}) \oplus (\bigoplus_{i=1}^{\ell} R\partial_i)$. Since

$$R\tau(\mathfrak{g}) \subseteq \{D \in \text{Der } R : D(u_j) = 0 \text{ for } 1 \leq j \leq \ell\} = \bigoplus_{i=\ell+1}^n R\partial_i,$$

the result follows. \square

Theorem 3.2. *Let A and \mathcal{C} be given the filtrations induced from that on $\mathcal{D}(\mathfrak{g})$. Then $\text{gr } A = \text{gr } \mathcal{C} \cong \text{Sym}(E)$.*

Proof. Since $\text{gr } \mathcal{C} \subset \text{gr } \mathcal{D}(\mathfrak{g}) \cong \mathcal{O}(T^*\mathfrak{g})$, certainly $\text{gr } A \subseteq \text{gr } \mathcal{C}$ are domains. Also, as $\tau(\mathfrak{g})$ consists of derivations, we may regard $\tau(\mathfrak{g}) \subseteq \text{Der } \mathcal{O} \subseteq \text{gr } \mathcal{D}(\mathfrak{g})$. Hence the ring $\mathcal{O}[\tau(\mathfrak{g})]$ is contained in $\text{gr } A$ and, by Corollary 2.5(i), the natural map $\pi : \text{Sym}(E) \rightarrow \mathcal{O}[\tau(\mathfrak{g})]$ is an isomorphism.

Let $x \in \mathfrak{g}$ be a regular point and let $S = \{f \in \mathcal{O} : f(x) \neq 0\}$. Given a ring C containing \mathcal{O} , we write C_x for the localization C_S (given that it exists). Then, we claim that

$$(3.1) \quad (\text{gr } A)_x = (\text{gr } \mathcal{C})_x \cong \text{Sym}(E)_x,$$

where the isomorphism is induced by π^{-1} .

By mimicking the proof of Richardson's Lemma [11, II.3.4], one can show that this suffices to prove the theorem. In more detail, assume that (3.1) is true. Since $\text{gr } \mathcal{C}$ and $\text{Sym}(E)$ are domains, (3.1) certainly implies that

$$\text{Sym}(E) \xrightarrow{\pi} \text{gr } A \subseteq \text{gr } \mathcal{C}$$

and that $\text{gr } \mathcal{C}$ and $\text{Sym}(E)$ have the same field of fractions. Moreover, $\{x \in \mathfrak{g} : (\text{gr } \mathcal{C})_x \neq \text{Sym}(E)_x\}$ is contained in the set of non-regular elements of \mathfrak{g} . By [10, Theorem 0.1], this is precisely the subspace $\mathcal{X}_{\ell-1}$ and, by Proposition 2.1 or [18, Theorem 4.12], $\text{codim } \mathcal{X}_{\ell-1} \geq 3$. Thus, for any $b \in \text{gr } \mathcal{C}$, there exists an ideal I of \mathcal{O} of height at least 3 such that $bI \subseteq \text{Sym}(E)$. By Proposition 2.4(iii), $\text{ht}_{\text{Sym}(E)} I \text{Sym}(E) \geq 2$. Hence, $b \in \text{Sym}(E)_{\mathfrak{p}}$ for every height one prime \mathfrak{p} of $\text{Sym}(E)$. Since $\text{Sym}(E)$ is Cohen-Macaulay, it satisfies the (S_2) -condition [12, p. 125], and therefore $b \in \text{Sym}(E)$.

Thus, it remains to prove (3.1). Let $R = \mathcal{O}_x = \mathcal{O}_{\mathfrak{g},x}$ and keep the notation of Lemma 3.1. It is immediate from that lemma that $D \in \mathcal{D}(\mathfrak{g})_x$ satisfies $[D, u_j] = 0$ if and only if $D \in R\langle \partial_j : j \neq i \rangle$. Consequently, $\mathcal{C}_x = A_x = R\langle \partial_{\ell+1}, \dots, \partial_n \rangle$.

Let $\bar{\partial}_k$ denote the image of ∂_k in $\text{gr } \mathcal{D}(\mathfrak{g})$. Obviously, Lemma 3.1 also implies that $R\tau(\mathfrak{g}) = \bigoplus_{k=\ell+1}^n R\bar{\partial}_k$, where $R\tau(\mathfrak{g})$ is now regarded as a subspace of $\text{Der } R \subset \text{gr } \mathcal{D}(\mathfrak{g})_x \cong R \otimes_{\mathbb{C}} S(\mathfrak{g})$. Thus,

$$\text{gr } \mathcal{C}_x = \text{gr } A_x = \text{gr } R\langle \partial_{\ell+1}, \dots, \partial_n \rangle = R[\bar{\partial}_{\ell+1}, \dots, \bar{\partial}_n] = R[\tau(\mathfrak{g})].$$

Since $\pi : \text{Sym}(E)_x \rightarrow R[\tau(\mathfrak{g})]$ is an isomorphism, this completes the proof of (3.1) and hence of the theorem. \square

The Gelfand-Kirillov dimension of a module M will be denoted $\text{GKdim } M$. If a Noetherian ring A has finite injective dimension, then A is called *Auslander-Gorenstein* if A satisfies the following condition: For any integers $0 \leq i < j$ and finitely generated (right) A -module M , one has $\text{Ext}_A^i(N, A) = 0$ for all (left) A -submodules N of $\text{Ext}_A^j(M, A)$. Set $j(M) = \min\{j : \text{Ext}_A^j(M, A) \neq 0\}$. The algebra A is *CM* if $j(M) + \text{GKdim } M = \text{GKdim } A$ holds for all finitely generated, non-zero A -modules M .

Corollary 3.3. (i) The commutant \mathcal{C} of \mathcal{O}^G in $\mathcal{D}(\mathfrak{g})$ is the ring generated by \mathcal{O} and $\tau(\mathfrak{g})$. Moreover, \mathcal{C} is an Auslander-Gorenstein, CM, Noetherian domain and a maximal order.

(ii) As a (left or right) \mathcal{O} -module, $\mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$ is generated by the elements

$$\{\tau(\xi_1)\tau(\xi_2)\cdots\tau(\xi_k) : \xi_i \in \mathfrak{g} \text{ and } k \leq m\}.$$

(iii) The centre of \mathcal{C} is $\mathcal{O}(\mathfrak{g})^G$.

Proof. (i) By Theorem 3.2, \mathcal{C} is generated by \mathcal{O} and $\tau(\mathfrak{g})$. By that theorem and Proposition 2.4, $\text{gr } \mathcal{C}$ satisfies the other conditions given in part (i). Let $M = \bigcup_{n \in \mathbb{N}} M_n$ be a filtered right \mathcal{C} -module such that $\text{gr } M$ is a finitely generated $\text{gr } \mathcal{C}$ -module. By [3, Theorem 3.9], \mathcal{C} is Auslander-Gorenstein and $\text{j}_{\mathcal{C}}(M) = \text{j}_{\text{gr } \mathcal{C}}(\text{gr } M)$. However, [13, Corollary 1.4] implies that $\text{GKdim } \text{gr } M = \text{GKdim } M$ and hence that \mathcal{C} is CM. Finally, [15] implies that \mathcal{C} is a maximal order.

(ii) This follows from the fact that, in $\text{gr } \mathcal{C} = \text{Sym}(E)$, a homogeneous element \bar{c} of degree m can be written $\bar{c} = \sum f_{i_1, \dots, i_m} \xi_{i_1} \cdots \xi_{i_m}$, for some $f_{i_1, \dots, i_m} \in \mathcal{O}$ and $\xi_{i_j} \in \tau(\mathfrak{g})$.

(iii) Let Z denote the centre of \mathcal{C} . Clearly both $\tau(\mathfrak{g})$ and \mathcal{O} commute with \mathcal{O}^G and so $\mathcal{O}^G \subseteq Z$. Conversely, Z is contained in the commutant, in $\mathcal{D}(\mathfrak{g})$, of \mathcal{O} . Hence, $Z \subseteq \mathcal{O}$. Since \mathcal{O}^G is the commutant, in \mathcal{O} , of $\tau(\mathfrak{g})$, the result follows. \square

Corollary 3.4. *Both \mathcal{C} and $\text{Sym}(E)$ are free (left or right) modules over $\mathcal{O}(\mathfrak{g})^G$.*

Proof. Set $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ and $S = \text{Sym}(E) = \bigoplus_{m=0}^{\infty} \text{Sym}_m(E)$. We first prove the result for \mathcal{C} , assuming that $\text{Sym}_m(E)$ is a free \mathcal{O}^G -module for all $m \in \mathbb{N}$. Note that the isomorphism $\text{gr } \mathcal{C} \cong S$ of Theorem 3.2 is a graded isomorphism of \mathcal{O} -algebras, for the natural graded structure of the two objects. In other words $\mathcal{C}_m/\mathcal{C}_{m-1} \cong \text{Sym}_m(E)$, for all m , where $\mathcal{C}_m = \mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$. Hence, each $\mathcal{C}_m/\mathcal{C}_{m-1}$ is a free \mathcal{O}^G -module; it follows routinely that \mathcal{C} is also free over \mathcal{O}^G .

We now prove the result for $\text{Sym}_m(E)$. Note, first, that S is a quotient of the polynomial ring

$$T = \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes \mathfrak{g}) \cong \mathcal{O}[y_1, \dots, y_n] \cong \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n],$$

which we now grade by giving each generator x_i and y_j degree one. Since the u_i are homogeneous in \mathcal{O} , the $\nabla_i = \text{grad}(u_i)$ are homogeneous in T and so, by Corollary 2.3, $\text{Sym}_m(E)$ is a graded \mathcal{O}^G -module.

Set $P = \sum_{i=1}^{\ell} u_i S$. By [6, Proposition 2.16] and its proof (which depends upon a case by case analysis), S/P is a domain of dimension $2n - 2\ell = \dim S - \ell$. Hence, the u_j form a regular sequence in S , and therefore in each module $\text{Sym}_m(E)$. Thus, by [4, § 8, Proposition 8 and § 9, Corollaire 2], $\text{Sym}_m(E)$ is a graded free \mathcal{O}^G -module. \square

Corollary 3.3 and Corollary 3.4 should be compared with [9] which (as a very special case) shows that the commutant of $\mathcal{D}(\mathfrak{g})^G$ is simply $\mathbb{C}\langle \tau(\mathfrak{g}) \rangle$ ($\cong U(\mathfrak{g})$ when \mathfrak{g} is semisimple). Moreover, both rings are free modules over the centre of $\mathcal{D}(\mathfrak{g})^G$ (which is also the centre of $\mathbb{C}\langle \tau(\mathfrak{g}) \rangle$).

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