DIFFERENTIAL OPERATORS AND COHOMOLOGY GROUPS ON THE BASIC AFFINE SPACE

T. LEVASSEUR AND J. T. STAFFORD

This paper is dedicated to Tony Joseph on the occasion of his 60^{th} *birthday.*

Abstract. We study the ring of differential operators $\mathcal{D}(\mathbf{X})$ on the basic affine space $\mathbf{X} = G/U$ of a complex semisimple group G with maximal unipotent subgroup U. One of the main results shows that the cohomology group $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ decomposes as a finite direct sum of non-isomorphic simple $\mathcal{D}(\mathbf{X})$-modules, each of which is isomorphic to a twist of $\mathcal{O}(\mathbf{X})$ by an automorphism of $\mathcal{D}(\mathbf{X})$.

We also use $\mathcal{D}(\mathbf{X})$ to study the properties of $\mathcal{D}(\mathbf{Z})$ for highest weight varieties $\mathbf{Z}$. For example, we prove that $\mathbf{Z}$ is $\mathcal{D}$-simple in the sense that $\mathcal{O}(\mathbf{Z})$ is a simple $\mathcal{D}(\mathbf{Z})$-module and produce an irreducible G-module of differential operators on $\mathbf{Z}$ of degree -1 and specified order.

Contents

2000 *Mathematics Subject Classification.* 13N10, 14L30, 16S32, 17B56, 20G10.

Key words and phrases. semisimple Lie group, basic affine space, highest weight variety, rings of differential operators, D-simplicity.

The second author was supported in part by the NSF through the grants DMS-9801148 and DMS-0245320. Part of this work was done while he was visiting the Mittag-Leffler Institute and he would like to thank the Institute for its financial support and hospitality.

1. Introduction

Fix a complex semisimple, connected and simply connected Lie group G with maximal unipotent subgroup U and Lie algebra $\mathfrak{g}$. Then the *basic affine space* is the quasi-affine variety $\mathbf{X} = G/U$. The ring of global differential operators $\mathcal{D}(\mathbf{X})$ has a long history, going back to the work [GK] of Gelfand and Kirillov in the late 60's who used this space to formulate and partially solve their conjecture that the quotient division ring of the enveloping algebra $U(\mathfrak{g})$ should be isomorphic to a Weyl skew field.

The variety $\mathbf{X}$ is only quasi-affine and, when $\mathfrak{g}$ is not isomorphic to a direct sum of copies of $\mathfrak{sl}(2)$, the affine closure $\overline{\mathbf{X}}$ of $\mathbf{X}$ is singular. In general, rings of differential operators on a singular variety $\mathbf{Z}$ can be quite unpleasant; for example, and in contrast to the case of a smooth affine variety, $\mathcal{D}(\mathbf{Z})$ need not be noetherian, finitely generated or simple and (conjecturally) it will not be generated by the derivations $\mathrm{Der}_{\mathbb{C}}(\mathbf{Z})$. Moreover, the canonical module $\mathcal{O}(\mathbf{Z})$ need not be a simple $\mathcal{D}(\mathbf{Z})$-module. Recently, Bezrukavnikov, Braverman and Positselskii [BBP] proved a remarkable result on the structure of $\mathcal{D}(\mathbf{X})$ which shows that it actually has very pleasant properties. Specifically, they proved that there exist automorphisms $\{F_w\}_w$, indexed by the Weyl group W of G, such that for any nonzero $\mathcal{D}(\mathbf{X})$-module M there exists $w \in W$ such that $\mathcal{D}_{\mathbf{X}} \otimes_{\mathcal{D}(\mathbf{X})} M^w \neq 0$, where $M^w = M^{F_w}$ is the twist of M by F_w. (The F_w should be thought of as analogues of partial Fourier transforms.) Since $\mathbf{X}$ is smooth this implies that, for any finite open affine cover $\{\mathbf{X}_i\}_i$ of $\mathbf{X}$, the ring $\bigoplus_{i,w} \mathcal{D}(\mathbf{X}_i)^w$ is a noetherian, faithfully flat overring of $\mathcal{D}(\mathbf{X})$. As is shown in [BBP], it follows easily that $\mathcal{D}(\mathbf{X})$ is a noetherian domain of finite injective dimension.

The aim of this paper is to extend and apply the results of [BBP]. Our first result, which combines Proposition 3.1, Theorem 3.3 and Theorem 3.8, further elucidates the structure of $\mathcal{D}(\mathbf{X})$.

Proposition 1.1. *Let $\mathbf{X} = G/U$ denote the basic affine space of G. Then:*

(1) $\mathcal{D}(\mathbf{X})$ *is a simple ring satisfying the Auslander-Gorenstein and Cohen-Macaulay conditions (see Section 3 for the definitions);*
(2) $\mathcal{D}(\mathbf{X})$ *is (finitely) generated, as a $\mathbb{C}$-algebra, by* $\{\mathcal{O}(\mathbf{X})^w : w \in W\} \cup \widehat{\mathfrak{h}}$.

Note that $\mathcal{D}(\mathbf{X})$ is quite a subtle ring; for example, if $G = \mathrm{SL}(3,\mathbb{C})$ then $\mathcal{D}(\mathbf{X}) \cong U(\mathfrak{so}(8))/J$, where J is the *Joseph ideal* as defined in [Jo1] (see Example 2.8).

The variety $\mathbf{X} = G/U$ has a natural left action of G and a right action of the maximal torus H for which $B = HU$ is a Borel subgroup. Differentiating these actions gives embeddings of $\mathfrak{g} = \mathrm{Lie}(G)$, respectively $\widehat{\mathfrak{h}} = \mathrm{Lie}(H)$ into $\mathrm{Der}_{\mathbb{C}}(\mathbf{X})$. It follows easily from the simplicity of $\mathcal{D}(\mathbf{X})$ that $\mathcal{O}(\mathbf{X}) = \mathrm{H}^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is a simple $\mathcal{D}(\mathbf{X})$-module. One of the main results of this paper extends this to describe the $\mathcal{D}(\mathbf{X})$-module structure of the full cohomology group $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$:

Theorem 1.2. [Theorem 4.11] *Let* $\mathbf{X} = G/U$ *and set* $\mathcal{M} = \mathcal{D}(\mathbf{X})/\mathcal{D}(\mathbf{X})\mathfrak{g}$.

(1) $\mathcal{M} \cong \mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$.
(2) *For each* i, $\mathrm{H}^i(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \cong \bigoplus \Big\{ \mathcal{O}(\mathbf{X})^w : w \in W : \ell(w) = i \Big\}$.
(3) *For* $w \neq v \in W$, *the* $\mathcal{D}(\mathbf{X})$*-modules* $\mathcal{O}(\mathbf{X})^w$ *and* $\mathcal{O}(\mathbf{X})^v$ *are simple and nonisomorphic.*

A result analogous to Theorem 1.2, but in the l-adic setting, has been proved in [Po, Lemma 12.0.1]. It is not clear to us what is the relationship between the two results.

A key point in the proof of Theorem 1.2 is that, by the Borel-Weil-Bott theorem, one has an explicit description of the G-module structure of the $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ and one then proves Theorem 1.2 by comparing that structure with the G-module structure of the $\mathcal{O}(\mathbf{X})^w$. Theorem 1.2 is rather satisfying since it relates the left ideal of $\mathcal{D}(\mathbf{X})$ generated by the derivations coming from $\mathfrak{g}$ to the only "obvious" simple $\mathcal{D}(\mathbf{X})$-modules $\mathcal{O}(\mathbf{X})^w$. In contrast, if one considers the left ideal generated by all the "obvious" derivations, $\mathcal{D}(\mathbf{X})\mathfrak{g} + \mathcal{D}(\mathbf{X})\widehat{\mathfrak{h}}$, then one obtains:

Proposition 1.3. [Theorem 4.6] *As left* $\mathcal{D}(\mathbf{X})$*-modules,*

$$\frac{\mathcal{D}(\mathbf{X})}{\mathcal{D}(\mathbf{X})\mathfrak{g} + \mathcal{D}(\mathbf{X})\widehat{\mathfrak{h}}} \;=\; \frac{\mathcal{D}(\mathbf{X})}{\mathcal{D}(\mathbf{X})\mathrm{Der}_{\mathbb{C}}(\mathbf{X})} \;\cong\; \mathcal{O}(\mathbf{X}).$$

This proposition is somewhat surprising since, at least when $\mathfrak{g}$ is not a direct sum of copies of $\mathfrak{sl}(2)$, one can show that $\mathcal{D}(\mathbf{X})$ is *not* generated by $\mathcal{O}(\mathbf{X})$ and $\mathrm{Der}_{\mathbb{C}}(\mathbf{X})$ as a $\mathbb{C}$-algebra (see Corollary 5.11). Of course, the analogue of Proposition 1.3 for smooth varieties is standard.

A natural class of varieties associated to G are *S-varieties*: closures $\overline{\mathbf{Y}}$ of a G-orbit $\mathbf{Y} = \mathbf{Y}_\Gamma = G.v_\Gamma$ where v_Γ is a sum of highest weight vectors in some finite dimensional G-module. In such a case there exists a natural surjection $\mathbf{X} \twoheadrightarrow \mathbf{Y} = G/S_\Gamma$, for the isotropy group S_Γ of v_Γ. This induces, by restriction of operators, a map $\psi_\Gamma : \mathcal{D}(\mathbf{X})^{S_\Gamma} \to \mathcal{D}(\overline{\mathbf{Y}})$ and allows us to use our structure results on $\mathcal{D}(\mathbf{X})$ give information on $\mathcal{D}(\mathbf{Y})$. For this to be effective we need the mild assumption that $\overline{\mathbf{Y}}$ is normal and $\mathrm{codim}_{\overline{\mathbf{Y}}}(\overline{\mathbf{Y}} \smallsetminus \mathbf{Y}) \geq 2$ or, equivalently, that $\mathcal{O}(\mathbf{Y}) = \mathcal{O}(\overline{\mathbf{Y}})$ (see Theorem 5.3 for further equivalent conditions).

Corollary 1.4. [Proposition 5.7] *Let* $\overline{\mathbf{Y}}$ *be an S-variety such that* $\mathcal{O}(\mathbf{Y}) = \mathcal{O}(\overline{\mathbf{Y}})$. *Then* $\mathcal{O}(\mathbf{Y})$ *is a simple* $\mathcal{D}(\mathbf{Y})$*-module.*

When $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_\Gamma$ is singular, this result says that there exists operators $D \in \mathcal{D}(\mathbf{Y})$ that cannot be constructed from derivations; these are "exotic" operators in the terminology of [AB]). As the name suggests, exotic operators can be hard to construct—see [AB] or [BK2], for example—but in our context their construction is easy; they arise as $\psi_\Gamma F_{w_0}(\mathcal{O}(\mathbf{Y}_{\Gamma^*}))$. In the special case of a highest weight variety (this is just an S-variety $\overline{\mathbf{Y}}_\gamma = \overline{\mathbf{Y}}_\Gamma$ for $\Gamma = \mathbb{N}\gamma$) we can be more precise about these operators.

Corollary 1.5. [Corollary 6.8] *Suppose that $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_\gamma$ is a highest weight variety. Then there exists an irreducible G-module $E \cong V(\gamma)$ of differential operators on $\mathbf{Y}$ of degree -1 and order $\langle \gamma, 2\rho^\vee \rangle$.*

Corollary 1.4 also proves the Nakai conjecture for S-varieties satisfying the hypotheses of that result and this covers most of the known cases of normal, singular varieties for which the conjecture is known. See Section 5 for the details.

A fundamental question in the theory of differential operators on invariant rings asks the following. Suppose that Q is a (reductive) Lie group acting on a finite dimensional vector space V such that the fixed ring $\mathcal{O}(V)^Q$ is singular. Then, is the natural map $\mathcal{D}(V)^Q \to \mathcal{D}(\mathcal{O}(V)^Q)$ surjective? A positive answer to this question is known in a number of cases and this has had significant applications to representation theory; see, for example, [Jo2, LS1, LS2, Sc] and Remark 6.10. It is therefore natural to ask when the analogous map $\psi_\Gamma : \mathcal{D}(\mathbf{X})^{S_\Gamma} \to \mathcal{D}(\mathbf{Y}_\Gamma)$ is surjective. Although we do not have a general answer to this question, we suspect that ψ_Γ will usually not be surjective. As evidence for this we prove that this is the case for one of the fundamental examples of an S-variety: the closure of the minimal orbit in a simple Lie algebra $\mathfrak{g}$.

Proposition 1.6. [Theorem 6.9] *If $\mathbf{Y}_\gamma$ is the minimal (nonzero) nilpotent orbit of a simple classical Lie algebra $\mathfrak{g}$, then ψ_γ is surjective if and only if $\mathfrak{g} = \mathfrak{sl}(2)$ or $\mathfrak{g} = \mathfrak{sl}(3)$.*

Acknowledgement. We would like to thank the referee for some helpful comments.

2. Preliminaries

In this section we describe the basic results and notation we need from the literature, notably the relevant results from [BBP]. The reader is also referred to [GK, HV, Sh1, Sh2] for the interrelationship between differential operators on the base affine space and the corresponding enveloping algebra.

We begin with some necessary notation. The base field will always be the field $\mathbb{C}$ of complex numbers. Let G be a connected simply-connected semisimple algebraic group of rank ℓ, B a Borel subgroup, $H \subset B$ a maximal torus, $U \subset B$ a maximal unipotent subgroup of G. The Lie algebra of an algebraic group is denoted by the corresponding gothic character; thus $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{u} = \mathrm{Lie}(U)$.

We will use standard Lie theoretic notation, as for example given in [Bo]. In particular, let Δ denote the root system of $(\mathfrak{g}, \mathfrak{h})$ and fix a set of positive roots Δ_+ such that $\mathfrak{u} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{u}_\alpha$. Denote by W the Weyl group of Δ. Let Λ be the weight lattice of Δ which we identify with the character group of H. The set of dominant weights is denoted by Λ^+. Fix a basis $\Sigma = \{\alpha_1, \dots, \alpha_\ell\}$ of Δ_+ and write $\{\varpi_1, \dots, \varpi_\ell\}$ for the fundamental weights; thus $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. Denote by ρ, respectively $\rho^\vee$, the half sum of the positive roots, respectively coroots. Let w_0 be the longest element of W and set $\lambda^* = -w_0(\lambda)$ for all $\lambda \in \Lambda$. Then $w_0(\rho) = -\rho$, $w_0(\rho^\vee) = -\rho^\vee$ and, by [Bo, Chapter 6, 1.10, Corollaire],

$$\langle \lambda^*, \rho^\vee \rangle = \langle \lambda, \rho^\vee \rangle = \sum_{i=1}^{\ell} m_i \qquad \text{if} \quad \lambda = \sum_{i=1}^{\ell} m_i \alpha_i.$$

For each $\omega \in \Lambda^+$ let $V(\omega)$ be the irreducible G-module of highest weight ω and $V(\omega)_\mu$ be the subspace of elements of weight $\mu \in \Lambda$; we will denote by v_ω a highest weight vector of $V(\omega)$. Recall that the G-module $V(\omega)^*$ identifies naturally with $V(\omega^*)$. If E is a locally finite $(G \times H)$-module, we denote by $E[\lambda]$ the isotypic G-component of type $\lambda \in \Lambda^+$ and by E^μ the μ-weight space for the action of $H \equiv \{1\} \times H$. Hence,

$$E = \bigoplus \left\{ E[\lambda]^\mu : \mu \in \Lambda,\, \lambda \in \Lambda^+ \right\}.$$

Let $\mathbf{Y}$ be an algebraic variety. We denote by $\mathcal{O}_{\mathbf{Y}}$ the structural sheaf of $\mathbf{Y}$ and by $\mathcal{O}(\mathbf{Y})$ its algebra of regular functions. The sheaf of differential operators on $\mathbf{Y}$ is denoted by $\mathcal{D}_{\mathbf{Y}}$ with global sections $\mathcal{D}(\mathbf{Y}) = \mathrm{H}^0(\mathbf{Y}, \mathcal{D}_{\mathbf{Y}})$. The $\mathcal{O}(\mathbf{Y})$-module of elements of order $\leq k$ in $\mathcal{D}(\mathbf{Y})$ is denoted by $\mathcal{D}_k(\mathbf{Y})$, and the order of $D \in \mathcal{D}(\mathbf{Y})$ will be written $\operatorname{ord} D$. Now suppose that $\mathbf{Y}$ is an irreducible quasi-affine algebraic variety, embedded as an open subvariety of an affine variety $\overline{\mathbf{Y}}$. We will frequently use the fact that, if $\overline{\mathbf{Y}}$ is normal with $\operatorname{codim}_{\overline{\mathbf{Y}}}(\overline{\mathbf{Y}} \smallsetminus \mathbf{Y}) \geq 2$, then $\mathcal{O}(\mathbf{Y}) = \mathcal{O}(\overline{\mathbf{Y}})$ and so $\mathcal{D}(\mathbf{Y}) = \mathcal{D}(\overline{\mathbf{Y}})$.

Let Q be an affine algebraic group. We say that an algebraic variety $\mathbf{Y}$ is a Q-variety if it is equipped with a rational action of Q. For such a variety, $\mathcal{O}(\mathbf{Y}), \mathcal{D}_k(\mathbf{Y})$ and $\mathcal{D}(\mathbf{Y})$ are locally finite Q-modules, with the action of $a \in Q$ on $\varphi \in \mathcal{O}(\mathbf{Y})$ and $D \in \mathcal{D}(\mathbf{Y})$ being defined by $a.\varphi(y) = \varphi(a^{-1}.y)$ for $y \in \mathbf{Y}$, respectively $(a.D)(\varphi) = a.D(a^{-1}.\varphi)$. If $\mathbf{Y}$ is also an affine variety such that $\mathcal{O}(\mathbf{Y})^Q$ is an affine algebra, we define the *categorical quotient* $\mathbf{Y}/\!\!/Q$ by $\mathcal{O}(\mathbf{Y}/\!\!/Q) = \mathcal{O}(\mathbf{Y})^Q$. We then have a restriction morphism

$$\psi : \mathcal{D}(\mathbf{Y})^Q \longrightarrow \mathcal{D}(\mathbf{Y}/\!\!/Q), \qquad \psi(D)(f) = D(f) \text{ for } f \in \mathcal{O}(\mathbf{Y})^Q.$$

Notice that $\psi(\mathcal{D}_k(\mathbf{Y})^Q) \subseteq \mathcal{D}_k(\mathbf{Y}/\!\!/Q)$ for all k. In many cases $\mathcal{O}(\mathbf{Y})$ will be a $\mathbb{Z}$-graded algebra, $\mathcal{O}(\mathbf{Y}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}^n$, in which case $\mathcal{D}(\mathbf{Y})$ has an induced $\mathbb{Z}$-graded structure, with the n^{th} graded piece being

$$\mathcal{D}(\mathbf{Y})^n = \{\theta \in \mathcal{D}(\mathbf{Y}) : \theta(\mathcal{O}^r) \subseteq \mathcal{O}^{r+n} \text{ for all } r \in \mathbb{Z}\}. \tag{2.1}$$

In this situation, $\mathcal{D}(\mathbf{Y})^n$ will be called operators of *degree* n.

Assume now that V is a $(G \times H)$-module and that $\mathbf{Y}$ is a $(G \times H)$-subvariety of V. Then

$$\mathcal{O}(\mathbf{Y}) = \bigoplus_{\substack{\mu \in \Lambda \\ \lambda \in \Lambda^+}} \mathcal{O}(\mathbf{Y})[\lambda]^\mu \;\subset\; \mathcal{D}(\mathbf{Y}) = \bigoplus_{\substack{\mu \in \Lambda \\ \lambda \in \Lambda^+}} \mathcal{D}(\mathbf{Y})[\lambda]^\mu.$$

It follows easily from the definitions that

$$\mathcal{D}(\mathbf{Y})^\mu = \{d \in \mathcal{D}(\mathbf{Y}) : d(\mathcal{O}(\mathbf{Y})^\lambda) \subseteq \mathcal{O}(\mathbf{Y})^{\lambda+\mu} \text{ for all } \lambda \in \Lambda\}. \tag{2.2}$$

One clearly has a surjection $S(V^*)[\lambda]^\mu \twoheadrightarrow \mathcal{O}(\mathbf{Y})[\lambda]^\mu$. Furthermore, if $V = V^{-\gamma^*}$ for some $0 \neq \gamma^* \in \Lambda$, we will identify $S^m(V^*)$ with $S(V^*)^{m\gamma^*}$ and obtain the surjective G-morphism $S^m(V^*)[\lambda] \twoheadrightarrow \mathcal{O}(\mathbf{Y})[\lambda]^{m\gamma^*}$. In this case, $\mathcal{O}(\mathbf{Y}) = \bigoplus_{m\in\mathbb{N}} \mathcal{O}(\mathbf{Y})^{m\gamma^*}$ and $\mathcal{D}(\mathbf{Y}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{D}(\mathbf{Y})^{m\gamma^*}$ are $\mathbb{Z}$-graded and (2.2) can be interpreted as saying that $\mathcal{D}(\mathbf{Y})^{m\gamma^*}$ is the set of differential operators of degree m on $\mathbf{Y}$.

The previous results apply in particular to the basic affine space $\mathbf{X} = G/U$. We need to collect here a few facts about the $(G \times H)$-variety $\mathbf{X}$ and its canonical affine closure $\overline{\mathbf{X}}$. For these assertions, see, for example, [GK, Gr, VP]. Set $V = V(\varpi_1) \bigoplus \cdots \bigoplus V(\varpi_\ell)$ and recall that there is an isomorphism

$$\mathbf{X} \xrightarrow{\sim} G.(v_{\varpi_1} \oplus \cdots \oplus v_{\varpi_\ell}) \subset V, \qquad \bar{g} \mapsto g.(v_{\varpi_1} \oplus \cdots \oplus v_{\varpi_\ell}),$$

where $\bar{g}$ denotes the class of $g \in G$ modulo U. We identify $\mathbf{X}$ with this G-orbit and write

$$\mathcal{O} = \mathcal{O}(\mathbf{X}) \quad \text{and} \quad \mathcal{D} = \mathcal{D}(\mathbf{X}).$$

Then the Zariski closure $\overline{\mathbf{X}}$ of $\mathbf{X}$ in V is a normal irreducible affine variety that satisfies $\operatorname{codim}_{\overline{\mathbf{X}}}(\overline{\mathbf{X}} \smallsetminus \mathbf{X}) \geq 2$; thus $\mathcal{O} = \mathcal{O}(\overline{\mathbf{X}})$, etc.

Identify w_0 with an automorphism of H; thus $\lambda^*(h) = \lambda(w_0(h^{-1}))$ for all $\lambda \in \Lambda$ and $h \in H$. Define the twisted (right) action of H on $\mathbf{X}$ by $r_h.\bar{g} = \overline{gw_0(h)}$. Endow the G-module V with the action of $H \equiv \{1\} \times H$ defined by

$$r_h.(u_1 \oplus \cdots \oplus u_\ell) = \varpi_1^*(h^{-1})u_1 \oplus \cdots \oplus \varpi_\ell^*(h^{-1})u_\ell.$$

Then the embedding $\mathbf{X} \hookrightarrow V$ is a morphism of $(G\times H)$-varieties. The induced (left) action of H on $\mathcal{O}$ is then given by $(r_h.f)(\bar{g}) = f(r_{h^{-1}}.\bar{g})$ for all $f \in \mathcal{O}, h \in H, \bar{g} \in \mathbf{X}$. It follows that $\mathcal{O}[\nu^*] = \mathcal{O}^{\nu^*}$, $\mathcal{O}^\mu = 0$ if $\mu \notin \Lambda^+$, and we can decompose the $(G\times H)$-module $\mathcal{O}$ as:

$$\mathcal{O} = \bigoplus_{\lambda\in\Lambda^+} \mathcal{O}^\lambda, \qquad \mathcal{O}^\lambda = \mathcal{O}[\lambda] \cong V(\lambda). \tag{2.3}$$

The final isomorphism in (2.3) comes from the fact that the G-action on $\mathcal{O}$ is multiplicity free [VP, Theorem 2]. Notice that the algebra $\mathcal{O}$ is (finitely) generated by the G-modules $\mathcal{O}^{\varpi_j}$, $1 \leq j \leq \ell$. Also, (2.2) implies that $\mathcal{D}^\mu(\mathcal{O}^\lambda) = 0$ when $\lambda+\mu$ is not dominant.

Notation 2.4. The differentials of the actions of G and H (via $h \mapsto r_h$) on $\mathbf{X}$ yield morphisms of algebras $\imath : U(\mathfrak{g}) \to \mathcal{D}$ and $\jmath : U(\mathfrak{h}) \to \mathcal{D}$. By [GK, Corollary 8.1 and Lemma 9.1], $\imath$ and $\jmath$ are injective; from now on we will identify $U(\mathfrak{g})$ with $\imath(U(\mathfrak{g}))$ but write $\widehat{\mathfrak{h}} = \jmath(\mathfrak{h})$ and $U(\widehat{\mathfrak{h}}) = \jmath(U(\mathfrak{h}))$, to distinguish these objects from their images under $\imath$.

Let $0 \neq M$ be a left $\mathcal{D}$-module and $\tau \in \mathrm{Aut}(\mathcal{D})$, the $\mathbb{C}$-algebra automorphism group of $\mathcal{D}$. Then the *twist of* M *by* τ is the $\mathcal{D}$-module M^τ defined as follows: $M^\tau = M$ as an abelian group but $a \cdot x = \tau(a)x$ for all $a \in \mathcal{D}, x \in M$. Recall also that the localization of M on $\mathbf{X}$ is

$$L(M) = \mathcal{D}_{\mathbf{X}} \otimes_{\mathcal{D}} M \cong \mathcal{O}_{\mathbf{X}} \otimes_{\mathcal{O}} M.$$

Thus $L(M)$ is a quasi-coherent left $\mathcal{D}_{\mathbf{X}}$-module. Clearly $L(M) = 0$ when M is supported on $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$ but, remarkably, one can obtain a nonzero localization by first twisting the module M:

Theorem 2.5. [BBP, Proposition 3.1 and Theorem 3.4] *Let $\mathcal{D} = \mathcal{D}(\mathbf{X})$. Then:*

(1) *There exists an injection of groups $F : W \hookrightarrow \mathrm{Aut}(\mathcal{D})$ written $w \mapsto F_w$.*
(2) *For each $\mathcal{D}$-module $M \neq 0$ there exists $w \in W$ such that $L(M^{F_w}) \neq 0$.* □

This theorem is also valid for right modules. The morphisms F_w can be regarded as variants of partial Fourier transforms, and more details on their structure and properties can be found in [BBP].

It will be convenient to reformulate Theorem 2.5, for which we need some notation. Since $\mathbf{X}$ is an open subset of the non singular locus of $\overline{\mathbf{X}}$, we can fix an open (and smooth) affine cover of $\mathbf{X}$:

$$\mathbf{X} = \bigcup_{j=1}^{k} \mathbf{X}_j, \quad \text{where} \quad \mathbf{X}_j = \{x \in \overline{\mathbf{X}} : f_j(x) \neq 0\}, \tag{2.6}$$

for the appropriate $f_i \in \mathcal{O}(\mathbf{X})$. Notice that $\mathcal{D}(\mathbf{X}_j) = \mathcal{D}[f_j^{-1}]$ for each j and so $\mathcal{D}(\mathbf{X}_j)$ is a flat $\mathcal{D}$-module. If M is $\mathcal{D}$-module and $w \in W$, let M^w denote the twist of M by F_w.

For each pair (w, j) with $w \in W$ and $1 \leq j \leq k$, we have an injective morphism of algebras $\phi_{wj} : \mathcal{D} \hookrightarrow \mathcal{D}(\mathbf{X}_j)$ given by $\phi_{wj}(d) = F_w^{-1}(d)$. We write $\mathcal{D}(\mathbf{X}_j)_w$ for the algebra $\mathcal{D}(\mathbf{X}_j)$ regarded as an overring of $\mathcal{D}$ under this embedding. The significance of this construction is that, for any left $\mathcal{D}$-module M, the map $d \otimes v \mapsto d \otimes v$ induces an isomorphism $\mathcal{D}(\mathbf{X}_j) \otimes_{\mathcal{D}} M^w \cong \mathcal{D}(\mathbf{X}_j)_w \otimes_{\mathcal{D}} M$ of left $\mathcal{D}(\mathbf{X}_j)$-modules. Theorem 2.5 can now be rewritten as follows.

Corollary 2.7. *Let $0 \neq M$ be a left $\mathcal{D}$-module. Then:*

(1) *There exists a pair (w, j) such that $\mathcal{D}(\mathbf{X}_j)_w \otimes_{\mathcal{D}} M \neq 0$.*
(2) *Set $R_w = \bigoplus_{j=1}^k \mathcal{D}(\mathbf{X}_j)_w$ and $R = \bigoplus_{w \in W} R_w$. Then R is a faithfully flat (left or right) overring of $\mathcal{D}$.*

Proof. Part (1) is just a reformulation of Theorem 2.5. This in turn implies that R is a faithful right $\mathcal{D}$-module. It is flat since $\mathcal{D}(\mathbf{X}_j)_w$ is isomorphic, as a $\mathcal{D}$-module, to the localization of $\mathcal{D}$ at the powers of $F_w(f_j)$. Since Theorem 2.5 also holds for right modules, the same argument shows that R is a faithful flat left $\mathcal{D}$-module. □

When $\mathfrak{g} = \mathfrak{sl}(2)$ it is easy to check that $\overline{\mathbf{X}} = \mathbb{C}^2$, and so there is no subtlety to the structure of either $\overline{\mathbf{X}}$ or $\mathcal{D}$. However, when $\mathfrak{g} \neq \mathfrak{sl}(2)^m$, $\overline{\mathbf{X}}$ will be singular and $\mathcal{D}$ will be rather subtle. The simplest example is:

Example 2.8. *Assume that* $\mathfrak{g} = \mathfrak{sl}(3)$ *and set* $\mathbf{X} = \mathrm{SL}(3)/U$. *Then* $\overline{\mathbf{X}}$ *is the quadric* $\sum_{i=1}^3 u_i y_i = 0$ *inside* $\mathbb{C}^6$. *Moreover,* $\mathcal{D} = \mathcal{D}(\mathbf{X}) \cong U(\mathfrak{so}(8))/J$, *where* J *is the Joseph ideal.*

An explicit set of generators of $\mathcal{D}$ *is given in* [LS3, (2.2)]. *The algebra* $F_{w_0}(\mathcal{O})$ *is generated by the operators* $\{\Phi_j, \Theta_j\}$ *of order* 2 *from* [LS3, (2.2)].

Proof. The proof of the first assertion is an elementary and classical computation, which is left to the reader. The second assertion then follows from [LSS, Remark 3.2(v) and Corollary A]. The claim in the second paragraph will not be used in this paper and so is left to the interested reader (the next proposition may prove useful). A second way of interpreting this example is given in Remark 6.10. □

In the main body of the paper we will need some more technical results from [BBP] about the automorphisms F_w and for the reader's convenience we record them in the following proposition. This summarizes Lemma 3.3, Corollary 3.10, Proposition 3.11 and the proof of Lemma 3.12 of that paper.

Proposition 2.9. *Let* $\eta \in \Lambda^+$, $\mu \in \Lambda$ *and* $w \in W$. *Then:*

(1) F_w *is* G*-linear and* $F_w(\mathcal{D}^\mu) = \mathcal{D}^{w(\mu)}$. *Thus* $F_w(\mathcal{O}^\eta) \cong V(\eta)$ *as* G*-modules.*

(2) $F_w(h) = w(h) + \langle w(h) - h, \rho \rangle$ *for all* $h \in \widehat{\mathfrak{h}}$.

(3) $\operatorname{ord} F_w(d) = \operatorname{ord} d + \langle \mu - w(\mu), \rho^\vee \rangle$ *for all* $0 \neq d \in \mathcal{D}^\mu$. *In particular, if* $0 \neq f \in \mathcal{O}^\eta$, *then* $\operatorname{ord} F_{w_0}(f) = \langle \eta, 2\rho^\vee \rangle$.

(4) *Let* $\{f_i\}_{1 \leq i \leq n} \subset \mathcal{O}^\eta$ *and* $\{g_i\}_{1 \leq i \leq n} \subset \mathcal{O}^{\eta^*} \cong (\mathcal{O}^\eta)^*$ *be dual bases such that (up to a constant)* $\sum_i f_i \otimes g_i$ *is the unique* G*-invariant element in* $\mathcal{O}^\eta \otimes \mathcal{O}^{\eta^*}$. *Then the* $(G \times H)$*-invariant operator* $P_\eta = \sum_{i=1}^n f_i F_{w_0}(g_i) \in U(\widehat{\mathfrak{h}})$ *is given by*

$$P_\eta = c_\eta \prod_{\alpha^\vee \in \Delta_+^\vee} \prod_{i=1}^{\langle \eta, \alpha^\vee \rangle} (\alpha^\vee + \langle \alpha^\vee, \rho \rangle - i), \tag{2.10}$$

for some $c_\eta \in \mathbb{C} \smallsetminus \{0\}$. *Moreover,* $\operatorname{ord} P_\eta = \langle \eta, 2\rho^\vee \rangle$.

(5) $U(\widehat{\mathfrak{h}}) = \sum_{w \in W} U(\widehat{\mathfrak{h}}) F_w(P_\eta)$. □

3. The structure of $\mathcal{D}(G/U)$.

In [BBP], the authors use Theorem 2.5 to prove that $\mathcal{D}$ is a noetherian ring of finite injective dimension. In this section we investigate other consequences of that result to the structure of $\mathcal{D}$. The notation from the last section will be retained; in particular, G is a connected, simply connected reductive algebraic group over $\mathbb{C}$, with basic affine space $\mathbf{X} = G/U$ and $\mathcal{D} = \mathcal{D}(\mathbf{X})$.

We begin with an easy application of the faithful flatness of the ring $R = \bigoplus_{w,j} \mathcal{D}(\mathbf{X}_j)_w$ defined in Corollary 2.7.

Proposition 3.1. *The ring $\mathcal{D}$ is simple and $\mathcal{O}$ is a simple left $\mathcal{D}$-module.*

Proof. Let J be a non zero ideal of $\mathcal{D}$. As in the proof of Corollary 2.7, $\mathcal{D}(\mathbf{X}_j)$ is a noetherian localization of $\mathcal{D}$ and so, by [MR, Proposition 2.1.16(vi)], each $\mathcal{D}(\mathbf{X}_j)_w \otimes_{\mathcal{D}} J \cong \mathcal{D}(\mathbf{X}_j) \otimes_{\mathcal{D}} J^w$ is an ideal of $\mathcal{D}(\mathbf{X}_j)$ and it is nonzero since $\mathcal{D}$ is a domain. But $\mathbf{X}_j$ is a smooth affine variety and so $\mathcal{D}(\mathbf{X}_j)$ is a simple ring. Thus $\mathcal{D}(\mathbf{X}_j)_w \otimes_{\mathcal{D}} J = \mathcal{D}(\mathbf{X}_j)_w$ for all $w \in W$ and $1 \leq j \leq k$. This means that the module $M = \mathcal{D}/J$ satisfies $R \otimes_{\mathcal{D}} M = 0$, hence $M = 0$ by Corollary 2.7(2). In other words, $J = \mathcal{D}$.

If $\mathcal{O}$ is not a simple $\mathcal{D}$-module, pick a proper factor module $\mathcal{O}/K$ and note that K is then an ideal of $\mathcal{O}$. But now the annihilator $\operatorname{ann}_{\mathcal{D}}(\mathcal{O}/K)$ of $\mathcal{O}/K$ as a $\mathcal{D}$-module is an ideal of $\mathcal{D}$ that contains K. This contradicts the simplicity of $\mathcal{D}$. $\square$

We now turn to the homological properties of $\mathcal{D}$. Two conditions that have proved very useful in applying homological techniques (see for example [Bj] or [LS1]) are the Auslander and Cohen-Macaulay conditions. These are defined as follows. A noetherian algebra A of finite injective dimension is called *Auslander-Gorenstein* if, for any finitely generated (left) A-module M and any right submodule $N \subseteq \operatorname{Ext}^i_A(M, A)$, one has $\operatorname{Ext}^j_A(N, A) = 0$, for $j < i$. The *grade* of M is $\mathrm{j}_A(M) = \inf\{j : \operatorname{Ext}^j_A(M, A) \neq 0\}$ (with the convention that $\mathrm{j}_A(0) = +\infty$). The *Gelfand-Kirillov dimension* of M will be denoted $\operatorname{GKdim}_A M$. We say that the algebra A is *Cohen-Macaulay* if

$$\operatorname{GKdim}_A M + \mathrm{j}_A(M) = \operatorname{GKdim} A \quad \text{for all } M \neq 0.$$

If $\mathbf{Z}$ is a smooth affine variety, then $\mathcal{D}(\mathbf{Z})$ is Auslander-Gorenstein and Cohen-Macaulay with $\operatorname{GKdim} \mathcal{D}(\mathbf{Z}) = 2 \dim \mathbf{Z}$ (see [Bj, Chapter 2, Section 7]). This applies, of course, when $\mathbf{Z} = \mathbf{X}_j$ for $1 \leq j \leq k$ and so $R = \bigoplus_{w,j} \mathcal{D}(\mathbf{X}_j)_w$ also satisfies these properties. As we show in Theorem 3.3, these properties descend to $\mathcal{D} = \mathcal{D}(\mathbf{X})$.

If M is a (finitely generated) $\mathcal{D}$-module, write

$$M_j^w = \mathcal{D}(\mathbf{X}_j) \otimes_{\mathcal{D}} M^w, \qquad \text{for} \quad 1 \leq j \leq k \text{ and } w \in W.$$

Lemma 3.2. *If M is a finitely generated left $\mathcal{D}$-module then*

$$\operatorname{GKdim}_{\mathcal{D}} M = \max\{\operatorname{GKdim}_{\mathcal{D}(\mathbf{X}_j)} M_j^w : 1 \leq j \leq k, w \in W\}.$$

Proof. By definition, $R \otimes_{\mathcal{D}} M = \bigoplus_{j=1}^k \bigoplus_{w \in W} M_j^w$ and so

$$\operatorname{GKdim}_R R \otimes_{\mathcal{D}} M = \max\{\operatorname{GKdim}_{\mathcal{D}(\mathbf{X}_j)} M_j^w : 1 \leq j \leq k, w \in W\}.$$

By faithfully flatness (Corollary 2.7), the natural map $M \to R \otimes_{\mathcal{D}} M$ is injective and it follows that $\operatorname{GKdim}_{\mathcal{D}} M \leq \operatorname{GKdim}_R R \otimes_{\mathcal{D}} M$. Conversely, since M_j^w is the localization of M^w at the Ore subset $\{f_j^s : s \in \mathbb{N}\}$, it follows from [Lo, Theorem 3.2] that $\operatorname{GKdim}_{\mathcal{D}(\mathbf{X}_j)} M_j^w \leq \operatorname{GKdim}_{\mathcal{D}} M^w$. Since $\operatorname{GKdim}_{\mathcal{D}} M^w = \operatorname{GKdim}_{\mathcal{D}} M$, this implies that $\operatorname{GKdim}_R R \otimes_{\mathcal{D}} M \leq \operatorname{GKdim}_{\mathcal{D}} M$ and the lemma is proved. □

Theorem 3.3. *The algebra $\mathcal{D}$ is Auslander-Gorenstein and Cohen-Macaulay.*

Proof. Let M be a finitely generated (left) $\mathcal{D}$-module and N is a (right) submodule of $\operatorname{Ext}^p_{\mathcal{D}}(M, \mathcal{D})$. As R is a flat $\mathcal{D}$-module,

$$R \otimes_{\mathcal{D}} \operatorname{Ext}^i_{\mathcal{D}}(N, \mathcal{D}) \cong \operatorname{Ext}^i_R(N \otimes_{\mathcal{D}} R, R)$$

and $N \otimes_{\mathcal{D}} R$ is a submodule of $\operatorname{Ext}^p_R(R \otimes_{\mathcal{D}} M, R)$. Since R is Auslander-Gorenstein, this implies that $R \otimes_{\mathcal{D}} \operatorname{Ext}^q_{\mathcal{D}}(N, \mathcal{D}) = 0$, for any $q < p$. Since R is a faithful $\mathcal{D}$-module, it follows that $\operatorname{Ext}^q_{\mathcal{D}}(N, \mathcal{D}) = 0$. Thus $\mathcal{D}$ is Auslander-Gorenstein.

From $\operatorname{Ext}^i_R(R \otimes_{\mathcal{D}} M, R) \cong \operatorname{Ext}^i_{\mathcal{D}}(M, \mathcal{D}) \otimes_{\mathcal{D}} R$ and the faithful flatness of $R_{\mathcal{D}}$ we get that $\operatorname{Ext}^i_R(R \otimes_{\mathcal{D}} M, R) = 0 \iff \operatorname{Ext}^i_{\mathcal{D}}(M, \mathcal{D}) = 0$. Thus, $\mathrm{j}_R(R \otimes_{\mathcal{D}} M) = \mathrm{j}_{\mathcal{D}}(M)$. But,

$$\operatorname{Ext}^p_R(R \otimes_{\mathcal{D}} M, R) \;\cong\; \bigoplus_{j,w} \operatorname{Ext}^p_{\mathcal{D}(\mathbf{X}_j)}(M_j^w, \mathcal{D}(\mathbf{X}_j))$$

and so $\mathrm{j}_{\mathcal{D}}(M) = \mathrm{j}_R(R \otimes_{\mathcal{D}} M) = \min\{\mathrm{j}_{\mathcal{D}(\mathbf{X}_j)}(M_j^w) : 1 \leq j \leq k, w \in W\}$.

Each $\mathbf{X}_j$ is a smooth affine variety of dimension $\dim \mathbf{X}$ and so each $\mathcal{D}(\mathbf{X}_j)$ is Cohen-Macaulay. It therefore follows from Lemma 3.2 that

$$\begin{aligned}
\operatorname{GKdim}_{\mathcal{D}} M &= \max\{\operatorname{GKdim}_{\mathcal{D}(\mathbf{X}_j)} M_j^w : 1 \leq j \leq k, w \in W, M_j^w \neq 0\} \\
&= \max\{2 \dim \mathbf{X} - \mathrm{j}_{\mathcal{D}(\mathbf{X}_j)}(M_j^w) : 1 \leq j \leq k, w \in W, M_j^w \neq 0\} \\
&= 2 \dim \mathbf{X} - \min\{\mathrm{j}_{\mathcal{D}(\mathbf{X}_j)}(M_j^w) : 1 \leq j \leq k, w \in W\} \\
&= 2 \dim \mathbf{X} - \mathrm{j}_{\mathcal{D}}(M),
\end{aligned}$$

as required. □

Remark 3.4. The last result should be compared with [YZ, Corollary 0.3] which shows that a simple noetherian $\mathbb{C}$-algebra with an affine commutative associated graded ring is automatically Auslander-Gorenstein and Cohen-Macaulay. It is not clear whether that result applies to $\mathcal{D}$, since we do not have a good description of the associated graded ring of $\mathcal{D}$.

When $\mathbf{Z}$ is a smooth affine variety, $\mathcal{D}(\mathbf{Z})$ is a finitely generated $\mathbb{C}$-algebra, simply because it is generated by $\mathcal{O}(\mathbf{Z})$ and $\operatorname{Der}_{\mathbb{C}}(\mathbf{Z})$. When $\mathbf{Z}$ is singular, it can easily happen that $\mathcal{D}(\mathbf{Z})$ is not affine (see, for example [BGG]). However, as we will show in Theorem 3.8, the $\mathbb{C}$-algebra $\mathcal{D}$ is finitely generated. We begin with some lemmas.

Lemma 3.5. *Let $\gamma \in \Lambda^+$. There exists a nondegenerate pairing of G-modules*

$$\tau : F_{w_0}(\mathcal{O}^\gamma) \times \mathcal{O}^{\gamma^*} \longrightarrow \mathbb{C}, \quad (F_{w_0}(f), g) \mapsto F_{w_0}(f)(g).$$

Proof. By Proposition 2.9 and (2.2), $F_{w_0}(\mathcal{O}^\gamma)(\mathcal{O}^{\gamma^*}) \subseteq \mathcal{O}^{w_0(\gamma)+\gamma^*} = \mathcal{O}^0 = \mathbb{C}$. Also,

$$(a.F_{w_0}(f))(a.g) = a.F_{w_0}(f)(a^{-1}.(a.g)) = a.F_{w_0}(f)(g) = F_{w_0}(f)(g),$$

for all $a \in G$. Therefore, τ is a well defined pairing of G-modules and it induces a G-linear map $F_{w_0}(\mathcal{O}^\gamma) \to (\mathcal{O}^{\gamma^*})^*$. Since $F_{w_0}(\mathcal{O}^\gamma) \cong (\mathcal{O}^{\gamma^*})^* \cong V(\gamma)$, it suffices to show that τ is non zero in order to show that it is non-degenerate. In the notation of Proposition 2.9(4), we will show that $F_{w_0}(f_i)(g_i) \neq 0$ for some i.

By Proposition 2.9(2),

$$F_{w_0}(\alpha^\vee) \;=\; w_0(\alpha^\vee) + \langle w_0(\alpha^\vee) - \alpha^\vee, \rho\rangle \;=\; w_0(\alpha^\vee) - 2\langle \alpha^\vee, \rho\rangle$$

and so Proposition 2.9(4) implies that

$$F_{w_0}(P_\gamma) \;=\; c_\gamma \prod_{\alpha^\vee \in \Delta_+^\vee} \prod_{i=1}^{\langle \gamma, \alpha^\vee\rangle} (w_0(\alpha^\vee) - \langle \alpha^\vee, \rho\rangle - i).$$

Since $w_0(\alpha^\vee) \in \widehat{\mathfrak{h}}$ is a vector field on $\mathbf{X}$, we have $w_0(\alpha^\vee)(1) = 0$ and so

$$F_{w_0}(P_\gamma)(1) \;=\; c_\gamma \prod_{\alpha^\vee \in \Delta_+^\vee} (-1)^{\langle \gamma, \alpha^\vee\rangle} \prod_{i=1}^{\langle \gamma, \alpha^\vee\rangle} (\langle \alpha^\vee, \rho\rangle + i) \;\neq\; 0,$$

since $\langle \alpha^\vee, \rho\rangle > 0$ for all $\alpha^\vee$. By Proposition 2.9(4), $F_{w_0}(P_\gamma)(1) = \sum_i F_{w_0}(f_i)(g_i)$ and so $F_{w_0}(f_i)(g_i) \neq 0$ for some $1 \leq i \leq n$. □

Lemma 3.6. *The $\mathcal{O}(\mathbf{X})$-module map $\mathsf{m} : \mathcal{O}(\mathbf{X}) \otimes F_{w_0}(\mathcal{O}(\mathbf{X})) \to \mathcal{D}(\mathbf{X})$ given by $g \otimes F_{w_0}(f) \mapsto gF_{w_0}(f)$ is injective.*

Proof. Let $t \in \operatorname{Ker}(\mathsf{m})$ and write $t = \sum_{i,j} h_{ij} \otimes F_{w_0}(f_{ij})$ where $\{f_{ij}\}_j$ is a basis of $\mathcal{O}^{\mu_i}$, for some $\mu_1, \ldots, \mu_s \in \Lambda^+$, and $h_{ij} \in \mathcal{O}$ for all i, j. We may assume that s is minimal; thus, for each $1 \leq i \leq s$, some $h_{ij} \neq 0$. Partially order Λ by $\omega^* \geqslant \lambda^*$ if $\omega^* - \lambda^* \in \Lambda^+$ and assume that μ_1^* is minimal among the μ_i^*'s for this ordering. By Lemma 3.5, for each i there exists a basis $\{g_{ij}\}_j$ of $\mathcal{O}^{\mu_i^*}$ such that $F_{w_0}(f_{ij})(g_{ik}) = \delta_{jk}$. When $i > 1$ we have $F_{w_0}(f_{ij})(g_{1k}) \subseteq \mathcal{O}^{\mu_1^* - \mu_i^*} = 0$, since $\mu_1^* - \mu_i^* \notin \Lambda^+$. Therefore, for each k, we have

$$0 = \mathsf{m}(t)(g_{1k}) = \sum_{i,j} h_{ij} F_{w_0}(f_{ij})(g_{1k}) = \sum_j h_{1j} F_{w_0}(f_{1j})(g_{1k}) = h_{1k},$$

contradicting the minimality of s. □

Define a finitely generated subalgebra of $\mathcal{D}(\mathbf{X})$ by

$$\mathcal{S} = \mathbb{C}\langle \mathcal{O}, F_{w_0}(\mathcal{O})\rangle = \mathbb{C}\langle \mathcal{O}^{\varpi_i}, F_{w_0}(\mathcal{O}^{\varpi_j})\,;\, 1 \leq i, j \leq \ell\rangle,$$

where $\ell = \operatorname{rank}(\mathfrak{g})$. The elements of $\mathcal{O}$ act locally nilpotently on $\mathcal{D}$, and therefore on $\mathcal{S}$. Thus $C = \mathcal{O} \smallsetminus \{0\}$ is an Ore subset of $\mathcal{S}$. Let $\mathbb{K} = C^{-1}\mathcal{O}$ denote the field of fractions of $\mathcal{O}$. Recall that $C^{-1}\mathcal{D}(\mathbf{X}) = \mathcal{D}(\mathbb{K})$ is the ring of differential operators on $\mathbb{K}$ and that $\mathcal{D}_r(\mathbf{X}) = \mathcal{D}_r(\mathbb{K}) \cap \mathcal{D}(\mathbf{X})$ for all $r \in \mathbb{N}$ (see, for example, [MR, Theorem 15.5.5]).

Lemma 3.7. *We have $C^{-1}\mathcal{S} = \mathcal{D}(\mathbb{K})$. In particular, for any finite dimensional subspace $E \subset \mathcal{D}(\mathbf{X})$, there exists $0 \neq f \in \mathcal{O}(\mathbf{X})$ such that $fE \subset \mathcal{S}$.*

Proof. The aim of the proof is to apply [LS2, Lemma 8].

Applying the exact functor $\mathbb{K}\otimes_{\mathcal{O}}-$ to the injective map m of Lemma 3.6 yields the $\mathbb{K}$-linear injection $\mathsf{m} : \mathbb{K} \otimes F_{w_0}(\mathcal{O}) \hookrightarrow A = C^{-1}\mathcal{S}$. Since $F_{w_0}(\mathcal{O})$ is a commutative algebra of dimension $n = \dim \mathbf{X} = \operatorname{trdeg}_{\mathbb{C}} \mathbb{K}$, we may pick $u_1, \dots, u_n \in F_{w_0}(\mathcal{O})$ algebraically independent over $\mathbb{C}$ and set $P = \mathbb{C}[u_1, \dots, u_n]$. For any $q \in \mathbb{N}$, denote by P_q the subspace of polynomials of degree at most q and define

$$\Theta_q = \{d \in \mathbb{K} \otimes P : \operatorname{ord} \mathsf{m}(d) \leq q\} \quad \text{and} \quad k = \max\{\operatorname{ord} \mathsf{m}(u_i) : 1 \leq i \leq n\}.$$

Observe that $\{\Theta_q\}_q$ and $\{\mathbb{K} \otimes P_q\}_q$ are two increasing filtrations on the $\mathbb{K}$-vector space $\mathbb{K} \otimes P$ and that the map $q \mapsto \dim_{\mathbb{K}}(\mathbb{K} \otimes P_q)$ is a polynomial function of degree n. Furthermore, since $\mathsf{m}(u_iu_j) = \mathsf{m}(u_i)\mathsf{m}(u_j)$, we have $\mathbb{K} \otimes P_q \subseteq \Theta_{kq}$ for all q. Hence, $\dim_{\mathbb{K}} \Theta_q \geq p(q)$ for some polynomial p of degree n. If $A_q = \mathcal{D}_q(\mathbb{K}) \cap A$, then $\mathsf{m}(\Theta_q) \subseteq A_q$ and so $\dim_{\mathbb{K}} A_q \geq p(q)$. It follows that

$$\limsup_{q\to\infty}\{\log_q(\dim_{\mathbb{K}} A_q/A_{q-1})\} \geq n - 1.$$

The hypotheses of [LS2, Lemma 8] are now satisfied by the pair $A \subseteq \mathcal{D}(\mathbb{K})$ and, by that result, $\mathcal{D}(\mathbb{K}) = A = C^{-1}\mathcal{S}$. The final assertion of the lemma follows by clearing denominators. $\square$

We can now describe a generating set for the $\mathbb{C}$-algebra $\mathcal{D}$, for which we recall the definition of $\widehat{\mathfrak{h}}$ from Notation 2.4.

Theorem 3.8. *As a $\mathbb{C}$-algebra, $\mathcal{D}$ is generated by $\widehat{\mathfrak{h}}$ and the $F_w(\mathcal{O})$, for $w \in W$.*

Proof. Set $\mathcal{B} = \mathbb{C}\langle \widehat{\mathfrak{h}}, F_w(\mathcal{O})\,;\, w \in W\rangle$; thus $\mathcal{B}$ is a G-submodule of $\mathcal{D}$ containing both $\mathcal{S}$ and $U(\widehat{\mathfrak{h}})$. Moreover, as $F_w(U(\widehat{\mathfrak{h}})) = U(\widehat{\mathfrak{h}})$ (see Proposition 2.9(2)), $F_w(\mathcal{B}) = \mathcal{B}$ for all $w \in W$. As $\mathcal{D}$ is a locally finite G-module, it suffices to show that $E \subset \mathcal{B}$ for all finite dimensional G-submodules E of $\mathcal{D}$. For such a module E, set $L = \{b \in \mathcal{B} : bE \subset \mathcal{B}\}$ and $I_w(E) = \{g \in \mathcal{O} : F_w(g)E \subset \mathcal{B}\}$ for $w \in W$. We aim to show that the left ideal L of $\mathcal{B}$ contains 1.

Clearly, $I_w(E)$ is an ideal of $\mathcal{O}$. It is also a G-submodule since

$$F_w(a.g)E = F_w(a.g)a.E = a.(F_w(g)E) \subset a.\mathcal{B} = \mathcal{B}$$

for all $a \in G$ and $g \in I_w(E)$. Since $\mathcal{S} \subseteq \mathcal{B}$, Lemma 3.7 implies that $I_1(E) \neq 0$. For each w, $F_{w^{-1}}(E)$ is a G-submodule of $\mathcal{D}$ (isomorphic to E) and

$$g \in I_w(E) \iff gF_{w^{-1}}(E) \subset \mathcal{B} \iff g \in I_1(F_{w^{-1}}(E)).$$

Thus, $I_w(E) \neq 0$ for all $w \in W$. Since $\mathcal{O}$ is a domain, it follows that $I = \bigcap_{w\in W} I_w(E)$ is a non-zero G-submodule of $\mathcal{O}$. Now, $\mathcal{O} = \bigoplus_{\lambda\in\Lambda^+} \mathcal{O}^\lambda$ and so $\mathcal{O}^\gamma \subset I$ for some $\gamma \in \Lambda^+$. By Proposition 2.9(4), $F_w(P_{\gamma^*}) = \sum_i F_w(f_i)F_{ww_0}(g_i)$

for some $f_i \in \mathcal{O}^{\gamma^*}$ and $g_i \in \mathcal{O}^{\gamma}$. By the definition of I and the choice of γ, we have $F_{ww_0}(g_i)E \subset \mathcal{B}$; that is, $F_{ww_0}(g_i) \in L$, for all $w \in W$. Since $F_w(f_i) \in F_w(\mathcal{O}) \subset \mathcal{B}$ we obtain that $F_w(P_{\gamma^*}) \in L$ for all $w \in W$. Finally, as $U(\widehat{\mathfrak{h}}) \subset \mathcal{B}$, Proposition 2.9(5) implies that $1 \in L$ and hence that $E \subset \mathcal{B}$. □

We conjecture that $\widehat{\mathfrak{h}}$ is unnecessary in the last theorem; i.e., we conjecture that

$$\mathcal{D} = \mathbb{C}\langle F_w(\mathcal{O}) : w \in W\rangle.$$

Using Example 2.8 the authors can prove this for $\mathfrak{g} = \mathfrak{sl}(3)$; indeed we can even show that $\mathcal{D} = \mathbb{C}\langle \mathcal{O}, F_{w_0}(\mathcal{O})\rangle$ in this case. However, the argument heavily uses facts about $\mathfrak{so}(8)$ and so it may not be a good guide to the general case.

4. The $\mathcal{D}(\mathbf{X})$-module $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$

As before, set $\mathbf{X} = G/U$ with associated rings $\mathcal{O} = \mathcal{O}(\mathbf{X})$ and $\mathcal{D} = \mathcal{D}(\mathbf{X})$. In this section we give a complete description of $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = \bigoplus_{i\in\mathbb{N}} \mathrm{H}^i(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ as a $\mathcal{D}$-module. Specifically, we will show that $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is simply the direct sum of the twists $\mathcal{O}^w = \mathcal{O}^{F_w}$ of $\mathcal{O}$ by $w \in W$.

The first cohomology group to consider is $\mathrm{H}^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = \mathcal{O}$. As a $\mathcal{D}$-module, $\mathcal{O} \cong \mathcal{D}/I$ where $I = \{d \in \mathcal{D} : d(1) = 0\}$ is a maximal left ideal of $\mathcal{D}$ (Proposition 3.1). Clearly, $I \supseteq \mathcal{D}\mathfrak{g} + \mathcal{D}\widehat{\mathfrak{h}}$, in the terminology of Notation 2.4. The first main result of this section, Theorem 4.6, shows that this is actually an equality and, further, that $\mathcal{D}/\mathcal{D}\mathfrak{g} \cong \bigoplus_{w\in W} \mathcal{O}^w$.

Before proving this theorem, we need some preliminary notation and lemmas. As in (2.6), we cover $\mathbf{X}$ by affine open subsets $\mathbf{X}_j = \{x \in \overline{\mathbf{X}} : f_j(x) \neq 0\}$ and let $\mathcal{C}_j = \{f_j^s : s \in \mathbb{N}\}$, for $1 \le j \le k$, denote the associated Ore subsets in $\mathcal{D}$. If M is a left $\mathcal{D}$-module, the kernel of the localization map $M \to M_{f_j} = \mathcal{D}(\mathbf{X}_j) \otimes_{\mathcal{D}} M$ is

$$T_j(M) = \{v \in M : f_j^s v = 0 \text{ for some } s > 0\}.$$

Lemma 4.1. *For all $w \in W$ and $x \in \mathfrak{g}$, one has $F_w(x) = x$.*

Proof. By Proposition 2.9(1), F_w is a G-linear automorphism of $\mathcal{D}$ and hence is $\mathfrak{g}$-linear, where $\mathfrak{g}$ acts by the adjoint action. Therefore, for any $\theta \in \mathcal{D}$ and $x \in \mathfrak{g}$,

$$[F_w(x), F_w(\theta)] = F_w([x, \theta]) = [x, F_w(\theta)].$$

Thus $[F_w(x), y] = [x, y]$ for all $y \in \mathcal{D}$. If $y \in \mathfrak{g}$, then $\mathfrak{g}$-linearity also implies that $[F_w(x), y] = F_w([x, y])$ and hence that $[x, y] = [F_w(x), y] = F_w([x, y])$. As $\mathfrak{g}$ is semisimple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and so this implies $F_w(z) = z$ for all $z \in \mathfrak{g}$. □

The next lemma shows why we should expect all the $\mathcal{O}^w$ to appear in a decomposition of $\mathcal{D}/\mathcal{D}\mathfrak{g}$.

Lemma 4.2. (1) *Each $\mathcal{D}$-module $\mathcal{O}^w$ is a factor of $\mathcal{M} = \mathcal{D}/\mathcal{D}\mathfrak{g}$.*
(2) *Set $\mathcal{N} = I/\mathcal{D}\mathfrak{g}$. Then, $\mathcal{N} = T_j(\mathcal{M})$ for $1 \le j \le k$ and $\mathcal{D}/I$ is the unique factor of $\mathcal{M}$ isomorphic to $\mathcal{O}$ as a $\mathcal{D}$-module.*

Proof. (1) Since the action of $d \in \mathcal{D}$ on $\mathcal{O}^w$ is given by $d \cdot f = F_w(d)(f)$ for $f \in \mathcal{O}$, we have $\mathcal{O}^w \cong \mathcal{D}/F_w^{-1}(I)$. As we noted above, $I \supseteq \mathcal{D}\mathfrak{g} + \mathcal{D}\widehat{\mathfrak{h}}$ and so, by Lemma 4.1, $F_w^{-1}(I) \supseteq \mathcal{D}\mathfrak{g}$.

(2) Let $T_x\mathbf{X}$ denote the tangent space of $\mathbf{X}$ at $x \in \mathbf{X}$. Observe that, if $e = U/U \in \mathbf{X}$, then $\mathrm{Stab}_G(e) = U$ and so $\mathfrak{g}/\mathfrak{u}$ identifies naturally with $T_e\mathbf{X}$. Since $\mathbf{X}$ is a homogeneous space, the map $\imath : \mathfrak{g} \to \mathrm{Der}_{\mathbb{C}} \mathcal{O}$ induces an isomorphism $\mathfrak{g}/\mathfrak{u} \cong T_e\mathbf{X} \cong T_x\mathbf{X}$. As $\mathbf{X}_j \subseteq \mathbf{X}$ is affine, it follows that $\mathrm{Der}\, \mathcal{O}(\mathbf{X}_j) = \mathcal{O}(\mathbf{X}_j)\mathfrak{g}$ for each $j = 1, \dots, k$. Furthermore, since $\mathbf{X}_j$ is smooth, the $\mathcal{D}(\mathbf{X}_j)$-module $\mathcal{O}(\mathbf{X}_j)$ is simple and isomorphic to $\mathcal{D}(\mathbf{X}_j)/\mathcal{D}(\mathbf{X}_j)\, \mathrm{Der}\, \mathcal{O}(\mathbf{X}_j)$. Since $\mathcal{O} = \mathcal{M}/\mathcal{N}$, this implies that

$$\mathcal{O}(\mathbf{X}_j) = \mathcal{C}_j^{-1}(\mathcal{M}/\mathcal{N}) \cong \mathcal{D}(\mathbf{X}_j)/\mathcal{D}(\mathbf{X}_j)\mathfrak{g} = \mathcal{C}_j^{-1}\mathcal{M} \tag{4.3}$$

for all j. Therefore $\mathcal{C}_j^{-1}\mathcal{N} = 0$ and so $\mathcal{N} \subseteq T_j(\mathcal{M})$. The equality $\mathcal{N} = T_j(\mathcal{M})$ then follows from $T_j(\mathcal{O}) = 0$.

Now suppose that $\mathcal{O} \cong \mathcal{D}/L$ for some maximal left ideal $L \supseteq \mathcal{D}\mathfrak{g}$. Then $\mathcal{C}_j^{-1}L \supseteq \mathcal{D}(\mathbf{X}_j)\mathfrak{g}$ and $\mathcal{C}_j^{-1}\mathcal{O} = \mathcal{O}(\mathbf{X}_j) \cong \mathcal{D}(\mathbf{X}_j)/\mathcal{C}_j^{-1}L$ for all j. By (4.3), the left ideal $\mathcal{D}(\mathbf{X}_j)\mathfrak{g}$ is maximal and so $\mathcal{C}_j^{-1}L = \mathcal{D}(\mathbf{X}_j)\mathfrak{g}$ for all j. Therefore $L/\mathcal{D}\mathfrak{g} \subseteq T_j(\mathcal{M}) = \mathcal{N} = I/\mathcal{D}\mathfrak{g}$, which implies that $L \subseteq I$. Hence $L = I$. $\square$

As in [Bo, Théorème 2, Section VI.1.5], $\Sigma^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}$ defines a dominant chamber

$$C(\Sigma^\vee) = \{y \in \widehat{\mathfrak{h}}_{\mathbb{R}} : \langle y, \alpha_i \rangle > 0 \text{ for all } i = 1, \dots, \ell\}$$

in the root system $\Delta^\vee = \jmath(\Delta^\vee) \subset \widehat{\mathfrak{h}}_{\mathbb{R}} = \bigoplus_{i=1}^{\ell} \mathbb{R}\alpha_i^\vee \subset \widehat{\mathfrak{h}}$.

Lemma 4.4. *If $y \in C(\Sigma^\vee)$ and $w \in W \smallsetminus \{1\}$, then $F_w(y) \notin I$.*

Proof. Using [Bo, Ch. VI, Section 1.6, Corollaire de la Proposition 18], we have $0 \neq y - w(y) = \sum_{i=1}^{\ell} n_i\alpha_i^\vee$ with $n_i \in \mathbb{R}_+$. Hence, Proposition 2.9(2) implies that

$$F_w(y) - w(y) \;=\; \langle w(y) - y, \rho \rangle \;=\; -\textstyle\sum_i n_i \;\in\; \mathbb{C} \smallsetminus \{0\}.$$

Thus $F_w(y) - w(y) \notin I$. Since $w(y) \in \widehat{\mathfrak{h}} \subset I$, this implies that $F_w(y) \notin I$. $\square$

Lemma 4.5. *As $\mathcal{D}$-modules, $\mathcal{O}^w \cong \mathcal{O}^{w'}$ if and only if $w = w'$.*

Proof. Since $(\mathcal{O}^w)^v \cong \mathcal{O}^{wv}$ it suffices to prove the result when $w' = 1$. So, assume that $\mathcal{O}^w \cong \mathcal{O}$ for some $w \neq 1$ and set $\mathcal{N}_w = F_w^{-1}(I)/\mathcal{D}\mathfrak{g}$; thus $\mathcal{N}_1 = \mathcal{N}$. Then $\mathcal{O}^w = \mathcal{D}/F_w^{-1}(I) = \mathcal{M}/\mathcal{N}_w$ and Lemma 4.2 implies that $\mathcal{N}_w = \mathcal{N}_1$; equivalently, $F_{w^{-1}}(I) = I$. Now pick $y \in C(\Sigma^\vee)$. Since $y \in I$, this implies that $F_{w^{-1}}(y) \in I$, contradicting Lemma 4.4. $\square$

The following theorem gives a precise description of the $\mathcal{D}$-modules $\mathcal{O}$ and $\mathcal{M}$; it shows in particular that $\mathcal{M}$ is a multiplicity free, semisimple module of length $|W|$.

Theorem 4.6. *Write $\mathcal{O}(\mathbf{X}) \cong \mathcal{D}(\mathbf{X})/I$ for $I = \{d \in \mathcal{D}(\mathbf{X}) : d(1) = 0\}$ and define $\mathcal{M} = \mathcal{D}(\mathbf{X})/\mathcal{D}(\mathbf{X})\mathfrak{g}$. Then:*

(1) $\mathcal{M} \cong \bigoplus_{w\in W} \mathcal{O}(\mathbf{X})^w$ *as a* $\mathcal{D}(\mathbf{X})$*-module;*
(2) $I = \mathcal{D}(\mathbf{X})\mathfrak{g} + \mathcal{D}(\mathbf{X})\widehat{\mathfrak{h}} = \mathcal{D}(\mathbf{X})\mathfrak{g} + \mathcal{D}(\mathbf{X})y$ *for all* $y \in C(\Sigma^\vee)$.

Proof. (1) Set $\mathcal{N}_w = F_w^{-1}(I)/\mathcal{D}\mathfrak{g}$, for $w \in W$. By Lemmas 4.2(2) and 4.5, the $\mathcal{D}$-modules $\mathcal{O}^w \cong \mathcal{M}/\mathcal{N}_w$ are nonisomorphic, so the natural map $\mathcal{M} \to \bigoplus_{w\in W} \mathcal{O}^w$ is surjective with kernel $N = \bigcap_{w\in W} \mathcal{N}_w$. It therefore remains to prove that $N = 0$. By Lemma 4.2, if $x \in I$, then there exists $t \in \mathbb{N}$ such that $f_j^t x \in \mathcal{D}\mathfrak{g}$ for $j = 1, \dots, k$. Therefore $F_{w^{-1}}(f_j)^t F_{w^{-1}}(x) \in \mathcal{D}\mathfrak{g}$ for all j; equivalently, each element $[F_{w^{-1}}(x) + \mathcal{D}\mathfrak{g}] \in \mathcal{N}_w$ is torsion for $F_{w^{-1}}(\mathcal{C}_j)$. Consequently, if $v \in N$, then there exists $s \in \mathbb{N}$ such that $F_w(f_j)^s v = 0$ for all $1 \le j \le k$ and $w \in W$. In other words, $N_{f_j}^w = (0)$, for all such j and w. By faithful flatness (Corollary 2.7) this implies that $N = 0$.

(2) It suffices to prove that $I = \mathcal{D}\mathfrak{g} + \mathcal{D}y$ for $y \in C(\Sigma^\vee)$. Set $\mathcal{I} = (\mathcal{D}\mathfrak{g} + \mathcal{D}y)/\mathcal{D}\mathfrak{g}$. By part (1) and Lemma 4.2, the $\mathcal{N}_w$, $w \in W$, are the only maximal submodules of $\mathcal{M}$. Therefore, if $\mathcal{I} \subsetneq \mathcal{N}_1$ we must have $\mathcal{I} \subseteq \mathcal{N}_w$ for some $w \neq 1$. This implies that $F_w(\mathcal{D}\mathfrak{g} + \mathcal{D}y) \subseteq I$, hence $F_w(y) - w(y) \in I$, in contradiction with Lemma 4.4. Therefore $\mathcal{N}_1 = \mathcal{I}$ and $I = \mathcal{D}\mathfrak{g} + \mathcal{D}y$. $\square$

We will prove in Theorem 4.11 that $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is isomorphic to $\mathcal{M}$ as a $\mathcal{D}$-module. In order to prove this, we need to recall some results on the cohomology of line bundles over the flag variety $\mathbf{B} = G/B$.

We begin with some general remarks. Let $\mathbf{Z}$ be a smooth G-variety and write $\tau : \mathfrak{g} \to \operatorname{Der} \mathcal{O}_{\mathbf{Z}}$ for the differential of the G-action. Let $\mathcal{F}$ be a G-equivariant $\mathcal{O}_{\mathbf{Z}}$-module as defined, for example, in [Ka, Section 4.4]. This implies, in particular, that $\mathcal{F}$ is a compatible $(\mathfrak{g}, \mathcal{O}_{\mathbf{Z}})$-module in the sense that for any open subset $\Omega \subseteq \mathbf{Z}$, one has $\xi.(fv) = \tau(\xi)(f)v + f(\xi.v)$ for all $\xi \in \mathfrak{g}, f \in \mathcal{O}_{\mathbf{Z}}(\Omega)$ and $v \in \mathcal{F}(\Omega)$. In this setting, the cohomology group $\mathrm{H}^i(\mathbf{Z}, \mathcal{F})$ inherits a structure of compatible $(G, \mathcal{O}(\mathbf{Z}))$-module, cf. [Ke, Theorem 11.6]. If $\mathcal{F}$ is a coherent G-equivariant $\mathcal{D}_{\mathbf{Z}}$-module, then it follows from [Ka, Sections 4.10 and 4.11] that $\mathrm{H}^i(\mathbf{Z}, \mathcal{F})$ is endowed with a $\mathcal{D}(\mathbf{Z})$-module structure such that $\tau(\xi)v = \frac{d}{dt}_{|t=0}(e^{t\xi}.v)$ for all $v \in \mathrm{H}^i(\mathbf{Z}, \mathcal{F})$ and $\xi \in \mathfrak{g}$. We will apply these observations in two cases: one is when $\mathbf{Z} = \mathbf{X} = G/U$ and $\mathcal{F} = \mathcal{O}_{\mathbf{X}}$ is a G-equivariant $\mathcal{D}_{\mathbf{X}}$-module under left translation; the other is described next.

Let $\pi : G \twoheadrightarrow \mathbf{B} = G/B$, $\phi : G \twoheadrightarrow \mathbf{X}$ and $\varphi : \mathbf{X} \twoheadrightarrow \mathbf{B}$ be the natural projections, thus $\pi = \varphi \circ \phi$. For each $\lambda \in \Lambda$, the one dimensional H-module $\mathbb{C}_{-\lambda}$ can be viewed as a B-module with trivial action of U. As in [Ke, pp.333-335], define the G-equivariant $\mathcal{O}_{\mathbf{B}}$-module $\mathcal{L}(\lambda)$ to be the sections of the line bundle $G \times^B \mathbb{C}_{-\lambda}$. Since $G \times^B \mathbb{C}_{-\lambda} \cong \mathbf{X} \times^H \mathbb{C}_{-\lambda}$, one has

$$\Gamma(\Omega, \mathcal{L}(\lambda)) = \{f : \varphi^{-1}\Omega \to \mathbb{C} : f(\bar{g}h) = \lambda(h)f(\bar{g}) \text{ for } h \in H,\, \bar{g} \in \varphi^{-1}\Omega\}. \tag{4.7}$$

where $\Omega \subseteq \mathbf{B}$ is any open subset.

Recall from (2.3) that the decomposition $\mathcal{O} = \mathcal{O}(\mathbf{X}) = \bigoplus_{\gamma\in\Lambda^+} \mathcal{O}^\gamma$ is induced by the twisted action of H on $\mathbf{X}$. Hence, $h \in H$ acts on $f \in \Gamma(\Omega, \mathcal{L}(\lambda))$ via

$$(r_h.f)(\bar{g}) = f(\bar{g}w_0(h^{-1})) = \lambda(w_0(h)^{-1})f(\bar{g}) = \lambda^*(h)f(\bar{g}). \tag{4.8}$$

Passing to Čech cohomology, H therefore acts on $\mathrm{H}^i(\mathbf{B}, \mathcal{L}(\lambda))$ with weight λ^*.

The cohomology groups of the line bundle $\mathcal{L}(\lambda)$ are described as G-modules by the Borel-Weil-Bott theorem, that we now recall (see [Ja, Corollaries II.5.5 and II.5.6], up to a switch from B to the opposite Borel). The length of $w \in W$ is denoted by $\ell(w)$ and we will write

$$W(i) = \{w \in W : \ell(w) = i\}.$$

The "dot action" of $w \in W$ on $\xi \in \mathfrak{h}^*$ is defined by $w \cdot \xi = w(\xi + \rho) - \rho$.

Theorem 4.9. (Borel-Weil-Bott) *The G-module $\mathrm{H}^i(\mathbf{B}, \mathcal{L}(\lambda))$ is isomorphic to*

$$\begin{cases} V(\mu)^* & \text{if } \exists\,(w,\mu) \in W(i) \times \Lambda^+ \text{ such that } \lambda = w\cdot\mu; \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

The following standard proposition reduces the computation of the G-module $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_\mathbf{X})$ to the Borel-Weil-Bott theorem. We include a proof since we could not find an appropriate reference.

Proposition 4.10. *The morphism $\varphi : \mathbf{X} \to \mathbf{B}$ is affine and $\varphi_*\mathcal{O}_\mathbf{X} \cong \bigoplus_{\nu\in\Lambda} \mathcal{L}(\nu)$ as $(\mathfrak{g}, \mathcal{O}_\mathbf{B})$-modules. In particular, for each $i \in \mathbb{N}$, there is a G-module isomorphism*

$$\mathrm{H}^i(\mathbf{X}, \mathcal{O}_\mathbf{X}) \cong \bigoplus_{\nu\in\Lambda} \mathrm{H}^i(\mathbf{B}, \mathcal{L}(\nu)).$$

Proof. By the Bruhat decomposition, $\mathbf{B}$ is covered by the affine open subsets $\Omega_w = \pi(\dot{w}U^-B)$ where $\dot{w} \in N_G(H)$ is a representative of $w \in W$ and U^- is the opposite maximal unipotent subgroup of G, see [Ja, (II.1.10)]. As $\dot{w}U^-B \cong U^- \times H \times U$, the subset $\varphi^{-1}\Omega_w = \phi(\dot{w}U^-B)$ is affine and isomorphic to $U^- \times H$ as an H-variety. Thus φ is an affine morphism and $\mathrm{H}^i(\mathbf{X}, \mathcal{O}_\mathbf{X}) \cong \mathrm{H}^i(\mathbf{B}, \varphi_*\mathcal{O}_\mathbf{X})$ by [Ha, III, Exercise 4.1].

The affine algebra $\Gamma(\Omega_w, \varphi_*\mathcal{O}_\mathbf{X}) = \mathcal{O}_\mathbf{X}(\varphi^{-1}\Omega_w)$ is endowed with a regular action of H. Hence, $\Gamma(\Omega_w, \varphi_*\mathcal{O}_\mathbf{X})$ decomposes as $\bigoplus_{\nu\in\Lambda} \mathcal{O}_\mathbf{X}(\varphi^{-1}\Omega_w)_\nu$ with

$$\mathcal{O}_\mathbf{X}(\varphi^{-1}\Omega_w)_\nu = \{f : \varphi^{-1}\Omega_w \to \mathbb{C} : f(\bar{g}h) = \nu(h)f(\bar{g}) \text{ for all } h \in H,\ \bar{g} \in \varphi^{-1}\Omega_w\}.$$

Therefore, by (4.7), $\Gamma(\Omega_w, \varphi_*\mathcal{O}_\mathbf{X}) \cong \bigoplus_{\nu\in\Lambda} \Gamma(\Omega_w, \mathcal{L}(\nu))$ as $(\mathfrak{g}, \mathcal{O}_\mathbf{B}(\Omega_w))$-modules, and it follows that $\varphi_*\mathcal{O}_\mathbf{X} \cong \bigoplus_{\nu\in\Lambda} \mathcal{L}(\nu)$ as $(\mathfrak{g}, \mathcal{O}_\mathbf{B})$-modules. $\square$

One consequence of this proposition is that $\mathrm{H}^i(\mathbf{X}, \mathcal{O}_\mathbf{X}) = 0$ for $i > \dim \mathbf{B} = \ell(w_0) = |R^+|$. We can now give the promised description of $\mathrm{H}^*(\mathbf{X}, \mathcal{O}_\mathbf{X})$.

Theorem 4.11. *As $\mathcal{D}(\mathbf{X})$ modules,* $\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \cong \bigoplus_{w\in W(i)} \mathcal{O}(\mathbf{X})^w$, *for* $0 \le i \le \dim B$. *Moreover*

$$\mathrm{H}^*(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \;\cong\; \mathcal{D}(\mathbf{X})/\mathcal{D}(\mathbf{X})\mathfrak{g} \;\cong\; \bigoplus_{w\in W} \mathcal{O}(\mathbf{X})^w$$

is a direct sum of nonisomorphic simple $\mathcal{D}$*-modules.*

Proof. By Proposition 3.1 and Lemma 4.5, the $\mathcal{O}^w$ are nonisomorphic simple modules. Combining Theorem 4.9 with Proposition 4.10 gives

$$\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \;\cong\; \bigoplus_{\nu\in\Lambda} \mathrm{H}^i(\mathbf{B},\mathcal{L}(\nu)) \;\cong\; \bigoplus_{\substack{\mu\in\Lambda^+\\ w\in W(i)}} \mathrm{H}^i(\mathbf{B},\mathcal{L}(w\cdot\mu)).$$

Since $\mathrm{H}^{\ell(w)}(\mathbf{B},\mathcal{L}(w\cdot\mu)) \cong V(\mu^*)$, the multiplicity $[\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) : V(\lambda)]$ is equal to $|W(i)|$ for any $\lambda\in\Lambda^+$.

Fix $w \in W(i)$ and pick $0 \neq e_w$ in the trivial G-module $\mathrm{H}^{\ell(w)}(\mathbf{B},\mathcal{L}(w\cdot 0))$. As $\mathcal{O}_{\mathbf{X}}$ is a $(G\times H)$-equivariant $\mathcal{D}_{\mathbf{X}}$-module, $xe_w = \frac{d}{dt}_{|t=0}(e^{tx}.e_w) = 0$ for $x\in\mathfrak{g}$ and, by (4.8), $y \in \widehat{\mathfrak{h}}$ acts on e_w with weight

$$(w\cdot 0)^* = -w_0(w(\rho)-\rho) = w_0ww_0(\rho) - \rho.$$

Hence $ye_w = \langle w_0ww_0(\rho)-\rho, y\rangle e_w = \langle \rho, w_0w^{-1}w_0(y) - y\rangle e_w$, for all $y\in\widehat{\mathfrak{h}}$. Substituting this into the formula from Proposition 2.9(2) shows that $F_{w_0ww_0}(y)e_w = 0$.

Thus $\big(\mathcal{D}\mathfrak{g} + \mathcal{D}F_{w_0ww_0}(\widehat{\mathfrak{h}})\big)e_w = 0$ and so, by Theorem 4.6(2), $\mathcal{D}e_w \cong \mathcal{O}^{w_0w^{-1}w_0}$. Since the map $w\mapsto w_0w^{-1}w_0$ permutes $W(i)$ and the $\mathcal{O}^w$ are nonisomorphic simple modules, we conclude that $\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \supseteq \sum_{w\in W(i)}\mathcal{D}e_w \cong \bigoplus_{w\in W(i)}\mathcal{O}^w$. By Lemma 4.1, $\mathcal{O}^w \cong \mathcal{O} \cong \bigoplus_{\lambda\in\Lambda^+} V(\lambda)$ as G-modules. Thus, for each $\lambda\in\Lambda^+$, the first paragraph of the proof implies that

$$\Big[\bigoplus\{\mathcal{O}^w : w\in W(i)\} : V(\lambda)\Big] \;=\; |W(i)| \;=\; \big[\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) : V(\lambda)\big].$$

Consequently, $\mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \cong \bigoplus_{w\in W(i)}\mathcal{O}^w$. Theorem 4.6 then implies that

$$\mathrm{H}^*(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \;=\; \bigoplus_{i=0}^{\ell(w_0)} \mathrm{H}^i(\mathbf{X},\mathcal{O}_{\mathbf{X}}) \;\cong\; \bigoplus_{w\in W}\mathcal{O}^w \;\cong\; \mathcal{D}/\mathcal{D}\mathfrak{g},$$

which completes the proof. □

5. Differential operators on S-varieties

In this section we consider highest weight varieties and, more generally, S-varieties $\overline{\mathbf{Y}}$ in the sense of [VP]. These are natural generalizations of the closure $\overline{\mathbf{X}}$ of the basic affine space $\mathbf{X}$ and there is a natural map from $\mathcal{D}(\mathbf{X})$ to the ring of differential operators $\mathcal{D}(\overline{\mathbf{Y}})$ over such a variety. Although this map need not be surjective (see Theorem 6.9) it does carry enough information to prove, under mild assumptions, that $\overline{\mathbf{Y}}$ is $\mathcal{D}$-simple in the sense that $\mathcal{O}(\overline{\mathbf{Y}})$ is a simple $\mathcal{D}(\overline{\mathbf{Y}})$-module. We will continue to write $\mathcal{O} = \mathcal{O}(\mathbf{X})$ and $\mathcal{D} = \mathcal{D}(\mathbf{X})$.

Definition 5.1. An irreducible affine G-variety $\overline{\mathbf{Y}}$ is called an *S-variety* if it contains a dense orbit $\mathbf{Y} = G.v$ such that $U \subseteq \mathrm{Stab}_G(v)$, the stabilizer of v in G.

Remark 5.2. One important feature of S-varieties is that any affine spherical variety (i.e. an irreducible affine G-variety having a dense B-orbit) is a flat deformation of an S-variety (see, for example, [Gr, Theorem 22.3]).

The S-varieties have been completely described in [VP]. We begin with the relevant notation. Set $\Gamma = \sum_{j=1}^{s} \mathbb{N}\gamma_j$, where $\gamma_1, \dots, \gamma_s \in \Lambda^+$ are distinct dominant weights. Write $V_\Gamma = \bigoplus_{j=1}^{s} V(\gamma_j) \ni v_\Gamma = v_{\gamma_1} \oplus \cdots \oplus v_{\gamma_s}$ and define $\mathbf{Y}_\Gamma = G.v_\Gamma$. The following theorem summarizes the results of [VP, Section 3] that we need.

Theorem 5.3. (1) *The closures $\overline{\mathbf{Y}}_\Gamma$ give all the S-varieties.*
(2) *One has $\mathcal{O}(\overline{\mathbf{Y}}_\Gamma) = \bigoplus_{\gamma\in\Gamma^*} \mathcal{O}^\gamma \subseteq \mathcal{O}(\mathbf{Y}_\Gamma) = \bigoplus_{\mu\in\mathbb{Z}\Gamma^*\cap\Lambda^+} \mathcal{O}^\mu$.*
(3) *The following assertions are equivalent:*
 (i) $\mathcal{O}(\overline{\mathbf{Y}}_\Gamma) = \mathcal{O}(\mathbf{Y}_\Gamma)$;
 (ii) $\Gamma = \mathbb{Z}\Gamma \cap \Lambda^+$;
 (iii) $\overline{\mathbf{Y}}_\Gamma$ *is normal and* $\mathrm{codim}_{\overline{\mathbf{Y}}_\Gamma}(\overline{\mathbf{Y}}_\Gamma \smallsetminus \mathbf{Y}_\Gamma) \geq 2$. □

We will always assume that Γ satisfies the equivalent conditions from Theorem 5.3(3); that is:

$$\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma. \tag{5.4}$$

Hence, for an S-variety $\overline{\mathbf{Y}}_\Gamma$ satisfying (5.4) we have $\mathcal{D}(\mathbf{Y}_\Gamma) = \mathcal{D}(\overline{\mathbf{Y}}_\Gamma)$. Notice also that $\Gamma^* = \sum_{j=1}^{s} \mathbb{N}\gamma_j^*$ satisfies (5.4) if and only if Γ does. Natural examples of S-varieties satisfying (5.4) are the following and more examples can be found in [Gr, Ch. 2, §11]).

Examples 5.5. (1) *For $\Gamma = \Lambda^+$ we obtain the basic affine space $\mathbf{X} = \mathbf{Y}_{\Lambda^+}$.*
(2) *Let $\gamma \in \Lambda^+$ and $\Gamma = \mathbb{N}\gamma$. Then, $\mathbf{Y}_\Gamma$ will be denoted by $\mathbf{Y}_\gamma$ and is the orbit of a highest weight vector $v_\gamma \in V(\gamma)$. Its closure $\overline{\mathbf{Y}}_\gamma$ is called a* highest weight *or HV-variety* [VP, Section 1].
(3) *An important example of highest weight variety is the closure of the minimal (nonzero) orbit in a simple Lie algebra $\mathfrak{g}$: in this case $\gamma = \tilde{\alpha}$ is the highest root and $V(\tilde{\alpha}) \cong \mathfrak{g}$ is the adjoint representation.*

Set $S_\Gamma = \mathrm{Stab}_G(v_\Gamma)$; thus $\mathbf{Y}_\Gamma \cong G/S_\Gamma$. Since $S_\Gamma = \bigcap_{j=1}^{s} \mathrm{Stab}_G(v_{\gamma_j})$, we see that $\mathfrak{h}$ normalizes $\mathfrak{s}_\Gamma = \mathrm{Lie}(S_\Gamma)$. Let $\Delta(\mathfrak{h}, \mathfrak{s}_\Gamma)$ denote the set of roots of $\mathfrak{h}$ in the Lie algebra $\mathfrak{s}_\Gamma$. Then $S_\Gamma = S'_\Gamma Q_\Gamma$ where $Q_\Gamma = H \cap S_\Gamma$ and S'_Γ is generated by the one parameter groups U_α, $\alpha \in \Delta(\mathfrak{h}, \mathfrak{s}_\Gamma)$ (see [VP, p. 753] or [Gr, Corollary 3.5]). Clearly, $Q_\Gamma = \{h \in H : \forall\gamma \in \Gamma,\ \gamma(h) = 1\}$ is a diagonalizable subgroup of H with character group $\Lambda/\mathbb{Z}\Gamma$. Under the right action of the given groups on $\mathcal{O}(\mathbf{Y}_\Gamma)$, the proof of [Gr, Lemma 17.1(b)] shows that

$$\mathcal{O}(\mathbf{Y}_\Gamma) = \mathcal{O}(G)^{S_\Gamma} = \mathcal{O}(G)^{UQ_\Gamma} = \mathcal{O}^{Q_\Gamma}.$$

When (5.4) holds, this implies that $\mathcal{O}(\overline{\mathbf{Y}}_\Gamma) = \mathcal{O}(\mathbf{Y}_\Gamma) = \mathcal{O}^{Q_\Gamma}$ and $\overline{\mathbf{Y}}_\Gamma = \overline{\mathbf{X}}/\!\!/Q_\Gamma$.

If an algebraic group L acts on the right on some variety $\mathbf{Z}$ we denote by $\delta_g : z \mapsto z.g$ the right translation by $g \in L$ on $\mathbf{Z}$. It induces a right action on $f \in \mathcal{O}(\mathbf{Z})$ by $\delta_g.f(z) = f(z.g)$. This applies for example to S_Γ acting on G and Q_Γ acting on $\overline{\mathbf{X}}$. In this notation, $\mathcal{D}(\mathbf{X})^\mu = \{d \in \mathcal{D}(\mathbf{X}) : \forall h \in H,\, \delta_h.d = \mu^*(h)d\}$ for any $\mu \in \Lambda$.

When $\overline{\mathbf{Y}}_\Gamma = \overline{\mathbf{Y}}_\gamma$ is an HV-variety we will set $S_\gamma = S_\Gamma$ and $Q_\gamma = Q_\Gamma$, etc.

Proposition 5.6. *Suppose that $\overline{\mathbf{Y}}_\Gamma$ is an S-variety satisfying* (5.4)*. Then $\mathcal{D}^{Q_\Gamma} = \bigoplus_{\mu\in\mathbb{Z}\Gamma^*}\mathcal{D}^\mu$ and there is a natural morphism of algebras $\psi_\Gamma : \mathcal{D}^{Q_\Gamma} \to \mathcal{D}(\mathbf{Y}_\Gamma)$.*

Proof. Let $d = \sum_{\mu\in\Lambda} d_{\mu^*}$ with $d_{\mu^*} \in \mathcal{D}(\mathbf{X})^{\mu^*}$. It is clear that $d \in \mathcal{D}(\mathbf{X})^{Q_\Gamma}$ if and only if $\delta_h.d_{\mu^*} = d_{\mu^*}$, for all $h \in Q_\Gamma$. This condition is equivalent to $\mu(h)d_{\mu^*} = d_{\mu^*}$; that is, $\mu(h) = 1$ when $d_{\mu^*} \neq 0$. Since Q_Γ is a diagonalizable group, [Sp, Proposition 2.5.7(iii)] implies that $\mathbb{Z}\Gamma = \{\mu \in \Lambda : \forall h \in Q_\Gamma,\, \mu(h) = 1\}$. It follows that $\mathcal{D}(\mathbf{X})^{Q_\Gamma} = \bigoplus_{\mu\in\mathbb{Z}\Gamma^*}\mathcal{D}(\mathbf{X})^\mu$. The morphism ψ_Γ is simply the restriction morphism coming from the identification $\overline{\mathbf{Y}}_\Gamma = \overline{\mathbf{X}}/\!\!/Q_\Gamma$. □

An affine variety $\mathbf{Z}$ (respectively an algebra R) is called $\mathcal{D}$*-simple* if $\mathcal{O}(\mathbf{Z})$ (respectively R) is a simple $\mathcal{D}(\mathbf{Z})$-module (respectively $\mathcal{D}(R)$-module). This does not hold for arbitrary varieties; for example when $\mathbf{Z}$ is the cubic cone in $\mathbb{C}^3$, $\mathcal{O}(\mathbf{Z})$ does not even have finite length as a $\mathcal{D}(\mathbf{Z})$-module [BGG]. It is, however, important in many situations to know that a variety is $\mathcal{D}$-simple. We first note that for S-varieties satisfying (5.4) this is an easy consequence of Proposition 3.1:

Proposition 5.7. *Let $\overline{\mathbf{Y}}_\Gamma$ be an S-variety satisfying* (5.4)*. Then $\mathcal{O}(\mathbf{Y}_\Gamma)$ is a simple left $\mathcal{D}(\mathbf{Y}_\Gamma)$-module.*

Proof. Since $\mathcal{O}(\mathbf{Y}_\Gamma) = \mathcal{O}(\mathbf{X})^{Q_\Gamma}$ with Q_Γ reductive, $\mathcal{O}(\mathbf{Y}_\Gamma)$ is an $\mathcal{O}(\mathbf{Y}_\Gamma)$-module summand of $\mathcal{O}(\mathbf{X})$. The proposition is now a consequence of Proposition 3.1 and the following result. □

Proposition 5.8. ([Sm, Proposition 3.1]) *Let $R \hookrightarrow T$ be an inclusion of commutative $\mathbb{C}$-algebras and suppose that R is a direct summand of the R-module T. If T is $\mathcal{D}$-simple, then R is $\mathcal{D}$-simple.* □

We next refine Proposition 5.7 by showing that $\mathcal{O}(\mathbf{Y}_\Gamma)$ is a simple module over a rather explicit subring of $\mathcal{D}(\mathbf{Y}_\Gamma)$. Let $\mathcal{S}_\Gamma$ be the subalgebra of $\mathcal{D}(\mathbf{X})$ generated by the two finitely generated commutative subalgebras $\mathcal{O}(\mathbf{Y}_\Gamma)$ and $F_{w_0}(\mathcal{O}(\mathbf{Y}_{\Gamma^*}))$. Observe that $F_{w_0}(\mathcal{O}(\mathbf{Y}_{\Gamma^*}))$ is isomorphic to $\mathcal{O}(\mathbf{Y}_{\Gamma^*})$ (as both an algebra and a G-module) and so

$$\mathcal{S}_\Gamma = \mathbb{C}\langle\, \mathcal{O}^{\gamma_i^*}, F_{w_0}(\mathcal{O}^{\gamma_j})\,;\, 1 \le i,j \le s\rangle.$$

By Proposition 2.9(1), $F_{w_0}(\mathcal{O}^\gamma) \subseteq \mathcal{D}^{-\gamma^*}$ and so $\mathcal{S}_\Gamma \subseteq \mathcal{D}(\mathbf{X})^{Q_\Gamma}$, by Proposition 5.6. We will consider $\mathcal{O}(\mathbf{Y}_\Gamma)$ as an $\mathcal{S}_\Gamma$-module through the map ψ_Γ defined in the latter result.

Proposition 5.9. *Let $\overline{\mathbf{Y}}_\Gamma$ be an S-variety satisfying* (5.4)*. Then $\mathcal{O}(\mathbf{Y}_\Gamma)$ is a simple $\mathcal{S}_\Gamma$-module.*

Proof. As in Lemma 3.6, partially order Λ by $\omega^* \geqslant \lambda^*$ if $\omega^* - \lambda^* \in \Lambda^+$. Let $0 \neq g \in \mathcal{O}(\mathbf{Y}_\Gamma)$ and write $g = g_{\lambda_0^*} + \sum_{\lambda_j^* \ngeqslant \lambda_0^*} g_{\lambda_j^*}$ with $g_{\lambda_0^*} \neq 0$ and $g_{\lambda_i^*} \in \mathcal{O}^{\lambda_i^*} \subset \mathcal{O}(\mathbf{Y}_\Gamma)$ for all i. By Lemma 3.5 there exists $f \in \mathcal{O}^{\lambda_0}$ such that $F_{w_0}(f)(g_{\lambda_0^*}) = 1$. Hence Proposition 2.9(1) implies that $F_{w_0}(f)(g) = 1 + \sum_{\lambda_j^* \ngeqslant \lambda_0^*} F_{w_0}(f)(g_{\lambda_j^*})$ with $F_{w_0}(f)(g_{\lambda_j^*}) \in \mathcal{O}^{-\lambda_0^* + \lambda_j^*}$. But $\mathcal{O}^{-\lambda_0^* + \lambda_j^*} = 0$ when $\lambda_j^* \ngeqslant \lambda_0^*$. Thus $F_{w_0}(f)(g) = 1$ and $\mathcal{O}(\mathbf{Y}_\Gamma)$ is simple over $\mathcal{S}_\Gamma$. □

Let $\mathbf{Z}$ be an affine variety with $R = \mathcal{O}(\mathbf{Z})$ and consider the subalgebra

$$\Delta(R) \;=\; \Delta(\mathbf{Z}) \;=\; \mathbb{C}\langle\, \mathcal{O}(\mathbf{Z}), \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(\mathbf{Z})\,\rangle \;\subseteq\; \mathcal{D}(\mathbf{Z}).$$

It is known [MR, Corollary 15.5.6] that $\Delta(\mathbf{Z}) = \mathcal{D}(\mathbf{Z})$ when $\mathbf{Z}$ is smooth. The Nakai conjecture [Na] says that the converse should be true:

$$\Delta(\mathbf{Z}) = \mathcal{D}(\mathbf{Z}) \overset{?}{\Longrightarrow} R \text{ is regular.}$$

The reader can consult [Be, Tr], [Sc, Section 12.3] and the references therein for work related to this conjecture.

The following observation, which is implicit in [Tr], implies that many singular varieties, notably S-varieties satisfying (5.4), do satisfy the conclusion of Nakai's conjecture.

Lemma 5.10. *Assume that $\mathbf{Z}$ is irreducible and $\mathcal{O}(\mathbf{Z})$ is $\mathcal{D}$-simple. Then the Nakai conjecture holds for $\mathbf{Z}$.*

Proof. Suppose that $\mathcal{D}(R) = \Delta(R)$. Then R is simple as a $\Delta(R)$-module and the result follows from [MR, Theorem 15.3.8]. □

Corollary 5.11. (1) *The Nakai conjecture holds for S-varieties satisfying* (5.4)*.*
(2) *More generally, suppose that $R \subseteq T$ are finitely generated $\mathbb{C}$-algebras such that R is a summand of the R-module T and that T is $\mathcal{D}$-simple. Then the Nakai conjecture holds for R.*

Proof. Part (1) is immediate from Lemma 5.10 combined with Proposition 5.7. Similarly, part (2) follows from Lemma 5.10 and Proposition 5.8. □

One significance of the Nakai conjecture is that it implies the Zariski-Lipman conjecture: $\mathbf{Z}$ is smooth whenever $\mathrm{Der}_{\mathbb{C}}\, \mathcal{O}(\mathbf{Z})$ is a projective $\mathcal{O}(\mathbf{Z})$-module. Part (1) of Corollary 5.11 does not give new information about that conjecture; indeed since S-varieties always have graded coordinate rings, the Zariski-Lipman conjecture for these varieties already follows from [Ho]. It is not clear whether part (2) of Corollary 5.11 has significant applications in this direction.

A natural situation where Corollary 5.11 applies is for invariant rings:

Corollary 5.12. *Let Q be an affine algebraic group and $\mathbf{V}$ be an irreducible affine Q-variety. Suppose that $\mathcal{O}(\mathbf{V})$ is $\mathcal{D}$-simple (for example, when $\mathbf{V}$ is smooth). Then the Nakai conjecture holds for $\mathbf{V}/\!\!/Q$ in the following two cases:*

(a) *Q is reductive;*

(b) *$\mathbf{V}$ is a G-variety and $Q = U$.*

Proof. (a) As in the proof of Proposition 5.7, $\mathcal{O}(\mathbf{V}/\!\!/Q)$ is a summand of $\mathcal{O}(\mathbf{V})$. Thus Corollary 5.11(2) applies.

(b) Observe that $\mathcal{D}(\overline{\mathbf{X}} \times \mathbf{V}) \cong \mathcal{D}(\mathbf{X}) \otimes \mathcal{D}(\mathbf{V})$; therefore, by hypothesis and Proposition 3.1 (or Proposition 5.9), $\mathcal{O}(\overline{\mathbf{X}} \times \mathbf{V}) \cong \mathcal{O}(\mathbf{X}) \otimes \mathcal{O}(\mathbf{V})$ is $\mathcal{D}$-simple. By [Kr, III.3.2, p. 191]), $(\overline{\mathbf{X}} \times \mathbf{V})/\!\!/G \cong \mathbf{V}/\!\!/U$ where G acts componentwise on $\overline{\mathbf{X}} \times \mathbf{V}$. Thus the result follows from (a) applied to G acting on $\overline{\mathbf{X}} \times \mathbf{V}$. □

Surprisingly, and despite the simplicity of its proof, only special cases of Corollary 5.12 have appeared before in the literature and these have typically required substantially harder proofs. See, for example, [Is, Theorem 2.3] and [Sc, Section 12.3].

6. Exotic differential operators

The results from the last section raise two questions which we study in this section. First, can one say more about the structure, notably the order, of the exotic differential operators in $\mathcal{D}(\mathbf{Y}_\Gamma)$ induced from the ring $\mathcal{S}_\Gamma$ of Proposition 5.9? This is answered by Theorem 6.7 and Corollary 6.8 and proves Corollary 1.5 from the introduction.

The second question concerns the following basic question in the theory of rings of differential operators. If V is a finite dimensional representation of a reductive group K, then restriction of operators induces a ring homomorphism $\mathcal{D}(V)^K \to \mathcal{D}(V/\!\!/K)$. When is this map surjective? The conjectural answer is that this happens if and only if $V/\!\!/K$ is singular. Positive answers to this question have been found in many cases and these solutions had significant applications to Lie theory (see, for example, [Jo2, Jo3, LS1, Sc]). These results have almost always been in situations where $V/\!\!/K$ is a highest weight variety. Now let $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_\Gamma$ be any HV-variety, or even S-variety satisfying (5.4). It is natural to ask whether the resulting map $\psi_\Gamma : \mathcal{D}(\mathbf{X})^{Q_\Gamma} \to \mathcal{D}(\mathbf{Y})$ is surjective. As we will show in Theorem 6.9, this even fails for the minimal orbit $\mathbf{O}_{min} = \mathbf{Y}_{\tilde{\alpha}}$ of a simple classical Lie algebra $\mathfrak{g}$. This proves Proposition 1.6 from the introduction.

The idea behind the proof of Theorem 6.9 is as follows. Let $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_\gamma$ be an HV-variety. Then $\mathcal{O}(\mathbf{Y}) = \bigoplus_{p \in \mathbb{N}} \mathcal{O}^{p\gamma^*}$ is an $\mathbb{N}$-graded algebra and so $\mathcal{D}(\mathbf{Y})$ is $\mathbb{Z}$-graded by (2.1). There exist examples of HV-varieties $\overline{\mathbf{Y}}$, notably the closures of minimal orbits $\mathbf{Y}_{\tilde{\alpha}}$, for which one can find an irreducible G-module E consisting of "exotic" operators of degree -1 and order at most 4, see [AB, BK2, LS3]. On the other hand, by combining (2.2) with Propositions 2.9(1,3) and 5.6 the only obvious operators of order -1 in $\mathrm{Im}(\psi_\gamma)$ and small order are those in $\psi_\gamma(F_{w_0}(\mathcal{O}^\gamma))$. But their order is only bounded above by $k(\gamma)$, where we write

$$k(\lambda) = \langle \lambda, 2\rho^\vee \rangle = 2\sum_i m_i \qquad \text{for} \quad \lambda = \sum_i m_i\alpha_i \in \Lambda^+. \tag{6.1}$$

Typically $k(\gamma)$ is significantly larger than 4 (see the table at the end of the section). The aim of the proof is therefore to show that for the minimal orbit the bound $k(\gamma)$ is attained and hence that the G-module E cannot lie in $\mathrm{Im}(\psi_\gamma)$.

We begin with some technical lemmas, the first of which is a mild generalization of [BBP, Lemma 3.8]. The subalgebra $U(\widehat{\mathfrak{h}}) \cong S(\mathfrak{h})$ of $\mathcal{D}$ can be identified with $\mathcal{O}(\mathfrak{h}^*)$ and it follows from the definitions that $u(g) = u(\lambda)g$ for $u \in U(\widehat{\mathfrak{h}})$ and $g \in \mathcal{O}^\lambda$. We denote by $\{U_m(\widehat{\mathfrak{h}})\}_{m\in\mathbb{N}}$ the standard filtration on the enveloping algebra $U(\widehat{\mathfrak{h}})$.

Lemma 6.2. *Let $f_1, \ldots, f_n \in \mathcal{O}$ be linearly independent and pick $D \in \mathcal{D}_p$. Suppose that there exist functions $c_i : \Lambda^+ \to \mathbb{C}$ such that $D(g) = \sum_{i=1}^n c_i(\mu)f_ig$ for all $g \in \mathcal{O}^\mu$ and $\mu \in \Lambda^+$. Then $D = \sum_{i=1}^n f_i\tilde{c}_i$ for some $\tilde{c}_1, \ldots, \tilde{c}_n \in U_p(\widehat{\mathfrak{h}})$.*

Proof. For any function $c : \Lambda^+ \to \mathbb{C}$ and $\lambda \in \Lambda^+$, define $T_\lambda(c) : \Lambda^+ \to \mathbb{C}$ by $T_\lambda(c)(\mu) = c(\lambda+\mu) - c(\mu)$. Since $\operatorname{ord} D \leq p$, we have $[g_{p+1}, [g_p, [\ldots, [g_1, D]] \ldots] = 0$ for all $0 \neq g_j \in \mathcal{O}^{\lambda_j}$. This easily implies that $T_{\lambda_1} \circ T_{\lambda_2} \circ \cdots \circ T_{\lambda_p} \circ T_{\lambda_{p+1}}(c_i) = 0$ for all $\lambda_i \in \Lambda^+$. Since T_λ is a difference operator, it follows that c_i is a polynomial function of degree $\leq p$ on Λ^+ and it is clear that there exist $\tilde{c}_i \in U(\mathfrak{h})$ (of degree $\leq p$) such that $\tilde{c}_i(\lambda) = c_i(\lambda)$ for all $\lambda \in \Lambda^+$. Obviously, $D(g) = \left(\sum_{i=1}^n f_i\tilde{c}_i\right)(g)$ for all $g \in \mathcal{O}^\lambda$, hence the result. $\square$

[Sh2, Theorem 1] shows that, for all $\gamma \in \Lambda^+$ and $w \in W$, one has $\mathcal{D}[\gamma]^{w(\gamma)} \cong U(\widehat{\mathfrak{h}}) \otimes E$, where E is a $G \times H$-module of dimension $\dim V(\gamma)$. The next lemma provides an explicit $G \times H$-module E with this property.

Lemma 6.3. *Let $\gamma \in \Lambda^+$. Then, via the multiplication map $\mathsf{m} : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$, we have*

$$\mathcal{D}_p[\gamma]^{w(\gamma)} = F_w(\mathcal{O}^\gamma) \otimes U_p(\widehat{\mathfrak{h}}) = U_p(\widehat{\mathfrak{h}}) \otimes F_w(\mathcal{O}^\gamma)$$

for all $w \in W$ and $p \in \mathbb{N}$. In particular, $\mathcal{D}[\gamma]^{w(\gamma)}$ is a free $U(\widehat{\mathfrak{h}})$-module with basis being any $\mathbb{C}$-basis of $F_w(\mathcal{O}^\gamma)$.

Proof. It is sufficient to prove the lemma for $w = 1$; indeed, applying F_w to the equalities $\mathcal{D}_p[\gamma]^\gamma = \mathcal{O}^\gamma \otimes U_p(\widehat{\mathfrak{h}}) = U_p(\widehat{\mathfrak{h}}) \otimes \mathcal{O}^\gamma$ and appealing to Proposition 2.9(1) gives the general result.

By [Sh2, Theorem 1], $n = \mathrm{rk}_{U(\widehat{\mathfrak{h}})}\, \mathcal{D}[\gamma]^\gamma = \dim V(\gamma)$. Let $\{D_1, \ldots, D_n\}$ be a basis of the right $U(\widehat{\mathfrak{h}})$-module $\mathcal{D}[\gamma]^\gamma$, and fix a basis $\{f_1, \ldots, f_n\}$ of the G-module $\mathcal{O}^\gamma$. It is easy to see that $U_p(\widehat{\mathfrak{h}}) \otimes \mathcal{O}^\gamma \cong U_p(\widehat{\mathfrak{h}})\mathcal{O}^\gamma = \mathcal{O}^\gamma U_p(\widehat{\mathfrak{h}}) \cong \mathcal{O}^\gamma \otimes U_p(\widehat{\mathfrak{h}})$, and that this space is contained in $\mathcal{D}[\gamma]^\gamma$. For the converse, write $f_j = \sum_i D_i a_{ij}$ for $a_{ij} \in U(\widehat{\mathfrak{h}})$. Then, $f_jg = \sum_i a_{ij}(\mu)D_i(g)$ for all $0 \neq g \in \mathcal{O}^\mu$. Thus $\bigoplus_{j=1}^n \mathbb{C}f_jg \subseteq \sum_{j=1}^n \mathbb{C}D_j(g)$.

Since $\dim\big(\bigoplus_{j=1}^{n}\mathbb{C}f_jg\big) = n \geq \dim\big(\sum_{j=1}^{n}\mathbb{C}D_j(g)\big)$, we obtain that $\bigoplus_{j=1}^{n}\mathbb{C}f_jg = \bigoplus_{j=1}^{n}\mathbb{C}D_j(g)$. Therefore, for all $g \in \mathcal{O}^{\mu}$, there exist unique elements $c_{ij}(g,\mu) \in \mathbb{C}$ such that $D_j(g) = \sum_{i=1}^{n} c_{ij}(g,\mu)f_ig$.

We next show that the $c_{ij}(g,\mu)$'s depend only on μ. Indeed, it follows from

$$f_jg = \sum_k a_{kj}(\mu)D_k(g) = \sum_i\sum_k a_{kj}(\mu)c_{ik}(g,\mu)f_ig$$

that $\sum_k c_{ik}(g,\mu)a_{kj}(\mu) = \delta_{ij}$ for all $1 \leq i,j \leq n$. In other words, the matrix $[c_{ik}(g,\mu)]_{ik}$ is the inverse of the matrix $[a_{kj}(\mu)]_{kj}$. Since the a_{kj}'s do not depend on g, nor do the $c_{ik}(g,\mu)$.

We can therefore write $D_j(g) = \sum_{i=1}^{n} c_{ij}(\mu)f_ig$ for any $g \in \mathcal{O}^{\mu}$. By Lemma 6.2 we deduce that $D_j = \sum_{i=1}^{n} f_i\tilde{c}_{ij}$ with $\tilde{c}_{ij} \in U_p(\widehat{\mathfrak{h}})$. □

Return to an S-variety $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_{\Gamma}$ satisfying (5.4) and define $\psi = \psi_{\Gamma} : \mathcal{D}(\mathbf{X})^{Q_{\Gamma}} \to \mathcal{D}(\mathbf{Y})$ by Proposition 5.6. Recall that $\Gamma = \sum_{i=1}^{s}\mathbb{N}\gamma_i$ for some $\gamma_i \in \Lambda^+$ and order the γ_i so that $\mathbb{Q}\Gamma = \bigoplus_{i=1}^{r}\mathbb{Q}\gamma_i$. Set $\mathfrak{t} = \{h \in \widehat{\mathfrak{h}} : \langle \Gamma^*, h\rangle = 0\}$; thus $\mathfrak{t} \cong \mathrm{Lie}(Q_{\Gamma})$ has dimension $\mathrm{rank}(\mathfrak{g}) - r$. Pick $x_1,\dots,x_r \in \widehat{\mathfrak{h}}$ such that $\widehat{\mathfrak{h}} = \big(\bigoplus_{i=1}^{r}\mathbb{C}x_i\big) \oplus \mathfrak{t}$ and $\langle \gamma_i^*, x_j\rangle = \delta_{ij}$ for $1 \leq i,j \leq r$. Set $y_j = \psi(x_j) \in \mathcal{D}(\mathbf{Y})$ and let $u \in U(\widehat{\mathfrak{h}})$. From the identity $u(f) = \psi(u)(f) = u(\mu)f$ for $\mu \in \Gamma^*$ and $f \in \mathcal{O}^{\mu}$, one deduces easily that $\psi(u) = 0$ when $u \in \mathfrak{t}U(\widehat{\mathfrak{h}})$ and that ψ induces an isomorphism of polynomial algebras:

$$\psi : \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1,\dots,x_r] \xrightarrow{\sim} \mathbb{C}[\mathbf{y}] = \mathbb{C}[y_1,\dots,y_r].$$

For $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{N}^r$ we set $x^{\mathbf{m}} = \prod_{i=1}^{r} x_i^{m_i}$ and $y^{\mathbf{m}} = \prod_{i=1}^{r} y_i^{m_i}$. Note that if $u(y) \in \mathbb{C}[\mathbf{y}]$, the (total) degree of $u(y)$ coincides with its order as a differential operator on $\mathbf{Y}$. We recall the definition of $k(\lambda)$ from (6.1).

Proposition 6.4. *Let $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_{\Gamma}$ satisfy* (5.4) *and set $\psi = \psi_{\Gamma}$. Let $\gamma \in \Gamma$, $0 \neq f \in \mathcal{O}^{\gamma}$ and $u(y) \in \mathbb{C}[\mathbf{y}]$. Then,*

$$\mathrm{ord}\,\psi(F_{w_0}(f))u(y) = k(\gamma) + \deg u(y).$$

In particular, the elements $\{\psi(F_{w_0}(f_{\mathbf{m}}))y^{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^r\}$ are linearly independent for any $f_{\mathbf{m}} \in \mathcal{O}^{\gamma} \smallsetminus \{0\}$.

Proof. As the variety $\overline{\mathbf{Y}}$ is irreducible, the associated graded ring $\mathrm{gr}\,\mathcal{D}(\mathbf{Y}) = \bigoplus_k \mathcal{D}_k(\mathbf{Y})/\mathcal{D}_{k-1}(\mathbf{Y})$ is a domain and so $\mathrm{ord}\,ab = \mathrm{ord}\,a + \mathrm{ord}\,b$ for $a,b \in \mathcal{D}(\mathbf{Y})$. Since $\mathrm{ord}\,u(y) = \deg u(y)$, it therefore suffices to show that $\mathrm{ord}\,\psi(F_{w_0}(f)) = k(\gamma)$. By Proposition 2.9(3), we do have $\mathrm{ord}\,\psi(F_{w_0}(f)) \leq \mathrm{ord}\,F_{w_0}(f) = k(\gamma)$.

Consider the operator $P_{\gamma^*} = \sum_i g_iF_{w_0}(f_i)$ (where $g_i \in \mathcal{O}^{\gamma^*}$, $f_i \in \mathcal{O}^{\gamma}$) defined by Proposition 2.9(4). For each $\alpha^{\vee} \in \Delta_+^{\vee}$ we have $\alpha^{\vee} = \sum_{j=1}^{r}\langle \alpha^{\vee}, \gamma_j^*\rangle x_j + z$ with $z \in \mathfrak{t}$. Thus, applying ψ to (2.10) gives

$$\psi(P_{\gamma^*}) = c_{\gamma^*}\prod_{\alpha^{\vee}\in\Delta_+^{\vee}}\prod_{i=1}^{\langle\gamma^*,\alpha^{\vee}\rangle}\Big(\sum_j \langle \alpha^{\vee}, \gamma_j^*\rangle y_j + \langle \alpha^{\vee}, \rho\rangle - i\Big). \tag{6.5}$$

Write $\gamma^* = \sum_{j=1}^r m_j\gamma_j^*$ for some $m_j \in \mathbb{Q}$ and pick α such that $\langle \gamma^*, \alpha^\vee\rangle \neq 0$. Then $\sum_j m_j \langle \alpha^\vee, \gamma_j^*\rangle \neq 0$ and so $\langle \alpha^\vee, \gamma_j^*\rangle \neq 0$ for some $1 \leq j \leq r$. Hence $\deg \psi(\alpha^\vee) = \deg\big(\sum_j \langle \alpha^\vee, \gamma_j^*\rangle y_j\big) = 1$. Thus (6.5) implies that $\psi(P_{\gamma^*}) = \sum_i g_i \psi(F_{w_0}(f_i))$ has order $\sum_{\alpha^\vee \in \Delta_+^\vee} \langle \gamma^*, \alpha^\vee\rangle = k(\gamma)$. Therefore, there exists $i \in \{1,\dots,n\}$ such that $\operatorname{ord}\psi(F_{w_0}(f_i)) = k(\gamma)$. In particular, $\psi(F_{w_0}(\mathcal{O}^\gamma)) \neq 0$.

Consider the symbol map $\operatorname{gr}_{k(\gamma)} : \mathcal{D}_{k(\gamma)}(\mathbf{Y}) \to \mathcal{D}_{k(\gamma)}(\mathbf{Y})/\mathcal{D}_{k(\gamma)-1}(\mathbf{Y})$. Notice that $\operatorname{gr}_{k(\gamma)}$ is a morphism of G-modules and that, by the previous paragraph, $\operatorname{gr}_{k(\gamma)}(\psi(F_{w_0}(f_i))) \neq 0$ for some $f_i \in \mathcal{O}^\gamma$. Since $\psi(F_{w_0}(\mathcal{O}^\gamma)) \cong V(\gamma)$ is an irreducible G-module, we deduce that $\operatorname{gr}_{k(\gamma)}(\psi(F_{w_0}(f))) \neq 0$ for all $0 \neq f \in \mathcal{O}^\gamma$; that is to say, $\operatorname{ord}\psi(F_{w_0}(f)) = k(\gamma)$. This proves the first assertion, from which the second follows immediately. $\square$

Corollary 6.6. *Let $\overline{\mathbf{Y}}_\Gamma$ satisfy* (5.4)*, write $\psi = \psi_\Gamma$ and pick $\gamma \in \Gamma$. Then ψ induces an isomorphism of G-modules:*

$$F_{w_0}(\mathcal{O}^\gamma) \otimes \mathbb{C}[\mathbf{x}] \xrightarrow{\sim} \psi(\mathcal{D}[\gamma]^{-\gamma^*}) = \psi(F_{w_0}(\mathcal{O}^\gamma)) \otimes \mathbb{C}[\mathbf{y}].$$

Proof. We claim that $\mathcal{D}[\gamma]^{-\gamma^*} \cap \operatorname{Ker}\psi = F_{w_0}(\mathcal{O}^\gamma) \otimes \mathfrak{t}U(\widehat{\mathfrak{h}})$. Let $Q \in \mathcal{D}[\gamma]^{-\gamma^*} \cap \operatorname{Ker}\psi$. By Lemma 6.3, write $Q = P + T$, where $P = \sum_{\mathbf{m}\in\mathbb{N}^r} F_{w_0}(f_{\mathbf{m}})x^{\mathbf{m}}$ for some $f_{\mathbf{m}} \in \mathcal{O}^\gamma$, and $T \in F_{w_0}(\mathcal{O}^\gamma) \otimes \mathfrak{t}U(\widehat{\mathfrak{h}})$. Since $T \in \operatorname{Ker}\psi$, certainly $0 = \psi(P) = \sum_{\mathbf{m}} \psi(F_{w_0}(f_{\mathbf{m}}))y^{\mathbf{m}}$. By Proposition 6.4, this implies that $f_{\mathbf{m}} = 0$ for all $\mathbf{m}$. Hence $P = 0$ and $Q \in F_{w_0}(\mathcal{O}^\gamma) \otimes \mathfrak{t}U(\widehat{\mathfrak{h}})$. Since the opposite inclusion is clear, the claim is proven. As $U(\widehat{\mathfrak{h}}) = \mathbb{C}[\mathbf{x}] \bigoplus \mathfrak{t}U(\widehat{\mathfrak{h}})$, it follows that ψ induces the required isomorphism. $\square$

Theorem 6.7. *Let $\overline{\mathbf{Y}}_\Gamma$ be an S-variety satisfying* (5.4)*, set $\psi = \psi_\Gamma$ and pick $\gamma \in \Gamma$. Let $E \subset \psi(\mathcal{D}[\gamma]^{-\gamma^*})$ be an irreducible G-module. Then $E = \psi(F_{w_0}(\mathcal{O}^\gamma))u(y)$ for some $0 \neq u(y) \in \mathbb{C}[\mathbf{y}]$ and $\operatorname{ord} q = k(\gamma) + \deg u(y)$ for all $0 \neq q \in E$.*

In particular, if ψ is surjective then $\operatorname{ord} q \geq k(\gamma)$ for all $0 \neq q \in \mathcal{D}(\mathbf{Y})[\gamma]^{-\gamma^*}$.

Proof. Let $q \in E \smallsetminus \{0\}$; by Corollary 6.6 we may write $q = \sum_i p_i u_i(y)$ where the $p_i \in \psi(F_{w_0}(\mathcal{O}^\gamma))$ are linearly independent and $0 \neq u_i(y) \in \mathbb{C}[\mathbf{y}]$. Since $\psi(F_{w_0}(\mathcal{O}^\gamma))$ is an irreducible G-module, the Jacobson Density Theorem produces an element $a \in \mathbb{C}G$ such that $a.p_i = \delta_{i,1}p_1$ for all i. Recall that G acts trivially on $\widehat{\mathfrak{h}}$, hence $a.q = p_1u_1(y) \in E \smallsetminus \{0\}$. Therefore, $E = \mathbb{C}G.(a.q) = \psi(F_{w_0}(\mathcal{O}^\gamma))u_1(y)$. This proves the first claim and the second is then a consequence of Proposition 6.4.

Suppose that ψ is surjective. Since ψ is a $(G \times H)$-module map, we must have $\mathcal{D}(\mathbf{Y})[\gamma]^{-\gamma^*} = \psi(\mathcal{D}[\gamma]^{-\gamma^*})$ and the final claim follows from the previous ones. $\square$

Assume now that $\overline{\mathbf{Y}}_\Gamma = \overline{\mathbf{Y}}_\gamma$ is an HV-variety and recall that, by Example 5.5(2), $\overline{\mathbf{Y}}_\gamma$ automatically satisfies (5.4). Then $\mathcal{O}(\mathbf{Y}_\gamma)$ is graded and from the discussion after (2.2) we know that $\mathcal{D}(\mathbf{Y}_\gamma)^{-m\gamma^*}$ identifies with the space of differential operators of degree $-m$ on the graded ring $\mathcal{O}(\mathbf{Y}_\gamma)$. In particular, we obtain the following explicit module of exotic differential operators:

Corollary 6.8. *Let $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}_\gamma$ be an HV-variety. Then $\mathcal{D}(\mathbf{Y})$ contains an irreducible G-module $E = \psi_\gamma(F_{w_0}(\mathcal{O}^\gamma)) \cong V(\gamma)$ of differential operators of degree -1 and order $k(\gamma) = \langle \gamma, 2\rho^\vee \rangle$.* $\square$

For a number of important HV-varieties, G-modules of differential operators of degree -1 have been constructed, but these constructions are typically quite subtle (see, for example, [AB, BK1, BK2]). The results of [AB] apply to the minimal nilpotent orbit. In our final result we will show that their operators almost never appear in the image of ψ and therefore that ψ is not surjective for those varieties.

Assume that G is simple. Then the minimal (nonzero) nilpotent orbit of $\mathfrak{g} = \mathrm{Lie}(G)$ is $\mathbf{O}_{min} = \mathbf{Y}_{\tilde\alpha}$, where $\tilde\alpha = \tilde\alpha^*$ is the highest root. In this case $k(\tilde\alpha)$ is easy to compute. Indeed, by [Bo, Ch. VI, 1.11, Proposition 31], $k(\tilde\alpha) = 2(h-1)$, where h is the Coxeter number of the root system Δ. These Coxeter numbers are described, for example, in [Bo, Planche I–IX] and we therefore obtain the following values for $k(\tilde\alpha)$.

Type of $\mathfrak{g}$	A_ℓ	$\mathsf{B}_\ell,\ \ell \geq 2$	$\mathsf{C}_\ell,\ \ell \geq 2$	$\mathsf{D}_\ell,\ \ell \geq 3$	E_6	E_7	E_8	F_4	G_2
$k(\tilde\alpha)$	2ℓ	$2(2\ell-1)$	$2(2\ell-1)$	$2(2\ell-3)$	22	34	58	22	10

It is now easy to prove Proposition 1.6 from the introduction.

Theorem 6.9. *Let $\mathbf{O}_{min} = \mathbf{Y}_{\tilde\alpha}$ be the minimal nonzero orbit in a simple classical Lie algebra $\mathfrak{g}$. Then the restriction map $\psi : \mathcal{D}(\mathbf{X})^{Q_{\tilde\alpha}} \to \mathcal{D}(\mathbf{Y}_{\tilde\alpha})$ is surjective if and only if $\mathfrak{g} = \mathfrak{sl}(2)$ or $\mathfrak{g} = \mathfrak{sl}(3)$.*

Proof. By Example 5.5(3), $\overline{\mathbf{Y}}_{\tilde\alpha}$ is an HV-variety and so it satisfies (5.4).

Suppose that ψ is surjective. As $\mathfrak{g}$ is classical, it follows from [AB, Theorem 3.2.3 and Equation 3] that $\mathcal{D}(\mathbf{Y}_{\tilde\alpha})$ contains a G-module $E \cong V(\tilde\alpha)$ of differential operators of degree -1 and order ≤ 4. As ψ is a $G \times H$-module map, and $\mathcal{D}(\mathbf{Y}_{\tilde\alpha})^{-\tilde\alpha^*}$ is the space of operators of degree -1, this forces $E \subseteq \mathcal{D}(\mathbf{Y}_{\tilde\alpha})[\tilde\alpha]^{-\tilde\alpha^*} = \psi(\mathcal{D}[\tilde\alpha]^{-\tilde\alpha^*})$. Therefore, Theorem 6.7 says the operators in E have order $4 \geq k(\tilde\alpha)$. The table shows that this is only possible when $\mathfrak{g} = \mathfrak{sl}(2)$ or $\mathfrak{sl}(3)$.

Conversely, if $\mathfrak{g} = \mathfrak{sl}(2)$, then $\mathcal{D} = \mathbb{C}[u, v, \partial_u, \partial_v]$ is the second Weyl algebra, $Q_{\tilde\alpha} = \{\pm \mathrm{id}\} \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{O}(\mathbf{Y}_{\tilde\alpha}) = \mathcal{O}^{Q_{\tilde\alpha}} = \mathbb{C}[u^2, v^2, uv]$. Thus ψ is just the isomorphism $\mathcal{D}^{Q_{\tilde\alpha}} = \mathbb{C}[u^2, v^2, uv, \partial_u^2, \partial_v^2, \partial_u\partial_v] \xrightarrow{\sim} \mathcal{D}(\mathbf{Y}_{\tilde\alpha})$.

Now suppose that $\mathfrak{g} = \mathfrak{sl}(3)$. Then $\overline{\mathbf{X}}$ is the quadratic cone $\{\sum_{i=1}^3 u_i y_i = 0\}$ in $\mathbb{C}^3 \times \mathbb{C}^3$, $Q_{\tilde\alpha} \cong \mathbb{C}^*$ and the natural map $\psi : \mathcal{D}^{Q_{\tilde\alpha}} \to \mathcal{D}(\mathbf{Y}_{\tilde\alpha})$ is surjective by [LS3, Lemma 1.1 and Theorem 2.14]. $\square$

Differential operators have also been extensively studied for lagrangian subvarieties of minimal orbits in [BK1, BK2, LS3]. The varieties discussed in those papers are HV-varieties for an appropriate Lie algebra (see, for example, [BK1, Table 1]) and they again have differential operators of order ≤ 4 and degree -1. As might be expected by analogy with Theorem 6.9, the corresponding map ψ does produce operators of the required order for Lie algebras of small rank but in large rank the map ψ is definitely not surjective. We omit the details of these assertions since they are rather technical.

The differential operators constructed in [BK1, BK2, AB] have a number of interesting properties, as is explained in those papers. It would be interesting to know whether the operators constructed for arbitrary HV-varieties by Corollary 6.8 also have distinctive properties.

Remark 6.10. It is instructive to compare the results of this section with those from [LSS, LS1]. One of the main aims of those papers was to construct $\mathcal{D}(\mathbf{Z})$ for certain specific singular affine varieties $\mathbf{Z}$. The typical situation is that $\mathbf{Z}$ is an irreducible component of $\overline{\mathbf{O} \cap \mathfrak{n}^+}$, where $\mathbf{O}$ is a nilpotent orbit of a simple Lie algebra $\widetilde{\mathfrak{g}}$ with triangular decomposition $\widetilde{\mathfrak{g}} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}^+$. Those papers then show that $\mathcal{D}(\mathbf{Z}) = U(\widetilde{\mathfrak{g}})/P$ for some primitive ideal P. However, $\mathbf{Z}$ will almost never be an S-variety for $\widetilde{\mathfrak{g}}$. Rather, $\mathbf{Z}$ will be contained in the nilradical of a parabolic subalgebra $\mathfrak{p}$ of $\widetilde{\mathfrak{g}}$ and, at least when $\mathbf{O}$ is the minimal orbit, $\mathbf{Z}$ then will be an HV-variety for a smaller Lie algebra $\mathfrak{g}$ contained in the Levi factor of $\mathfrak{p}$. For these examples one would not expect to find a group Q and a surjective map $\psi : \mathcal{D}(G/U)^Q \to \mathcal{D}(\mathbf{Z})$, simply because it is unlikely for the big Lie algebra $\widetilde{\mathfrak{g}}$ to lie in the image of such a map.

The reader is referred to [LSS, Theorem 5.2] and [LS1, Introduction] for explicit examples of this behaviour and to [Jo2] for a more general framework. One example is provided by Example 2.8, for which we take $\widetilde{\mathfrak{g}} = \mathfrak{so}(8)$. The parabolic $\mathfrak{p}$ is described explicitly in [LSS, Table 3.1 and Remark 3.2(v)], so suffice it to say that $\mathfrak{p}$ has radical $\mathfrak{r} \cong \mathbb{C}^6$ and Levi factor $\mathfrak{a} \oplus \mathbb{C}$, where $\mathfrak{a} \cong \mathfrak{so}(6)$. The variety $\mathbf{Z}$ is the quadric $\sum_{i=1}^3 x_i y_i = 0$ inside $\mathfrak{r}$. Since $\mathfrak{r}$ is the natural representation for $\mathfrak{a}$ it follows easily that $\mathbf{Z}$ is an HV-variety for $\mathfrak{a}$. Example 2.8 follows from this discussion by the lucky coincidence that the closure $\overline{\mathbf{X}}$ of the basic affine space for $\mathfrak{sl}(3)$ identifies with $\mathbf{Z}$ under an embedding of $\mathfrak{sl}(3)$ into $\mathfrak{a}$.

References

[AB] A. Astashkevich and R. Brylinski, Exotic differential operators on complex minimal nilpotent orbits, in "Advances in Geometry", Prog. Math. Vol. 172, Birkhäuser, Boston, 1999, 19-51.

[Be] J. Becker, Higher derivations and integral closure, *Amer. J. Math.*, **100** (1978), 495-521.

[BGG] J. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Differential operators on the cubic cone, *Russian Math. Surveys*, **27** (1972), 466-488.

[BBP] R. Bezrukavnikov, A. Braverman and L. Positselskii, Gluing of abelian categories and differential operators on the basic affine space, *J. Inst. Math. Jussieu*, **1** (2002), 543-557.

[Bj] J.-E. Björk, *Rings of Differential Operators*, North-Holland, Amsterdam, 1979.

[Bo] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4,5 et 6*, Masson, Paris, 1981.

[BK1] R. Brylinski and B. Kostant, Minimal representations, geometrical quantization and unitarity, *Proc. Natl. Acad. Sci.*, **91** (1994), 6026-6029.

[BK2] ______, Differential operators on conical Lagrangian manifolds, in "Lie Theory and Geometry: in honor of Bertram Kostant on the occasion of his 65th birthday", Prog. Math. Vol. 123, Birkhäuser, Boston, 1994, 65-96.

[GK] I. M. Gel'fand and A. A. Kirillov, The structure of the Lie field connected with a split semisimple Lie algebra, *Funct. Anal. Appl.*, **3** (1969), 6-21.

[Gr] F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics Vol. 1673, Springer-Verlag, Berlin Heidelberg, 1997.

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York Heidelberg Berlin, 1977.

[Ho] M. Hochster, The Zariski-Lipman conjecture in the graded case, *J. Algebra*, **47** (1977), 411-424.

[HV] A. van den Hombergh and H. de Vries, On the differential operators on the quasi-affine variety G/N, *Indag. Math.*, **40** (1978), 460-466.

[Is] Y. Ishibashi, Nakai's conjecture for invariant subrings, *Hiroshima Math. J.*, **15** (1985), 429-436.

[Ja] J. C. Jantzen, *Representations of Algebraic Groups, Second Ed.*, Math. Surveys and Monographs Vol. 107, Amer. Math. Soc., Providence, RI, 2003.

[Jo1] A. Joseph, The minimal orbit in a simple Lie algebra and its associated maximal ideal, *Ann. Sci. Éc. Norm. Sup.*, **9** (1976), 1-29.

[Jo2] ______, A surjectivity theorem for rigid highest weight modules, *Invent. Math.*, **92** (1988), 567-596.

[Jo3] ______, Annihilators and associated varieties of unitary highest weight modules, *Ann. Sci. Éc. Norm. Sup.*, **25** (1992), 1-45.

[Ka] M. Kashiwara, Representation theory and D-modules on flag varieties, *Astérisque*, **173-174** (1989), 55-109.

[Ke] G. Kempf, The Grothendieck-Cousin complex of an induced representation, *Advances in Mathematics*, **29** (1978), 310-396.

[Kr] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspekte der Mathematik, Vieweg Verlag, 1985.

[LSS] T. Levasseur, S. P. Smith and J. T. Stafford, The minimal nilpotent orbit, the Joseph ideal, and differential operators, *J. Algebra*, **116** (1988), 480-501.

[LS1] T. Levasseur and J. T. Stafford, *Rings of Differential Operators on Classical Rings of Invariants*, Mem. Amer. Math. Soc. 412, 1989.

[LS2] ______, Invariant differential operators and an homomorphism of Harish-Chandra, *J. Amer. Math. Soc.*, **8** (1995), 365-372.

[LS3] ______, Differential operators on some nilpotent orbits, *Representation Theory*, **3** (1999), 457-473.

[Lo] M. Lorenz, *Gelfand-Kirillov Dimension*, Cuadernos de Algebra, No. 7 (Grenada, Spain), 1988.

[MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Grad. Texts in Math. Vol. 30, Amer. Math. Soc., Providence, RI, 2000.

[Na] Y. Nakai, On the theory of differentials in commutative rings, *J. Math. Soc. Japan*, **13** (1961), 63-84.

[Po] A. Polishchuk, Gluing of perverse sheaves on the basic affine space. With an appendix by R. Bezrukavnikov and the author, *Selecta Math. (N.S.)*, **7** (2001), 83-147.

[Sc] G. W. Schwarz, Lifting differential operators from orbit spaces, *Ann. Sci. Éc. Norm. Sup.*, **28** (1995), 253-306.

[Sh1] N. N. Shapovalov, On a conjecture of Gel'fand-Kirillov, *Funct. Anal. Appl.*, **7** (1973), 165-6.

[Sh2] ———, Structure of the algebra of regular differential operators on a basic affine space, *Funct. Anal. Appl.*, **8** (1974), 37-46.

[Sm] K. E. Smith, The D-module structure of F-split rings, *Math. Research Letters*, **2** (1995), 377-386.

[Sp] T. A. Springer, *Linear Algebraic Groups*, Progress in Math. Vol. 9, Birkhäuser, Boston, MA., 1998.

[Tr] W. N. Traves, Nakai's conjecture for varieties smoothed by normalization, *Proc. Amer. Math. Soc.*, **127** (1999), 2245-2248.

[VP] E. B. Vinberg and V. L. Popov, On a class of quasi-homogeneous affine varieties, *Math. USSR Izvestija*, **6** (1972), 743-758.

[YZ] A. Yekutieli and J. J. Zhang, Dualizing complexes and tilting complexes over simple rings, *J. Algebra*, **256** (2002), 556-567.

Département de Mathématiques, Université de Brest, 29238 Brest cedex 3, France
E-mail address: `Thierry.Levasseur@univ-brest.fr`

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA
E-mail address: `jts@umich.edu`