ADJOINT VECTOR FIELDS ON THE TANGENT SPACE OF SEMISIMPLE SYMMETRIC SPACES

T. LEVASSEUR AND R. USHIROBIRA

ABSTRACT. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\vartheta \in \operatorname{Aut}\mathfrak{g}$ be an involution. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition associated to ϑ, define a Lie subalgebra of $\operatorname{End}\mathfrak{p}$ by $\tilde{\mathfrak{k}} = \{\mathtt{X} : \forall f \in S(\mathfrak{p}^*)^{\mathfrak{k}},\ \mathtt{X}.f = 0\}$. We prove that $\operatorname{ad}_{\mathfrak{p}}(\mathfrak{k}) = \tilde{\mathfrak{k}}$ if, and only if, each irreducible factor of rank one of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is isomorphic to $(\mathfrak{so}(q+1), \mathfrak{so}(q))$.

0. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with adjoint group G. Let $\vartheta \in \operatorname{Aut}(\mathfrak{g})$ be an involution and set $\mathfrak{k} = \operatorname{Ker}(\vartheta - I)$, $\mathfrak{p} = \operatorname{Ker}(\vartheta + I)$, hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The pair $(\mathfrak{g}, \vartheta)$, or $(\mathfrak{g}, \mathfrak{k})$, will be called a (semisimple) symmetric pair.

Let $\Theta(\mathfrak{p})$ be the Lie algebra of (algebraic) vector fields on $\mathfrak{p}$. Thus $\Theta(\mathfrak{p})$ identifies with $\operatorname{Der}_{\mathbb{C}} \mathcal{O}(\mathfrak{p})$, where $\mathcal{O}(\mathfrak{p}) = S(\mathfrak{p}^*)$. There exists a Lie algebra homomorphism $\tau : \mathfrak{gl}(\mathfrak{p}) \to \Theta(\mathfrak{p})$ defined by $(\tau(\mathtt{X}).f)(v) = \frac{d}{dt}_{|t=0} f(e^{-t\mathtt{X}}.v)$ for $v \in \mathfrak{p}$, $f \in \mathcal{O}(\mathfrak{p})$ and $\mathtt{X} \in \mathfrak{gl}(\mathfrak{p})$. This applies in particular to $\operatorname{ad}(\mathtt{X})$, $\mathtt{X} \in \mathfrak{k}$, and we still set $\tau(\mathtt{X}) = \tau(\operatorname{ad}(\mathtt{X}))$.

Let K be the connected algebraic subgroup of G such that $\operatorname{Lie}(K) = \mathfrak{k}$. Recall, cf. [7], that

$$\mathcal{O}(\mathfrak{p})^K = \{f \in \mathcal{O}(\mathfrak{p}) : \tau(\mathfrak{k}).f = 0\} = \mathbb{C}[u_1, \ldots, u_p]$$

is a polynomial ring. Here, p is the rank of $(\mathfrak{g}, \vartheta)$, i.e. the dimension of a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ for $(\mathfrak{g}, \vartheta)$. One defines a Lie subalgebra of $\mathfrak{gl}(\mathfrak{p})$, containing $\operatorname{ad}(\mathfrak{k})$, by setting

$$\tilde{\mathfrak{k}} = \left\{\mathtt{X} \in \mathfrak{gl}(\mathfrak{p}) : \tau(\mathtt{X}).f = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{p})^K\right\}.$$

The Lie algebra $\tilde{\mathfrak{k}}$ has been considered by various authors (see, e.g., [8, 10]), in relation with the description of spherical hyperfunctions, or eigendistributions, on $\mathfrak{p}$. Observe that if $\mathfrak{s} \subset \mathfrak{k}$ is an ideal of $\mathfrak{g}$, we have $\operatorname{ad}(\mathfrak{s}) = 0$ and $\operatorname{ad}(\mathfrak{k}) = \operatorname{ad}(\mathfrak{k}/\mathfrak{s})$. We will therefore assume that $\mathfrak{k}$ does not contain a nonzero ideal of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ decomposes as a direct product $\prod_{i=1}^t (\mathfrak{g}^i, \mathfrak{k}^i)$ where each factor $(\mathfrak{g}^i, \mathfrak{k}^i)$ is irreducible, see [4, VIII.5].

When $p = 1$, the invariant u_1 is (up to a non-zero scalar) the nondegenerate quadratic form on $\mathfrak{p}$ induced by the Killing form B of $\mathfrak{g}$. Then, $\tilde{\mathfrak{k}} = \mathfrak{so}(\mathfrak{p}, u_1)$ and $\tilde{\mathfrak{k}} \supsetneq \operatorname{ad}(\mathfrak{k})$, unless $(\mathfrak{g}, \mathfrak{k}) \cong (\mathfrak{so}(q+1, \mathbb{C}), \mathfrak{so}(q, \mathbb{C}))$. The main result of this note is the following theorem, which does not seem to have been noticed before.

Theorem. (Main Theorem) *Let* $(\mathfrak{g}, \vartheta)$ *be as above. Then* $\operatorname{ad}(\mathfrak{k}) = \tilde{\mathfrak{k}}$ *if, and only if, each irreducible factor of rank one of* $(\mathfrak{g}, \mathfrak{k})$ *is isomorphic to* $(\mathfrak{so}(q+1, \mathbb{C}), \mathfrak{so}(q, \mathbb{C}))$.

Date: December 3, 1997.

1991 *Mathematics Subject Classification.* 17B45, 17B66, 53C35.

Key words and phrases. semisimple Lie algebra, symmetric space, polar representation.

The proof of the theorem goes as follows. Let $\tilde{K}$ be the connected algebraic subgroup of $\mathrm{GL}(\mathfrak{p})$ such that $\mathrm{Lie}(\tilde{K}) = \tilde{\mathfrak{k}}$, we first prove that the representation $(\tilde{K} : \mathfrak{p})$ is polar (see [1, 2]). Now, using the results of [1] one can suppose that there exists a semisimple symmetric pair $(\tilde{\mathfrak{g}}, \tilde{\vartheta})$ with associated decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \mathfrak{p}$ and Cartan subspace $\mathfrak{a}$. Then, a case by case examination of the restricted root systems $\Sigma(\mathfrak{g}, \mathfrak{a})$ and $\Sigma(\tilde{\mathfrak{g}}, \mathfrak{a})$ enables us to conclude the proof.

Our interest in this theorem originates in the more general problem of describing the $\mathcal{O}(\mathfrak{p})$-module of vector fields on $\mathfrak{p}$ which annihilate $\mathcal{O}(\mathfrak{p})^K$. Set

$$\mathcal{E} = \left\{ d \in \Theta(\mathfrak{p}) : d.f = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{p})^K \right\}.$$

Then, $E = \mathcal{O}(\mathfrak{p})\tau(\mathfrak{k}) \subset \tilde{E} = \mathcal{O}(\mathfrak{p})\tau(\tilde{\mathfrak{k}}) \subset \mathcal{E}$ and we conjecture that $\mathcal{E} = \mathcal{O}(\mathfrak{p})\tau(\tilde{\mathfrak{k}})$. The equality $\mathcal{E} = \mathcal{O}(\mathfrak{p})\tau(\mathfrak{k})$ was established by J. Dixmier [3] in the diagonal case, that is to say when $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_1$, $\mathfrak{g}_1$ semisimple, $\vartheta(x, y) = (y, x)$. It is not difficult to prove that the same conclusion holds when $(\mathfrak{g}, \mathfrak{k})$ has maximal rank, i.e. $p = \mathrm{rk}\,\mathfrak{g}$ (this is also a very particular case of the results in [13]). Furthermore, the modules E, $\tilde{E}$ and $\mathcal{E}$ are graded $\mathcal{O}(\mathfrak{p})$-submodules of $\Theta(\mathfrak{p})$ whose degree zero parts are given by $E_0 = \tau(\mathfrak{k})$, $\tilde{E}_0 = \mathcal{E}_0 = \tau(\tilde{\mathfrak{k}})$. Therefore, the Main Theorem indicates in which case one has $E \subsetneq \tilde{E} = \mathcal{O}(\mathfrak{p})\mathcal{E}_0$.

1. Generalities

We retain the notation of the introduction. Furthermore, we set $n = \dim \mathfrak{p}$, $\mathrm{ad}(x).y = [x, y]$ and $g.x = \mathrm{Ad}(g).x$ for $x, y \in \mathfrak{g}$, $g \in G$. If $V \subset \mathfrak{g}$, we denote by V^x the subset of elements of V which commute with x.

By [7], $\dim \mathfrak{p} - \dim \mathfrak{k} = \dim \mathfrak{p}^v - \dim \mathfrak{k}^v$ for all $v \in \mathfrak{p}$. Define the set of regular elements in $\mathfrak{p}$ by

$$\mathfrak{p}^{\mathrm{reg}} = \{ v \in \mathfrak{p} : \dim K.v = n - p \} = \{ v \in \mathfrak{p} : \dim \mathfrak{p}^v = p \}.$$

Then, cf. [7], one has $p = \min_{v \in \mathfrak{p}} \dim \mathfrak{p}^v = \dim \mathfrak{a}$ and $\max_{v \in \mathfrak{p}} \dim K.v = \dim \mathfrak{p} - p$. One can write $\mathfrak{a} = \mathfrak{p}^x$ for a generic element x, i.e. $x \in \mathfrak{p}^{\mathrm{reg}}$ and x semisimple in $\mathfrak{g}$.

Recall (see [4, Proposition X.1.4] and [5, Lemma III.4.1]) that the symmetric pair $(\mathfrak{g}, \vartheta)$ is the complexification of a real symmetric pair $(\mathfrak{g}_0, \vartheta_0)$ where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Thus $\mathfrak{k}_0$ is a compactly embedded subalgebra of $\mathfrak{g}_0$ and the restriction of B to $\mathfrak{p}_0$ is a $\mathfrak{k}_0$-invariant scalar product. We then have $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{p} = \mathfrak{p}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\vartheta = \vartheta_0 \otimes_{\mathbb{R}} 1$ and

$$S(\mathfrak{p}_0^*)^{\mathfrak{k}_0} \otimes_{\mathbb{R}} \mathbb{C} = S(\mathfrak{p}^*)^{\mathfrak{k}} = \mathcal{O}(\mathfrak{p})^K = \mathbb{C}[u_1, \dots, u_p].$$

It follows that $S(\mathfrak{p}_0^*)^{\mathfrak{k}_0}$ is a polynomial ring in p variables and that we may choose the generators $u_1, \dots, u_p$ in $S(\mathfrak{p}_0^*)$, the first invariant u_1 being the nondegenerate quadratic form on $\mathfrak{p}_0$ induced by the restriction of B. We have $\mathfrak{gl}(\mathfrak{p}) = \mathfrak{gl}(\mathfrak{p}_0) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}(\mathfrak{p}_0) \oplus i\mathfrak{gl}(\mathfrak{p}_0)$ and, if $\mathtt{X} \in \mathfrak{gl}(\mathfrak{p}_0)$, the vector field $\tau(\mathtt{X})$ is a derivation of the polynomial ring $S(\mathfrak{p}_0^*)$. Notice that $\mathfrak{s}_0 = \{ \mathtt{X} \in \mathfrak{gl}(\mathfrak{p}_0) : \tau(\mathtt{X}).u_1 = 0 \}$ is the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{p}_0, u_1) \cong \mathfrak{so}(n, \mathbb{R})$.

Define [8, §4] a closed subgroup of $\mathrm{GL}(\mathfrak{p}_0)$ by

$$K_0' = \{ g \in \mathrm{GL}(\mathfrak{p}_0) : g.u_j = u_j \text{ for all } j = 1, \dots, p \}.$$

Since $K_0' \subset \mathrm{SO}(\mathfrak{p}_0, u_1)$, K_0' is a compact Lie group. Denote by $\tilde{K}_0$ its identity component and set $\tilde{\mathfrak{k}}_0 = \mathrm{Lie}(K_0') = \mathrm{Lie}(\tilde{K}_0)$. We have

$$\tilde{\mathfrak{k}}_0 = \left\{ \mathtt{X} \in \mathfrak{gl}(\mathfrak{p}_0) : \tau(\mathtt{X}).f = 0 \text{ for all } f \in S(\mathfrak{p}_0^*)^{\mathfrak{k}_0} \right\}$$

and $\mathrm{ad}(\mathfrak{k}_0) \subset \tilde{\mathfrak{k}}_0 \subset \mathfrak{s}_0$. Let $\tilde{K} = (\tilde{K}_0)_{\mathbb{C}} \subset \mathrm{GL}(\mathfrak{p})$ be the complexification of $\tilde{K}_0$ (see [11, Chap. 5, Theorem 12]). Then, $\tilde{K}$ is a reductive algebraic group and is the unique connected reductive subgroup of $\mathrm{GL}(\mathfrak{p})$ such that $\mathrm{Lie}(\tilde{K}) = \tilde{\mathfrak{k}}_0 \otimes_{\mathbb{R}} \mathbb{C}$. One verifies easily that $\tilde{\mathfrak{k}} = \tilde{\mathfrak{k}}_0 \otimes_{\mathbb{R}} \mathbb{C}$. It will be convenient to denote the $\tilde{K}_0$-module $\mathfrak{p}_0$ by $\tilde{\mathfrak{p}}_0$.

Recall that the pair $(\mathfrak{g}, \mathfrak{k})$ is said to be irreducible if $(\mathfrak{g}_0, \mathfrak{k}_0)$ is irreducible in the following sense [5, VIII.5]: $\mathfrak{k}_0$ does not contain a nonzero ideal of $\mathfrak{g}_0$ and the K_0-module $\mathfrak{p}_0$ is simple. Decompose $(\mathfrak{g}_0, \mathfrak{k}_0)$ as a finite direct sum of irreducible symmetric pairs $(\mathfrak{g}_0^i, \mathfrak{k}_0^i)$, $1 \leqslant i \leqslant t$. We can then define, in a similar way, $\tilde{\mathfrak{k}}_0^i \subset \mathfrak{gl}(\mathfrak{p}_0^i)$, $\tilde{K}^i \subset \mathrm{GL}(\mathfrak{p}^i)$ etc., for each $i = 1, \dots, t$.

Lemma 1.1. *We have* $\tilde{\mathfrak{k}}_0 = \tilde{\mathfrak{k}}_0^1 \times \cdots \times \tilde{\mathfrak{k}}_0^t$ *and* $\tilde{K}_0 = \tilde{K}_0^1 \times \cdots \times \tilde{K}_0^t$.

Proof. We write the proof for $t = 2$, the general case being similar. Let $\{e_i, x_i = e_i^*\}_i$ and $\{f_i, y_i = f_i^*\}_i$ be orthonormal coordinate systems (w.r.t. the Killing forms) on $\mathfrak{p}_0^1$ and $\mathfrak{p}_0^2$. Thus, $S(\mathfrak{p}_0^*)^{\mathfrak{k}_0} = S((\mathfrak{p}_0^1)^*)^{\mathfrak{k}_0^1} \otimes_{\mathbb{R}} S((\mathfrak{p}_0^2)^*)^{\mathfrak{k}_0^2}$. Let $\mathtt{X} \in \mathrm{End}\,\mathfrak{p}_0$ and write $\mathtt{X} = \left(\begin{smallmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{smallmatrix}\right)$ with $\mathtt{A} = [a_{ij}] \in \mathrm{End}\,\mathfrak{p}_0^1$, $\mathtt{B} = [b_{ij}] \in \mathrm{L}(\mathfrak{p}_0^2, \mathfrak{p}_0^1)$, $\mathtt{C} = [c_{ij}] \in \mathrm{L}(\mathfrak{p}_0^1, \mathfrak{p}_0^2)$, $\mathtt{D} = [d_{ij}] \in \mathrm{End}\,\mathfrak{p}_0^2$. Then,

$$\tau(\mathtt{X}) = \textstyle\sum_s (\mathtt{A}_s(x) + \mathtt{B}_s(y)) \frac{\partial}{\partial x_s} + \sum_q (\mathtt{C}_q(x) + \mathtt{D}_q(y)) \frac{\partial}{\partial y_q}$$

where $\mathtt{A}_s(x) = -\sum_u a_{su} x_u$, $\mathtt{B}_s(y) = -\sum_u b_{su} y_u$, $\mathtt{C}_q(x) = -\sum_u c_{qu} x_u$, $\mathtt{D}_q(y) = -\sum_u d_{qu} y_u$. Suppose that $\mathtt{X} \in \tilde{\mathfrak{k}}_0$ and let $f(x) \in S((\mathfrak{p}_0^1)^*)^{\mathfrak{k}_0^1}$. Then, from $\tau(\mathtt{X}).f = 0$ we deduce that $\sum_s \mathtt{A}_s(x) \frac{\partial f(x)}{\partial x_s} = -\sum_s \mathtt{B}_s(y) \frac{\partial f(x)}{\partial x_s}$, which forces $\sum_s \mathtt{A}_s(x) \frac{\partial f(x)}{\partial x_s} = \sum_s \mathtt{B}_s(y) \frac{\partial f(x)}{\partial x_s} = 0$. Similarly, $\sum_s \mathtt{C}_s(x) \frac{\partial g(y)}{\partial y_s} = \sum_s \mathtt{D}_s(y) \frac{\partial g(y)}{\partial y_s} = 0$ for all $g(y) \in S((\mathfrak{p}_0^2)^*)^{\mathfrak{k}_0^2}$. Now, taking $f(x) = \sum_s x_s^2$ we obtain $\sum_s \mathtt{B}_s(y) x_s = 0$ and therefore $\mathtt{B}_s(y) = 0$. Hence $\mathtt{B} = 0$ and, similarly, $\mathtt{C} = 0$ (use $g(y) = \sum_q y_q^2$). This proves that $\mathtt{X} = \mathtt{A} \times \mathtt{D}$ with $\mathtt{A} \in \tilde{\mathfrak{k}}_0^1$, $\mathtt{D} \in \tilde{\mathfrak{k}}_0^2$. The second assertion follows easily. □

Remark 1.2. The previous lemma shows that $\mathrm{ad}(\mathfrak{k}_0) = \tilde{\mathfrak{k}}_0$ if and only if $\mathrm{ad}(\mathfrak{k}_0^i) = \tilde{\mathfrak{k}}_0^i$ for all i. Therefore, to prove the theorem of the introduction, we may assume that the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is irreducible.

Lemma 1.3. *Suppose that* $(\mathfrak{g}, \mathfrak{k})$ *is irreducible and* $p = 1$. *Then,* $\mathrm{ad}(\mathfrak{k}_0) = \tilde{\mathfrak{k}}_0$ *if and only if* $(\mathfrak{g}, \mathfrak{k})$ *is isomorphic to* $(\mathfrak{so}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$.

Proof. Note that $\mathfrak{k}_0 \rightsquigarrow \mathrm{ad}(\mathfrak{k}_0) \subset \tilde{\mathfrak{k}}_0 = \mathfrak{s}_0$ with $\mathfrak{s}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{so}(n, \mathbb{C})$. Assume that $(\mathfrak{g}, \mathfrak{k}) \cong (\mathfrak{so}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$; then, $\dim \mathfrak{k}_0 = \dim_{\mathbb{C}} \mathfrak{k} = \dim \mathfrak{s}_0$ and therefore $\mathrm{ad}(\mathfrak{k}_0) = \mathfrak{s}_0$. Conversely, if $\mathrm{ad}(\mathfrak{k}_0) = \mathfrak{s}_0$, we obtain that $\mathfrak{k} \cong \mathfrak{so}(n, \mathbb{C})$ acting naturally on $\mathfrak{p} \cong \mathbb{C}^n$. It follows that $(\mathfrak{g}, \mathfrak{k}) \cong (\mathfrak{so}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$. □

Recall (for completeness) the following lemma, cf. [8, Corollary 4.4] for a proof in the analytic case.

Lemma 1.4. *Let* $V \subset \mathfrak{p}$ *be an affine open subset and* $f \in \mathcal{O}(V)$, *then*

$$\{\forall\, \mathtt{X} \in \tilde{\mathfrak{k}},\ \tau(\mathtt{X}).f = 0\} \iff \{\forall\, \mathtt{X} \in \mathfrak{k},\ \tau(\mathtt{X}).f = 0\}.$$

In particular, $\mathcal{O}(\mathfrak{p})^K = \mathcal{O}(\mathfrak{p})^{\tilde{K}}$ *and* $S(\mathfrak{p}_0^*)^{K_0} = S(\mathfrak{p}_0^*)^{\tilde{K}_0}$.

Proof. Let $\mathtt{X} \in \tilde{\mathfrak{k}}$ and let $f \in \mathcal{O}(V)$ be such that $\tau(\mathfrak{k}).f = 0$. By [9, Lemma 4.9] (or the proof of [8, Lemma 4.3]), there exists $0 \neq \psi \in \mathcal{O}(\mathfrak{p})$ such that $\psi\tau(\mathtt{X}) \in \mathcal{O}(\mathfrak{p})\tau(\mathfrak{k})$. Hence $(\psi\tau(\mathtt{X})).f = 0$, forcing $\tau(\mathtt{X}).f = 0$. The converse is obvious and the last assertions follow easily by taking $V = \mathfrak{p}$. □

Corollary 1.5. *Let $v \in \mathfrak{p}_0$. Then $K_0.v = \tilde{K}_0.v$.*

Proof. By Lemma 1.4, the invariant functions u_j separate both the K_0-orbits and the $\tilde{K}_0$-orbits, see e.g. [12, (0.4)]. We clearly have $K_0.v \subset \tilde{K}_0.v$. Suppose that $y \in \tilde{K}_0.v \setminus K_0.v$. Since $K_0.y \neq K_0.v$, we get that $u_j(y) \neq u_j(v)$ for some j. But this yields $\tilde{K}_0.v \neq \tilde{K}_0.y$ and a contradiction. □

Let $(L : E)$ be a finite dimensional representation of a compact group L. Fix an L-invariant scalar product B on E and set $\mathfrak{l} = \mathrm{Lie}(L)$. Recall [1] that $v \in E$ is said to be L-regular if $\dim L.v$ is maximal. The representation $(L : E)$ is called *polar* if, whenever $v, v' \in E$ are regular, there exists $k \in L$ such that $\mathfrak{a}_v = k.\mathfrak{a}_{v'}$, where $\mathfrak{a}_v$ is the orthogonal of $\mathfrak{l}.v$ with respect to B. A subspace of the form $\mathfrak{a}_v$, v regular, is called a Cartan subspace for $(L : E)$ and we define the rank of $(L : E)$ to be $\mathrm{rk}(L : E) = \dim \mathfrak{a}_v$. We then have $\max_{v \in E} \dim L.v = \dim E - \mathrm{rk}(L : E)$.

The representation $(K_0 : \mathfrak{p}_0)$ is known to be polar and is called a symmetric space representation, see [1]. In this case a Cartan subspace is provided by a maximal abelian Lie subalgebra $\mathfrak{a}_0$ contained in $\mathfrak{p}_0$; then, $\mathfrak{a} = \mathfrak{a}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subspace for $(\mathfrak{g}, \vartheta)$.

Proposition 1.6. *The representation $(\tilde{K}_0 : \tilde{\mathfrak{p}}_0)$ is polar.*

Proof. By Corollary 1.5, $v_0 \in \mathfrak{p}$ is K_0-regular if and only if it is $\tilde{K}_0$-regular and we have $\mathfrak{k}_0.v_0 = \tilde{\mathfrak{k}}_0.v_0$. Set $\mathfrak{a}_0 = \mathfrak{a}_{v_0} = (\mathfrak{k}_0.v_0)^\perp$. Let $v \in \mathfrak{p}_0$ be regular, we then have $\mathfrak{a}_0 = k.\mathfrak{a}_v = k.(\mathfrak{k}_0.v)^\perp = k.(\tilde{\mathfrak{k}}_0.v)^\perp$ for some $k \in K_0$. This implies that $(\tilde{K}_0 : \tilde{\mathfrak{p}}_0)$ is polar with $\mathfrak{a}_0$ as Cartan subspace. □

We need to recall a few facts from the theory of symmetric spaces [4, VI.3]. Let $\mathfrak{a}_0$ be a Cartan subspace for $(K_0 : \mathfrak{p}_0)$ and let $\lambda \in \mathfrak{a}_0^*$. One sets:

$$\begin{aligned} \mathfrak{g}_0^\lambda &= \{x \in \mathfrak{g}_0 : [a, x] = \lambda(a)x \text{ for all } a \in \mathfrak{a}_0\} \\ \Sigma &= \{\alpha \in \mathfrak{a}_0^* : \alpha \neq 0 \text{ and } \mathfrak{g}_0^\alpha \neq 0\} \\ \mathfrak{m}_0 &= \mathfrak{g}_0^0 \cap \mathfrak{k}_0 = \mathrm{cent}_{\mathfrak{k}_0}(\mathfrak{a}_0) \end{aligned}$$

Then Σ is a root system, possibly non reduced; we fix a choice Σ^+ of positive roots. Define the reduced associated root system by

$$\Sigma_{\mathrm{red}} = \{\lambda \in \Sigma : \lambda \notin 2\Sigma\}.$$

(If Σ is reduced we have $\Sigma_{\mathrm{red}} = \Sigma$; otherwise, in the irreducible case, Σ is of type $(\mathsf{BC})_p$ and $\Sigma_{\mathrm{red}} \cong \mathsf{B}_p$.)

If V is a real vector space we denote by $V_{\mathbb{C}}$ its complexification and if $\mathfrak{l}_0$ is a subspace of $\mathfrak{g}_0$, we set $\mathfrak{l} = (\mathfrak{l}_0)_{\mathbb{C}}$. With this notation the decomposition $\mathfrak{g}_0 = \bigoplus_{\lambda \in \Sigma \cup \{0\}} \mathfrak{g}_0^\lambda$ yields

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} = \textstyle\bigoplus_{\lambda \in \Sigma \cup \{0\}} \mathfrak{g}^\lambda \\ \mathfrak{m} &= \mathrm{cent}_{\mathfrak{k}}(\mathfrak{a}) \end{aligned}$$

Recall that the multiplicity of $\lambda \in \Sigma$ is $m_\lambda = \dim_{\mathbb{C}} \mathfrak{g}^\lambda = \dim \mathfrak{g}_0^\lambda$. Let $\lambda \in \Sigma^+$ and set

$$\begin{aligned} \mathfrak{k}_0^\lambda &= \{\mathtt{X} \in \mathfrak{k}_0 : \mathrm{ad}(a)^2.\mathtt{X} = \lambda(a)^2 \mathtt{X} \text{ for all } a \in \mathfrak{a}_0\} \\ \mathfrak{p}_0^\lambda &= \{v \in \mathfrak{p}_0 : \mathrm{ad}(a)^2.v = \lambda(a)^2 v \text{ for all } a \in \mathfrak{a}_0\}. \end{aligned}$$

Then, $\mathfrak{k}_0 = \mathfrak{m}_0 \oplus \big(\bigoplus_{\lambda\in\Sigma^+}\mathfrak{k}_0^\lambda\big)$, $\mathfrak{p}_0 = \mathfrak{a}_0 \oplus \big(\bigoplus_{\lambda\in\Sigma^+}\mathfrak{p}_0^\lambda\big)$. Furthermore, see [5, III.4], $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\lambda} = \mathfrak{k}^\lambda \oplus \mathfrak{p}^\lambda$. Let $v \in \mathfrak{a}$ be generic, i.e. $\lambda(v) \neq 0$ for all $\lambda \in \Sigma_{\mathrm{red}}$, then $\mathrm{ad}(v)$ induces an isomorphism $\mathfrak{p}^\lambda \xrightarrow{\sim} \mathfrak{k}^\lambda$. It follows in particular that $m_\lambda = \dim \mathfrak{g}^\lambda = \dim \mathfrak{k}^\lambda = \dim \mathfrak{p}^\lambda$.

Denote the set of generic elements in $\mathfrak{a}$ by

$$\mathfrak{a}' = \{v \in \mathfrak{a} : \alpha(v) \neq 0 \text{ for all } \alpha \in \Sigma\}$$

and let $\mathfrak{a}^{\mathrm{sing}} = \mathfrak{a} \setminus \mathfrak{a}'$ be the set of singular elements. We recall, for completeness, the following lemma.

Lemma 1.7. *Let $x \in \mathfrak{a}$. Then*

(i) $\mathfrak{k}^x = \mathfrak{m} \oplus \big(\bigoplus_{\{\lambda\in\Sigma^+:\lambda(x)=0\}}\mathfrak{k}^\lambda\big)$

(ii) *x generic* $\iff$ $\mathfrak{k}^x = \mathfrak{m}$ $\iff$ $\dim \mathfrak{k}^x$ *is minimal* $\iff$ $\dim \mathfrak{k}^x = \dim \mathfrak{p} - p$.

Proof. (i) follows from $\mathfrak{k} = \mathfrak{m} \oplus \big(\bigoplus_{\lambda\in\Sigma^+}\mathfrak{k}^\lambda\big)$ and $\mathrm{Ker}\,\mathrm{ad}(a)^2 = \mathrm{Ker}\,\mathrm{ad}(a)$ for $a \in \mathfrak{a}$ (since a is semisimple).

(ii) is consequence of (i) and the definitions. ☐

For $\alpha \in \Sigma^+_{\mathrm{red}}$ we set $\mathfrak{a}_\alpha = \mathrm{Ker}\,\alpha = \{a \in \mathfrak{a} : \alpha(a) = 0\}$. Therefore,

$$\mathfrak{a}^{\mathrm{sing}} = \textstyle\bigcup_{\alpha\in\Sigma^+_{\mathrm{red}}}\mathfrak{a}_\alpha \tag{1.1}$$

and the $\mathfrak{a}_\alpha$ are pairwise distinct hyperplanes. Set

$$\mathfrak{a}_0' = \mathfrak{a}' \cap \mathfrak{a}_0, \quad \mathfrak{a}_0^{\mathrm{sing}} = \mathfrak{a}_0 \cap \mathfrak{a}^{\mathrm{sing}}, \quad \mathfrak{a}_{0,\alpha} = \mathfrak{a}_0 \cap \mathfrak{a}_\alpha.$$

Since $\dim K_0.x = \dim_{\mathbb{C}} K.x$ for all $x \in \mathfrak{a}_0$, it follows from Lemma 1.7 that $\mathfrak{a}_0'$ is the set of regular elements in $\mathfrak{a}_0$.

2. Proof of $\mathrm{ad}(\mathfrak{k}) = \tilde{\mathfrak{k}}$

We continue with the notation of the previous sections. Recall that the proof of the Main Theorem reduces to the case when $(\mathfrak{g}_0, \mathfrak{k}_0)$ is irreducible, see Remark 1.2. From now on, we assume that this hypothesis holds. Since $\mathrm{ad} : \mathfrak{k}_0 \to \mathfrak{gl}(\mathfrak{p}_0)$ is injective, we will identify $\mathfrak{k}_0$ with the Lie subalgebra $\mathrm{ad}(\mathfrak{k}_0)$ of $\tilde{\mathfrak{k}}_0$, therefore $\mathfrak{k}$ is identified with $\mathrm{ad}(\mathfrak{k})$. Note that the representations $(K_0 : \mathfrak{p}_0)$ and $(\tilde{K}_0 : \tilde{\mathfrak{p}}_0)$ are irreducible and faithful.

From the classification of irreducible polar representations one can deduce the following result, see [1, Theorem 9, Theorem 10 and Proposition 6].

Proposition 2.1. *Let $(L_0 : V_0)$ be an irreducible faithful polar representation of a compact Lie group L_0. Then, there exists a semisimple symmetric pair $(\bar{\mathfrak{g}}_0, \bar{\mathfrak{k}}_0)$ such that (with obvious notation):*

(i) $\bar{\mathfrak{g}}_0 = \bar{\mathfrak{k}}_0 \oplus V_0$ *is the associated Cartan decomposition;*

(ii) $L_0 \subset \overline{K}_0$ *and $(L_0 : V_0)$ is the restriction of $(\overline{K}_0 : V_0)$;*

(iii) $S(V_0^*)^{\overline{K}_0} = S(V_0^*)^{L_0}$.

Corollary 2.2. *The representation $(\tilde{K}_0 : \tilde{\mathfrak{p}}_0)$ is an irreducible symmetric space representation.*

Proof. By Proposition 1.6 and Proposition 2.1, there exists a semisimple symmetric pair $(\bar{\mathfrak{g}}_0, \bar{\mathfrak{k}}_0)$ such that $\mathfrak{p}_0 = \tilde{\mathfrak{p}}_0 = \bar{\mathfrak{p}}_0$ (as vector spaces), $\mathfrak{k}_0 \subset \tilde{\mathfrak{k}}_0 \subset \bar{\mathfrak{k}}_0$ and $S(\mathfrak{p}_0^*)^{K_0} = S(\mathfrak{p}_0^*)^{\overline{K}_0}$. It follows then from the definition of $\tilde{\mathfrak{k}}_0$ that $\tilde{\mathfrak{k}}_0 = \bar{\mathfrak{k}}_0$. ☐

Remark. B. Kostant has informed us that Corollary 2.2 can also be deduced from the results contained in [6].

From the previous corollary we may suppose now that $(\tilde{K}_0 : \tilde{\mathfrak{p}}_0)$ is coming from a semisimple symmetric pair $(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{k}}_0)$. Without lost of generality we can assume that $\tilde{\mathfrak{g}}_0$ has Cartan decomposition $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{p}}_0$ and that, if $[\ ,\ ]^\sim$ is the bracket on $\tilde{\mathfrak{g}}_0$, $[\mathtt{X}, v] = [\mathtt{X}, v]^\sim$, $[\mathtt{X}, \mathtt{Y}] = [\mathtt{X}, \mathtt{Y}]^\sim$ for all $\mathtt{X}, \mathtt{Y} \in \mathfrak{k}_0$, $v \in \mathfrak{p}_0 = \tilde{\mathfrak{p}}_0$. Notice that if $\mathfrak{l}_0 \subset \tilde{\mathfrak{k}}_0 \subset \operatorname{End} \tilde{\mathfrak{p}}_0$ is an ideal of $\tilde{\mathfrak{g}}_0$, then $\mathfrak{l}_0.\tilde{\mathfrak{p}}_0 = [\mathfrak{l}_0, \tilde{\mathfrak{p}}_0]^\sim \subset \tilde{\mathfrak{k}}_0 \cap \tilde{\mathfrak{p}}_0 = 0$ and therefore $\mathfrak{l}_0 = 0$. Thus the symmetric pair $(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{k}}_0)$ is also irreducible. Recall that we have fixed the Cartan subspace $\mathfrak{a}_0$ and that we can take $\tilde{\mathfrak{a}}_0 = \mathfrak{a}_0$ as Cartan subspace for $(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{k}}_0)$, see Proposition 1.6. The associated Weyl groups will be denoted by W and $\tilde{W}$.

The notation given in §1 for $\mathfrak{k}_0, \mathfrak{p}_0, \mathfrak{a}_0, \mathfrak{g}_0$, etc. can be introduced for $\tilde{\mathfrak{k}}_0, \tilde{\mathfrak{p}}_0, \tilde{\mathfrak{a}}_0, \tilde{\mathfrak{g}}_0$, etc. If an object x is defined relatively to $(\mathfrak{g}_0, \mathfrak{k}_0)$ we denote by $\tilde{x}$ the corresponding one, relatively to $(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{k}}_0)$. Since there is only one degree two invariant in $S(\mathfrak{p}_0^*)^{K_0} = S(\tilde{\mathfrak{p}}_0^*)^{\tilde{K}_0}$, the scalar product B on $\mathfrak{p}_0$ is a positive scalar multiple of the scalar product $\tilde{B}$ on $\tilde{\mathfrak{p}}_0$ and we will suppose in the sequel that they are actually equal.

Proposition 2.3. (1) *There exists a bijection* $\mathtt{t} : \Sigma^+_{\rm red} \to \tilde{\Sigma}^+_{\rm red}$, $\alpha \mapsto \tilde{\alpha}$, *such that* $\mathfrak{a}_{0,\alpha} = \tilde{\mathfrak{a}}_{0,\tilde{\alpha}}$.

(2) $W = \tilde{W}$.

(3) *Let* $\alpha \in \Sigma^+_{\rm red}$ *and* $w \in W$ *be such that* $w.\alpha \in \Sigma^+_{\rm red}$. *Then* $\mathtt{t}(w.\alpha) = \pm w.\mathtt{t}(\alpha)$.

(4) *There exist* $c_1, c_2 \in \mathbb{R}^*$ *such that*

$$\tilde{\alpha} = \begin{cases} \pm c_1 \alpha & \text{if } \alpha \text{ short,} \\ \pm c_2 \alpha & \text{if } \alpha \text{ long.} \end{cases}$$

Proof. (1) By Corollary 1.5 we have $\mathfrak{a}_0^{\rm sing} = \tilde{\mathfrak{a}}_0^{\rm sing}$, hence we get from (1.1):

$$\bigcup\nolimits_{\alpha \in \Sigma^+_{\rm red}} \mathfrak{a}_{0,\alpha} = \bigcup\nolimits_{\beta \in \tilde{\Sigma}^+_{\rm red}} \tilde{\mathfrak{a}}_{0,\beta}$$

Since the hyperplanes occuring in each side of the previous equality are pairwise distinct, we obtain that

$$\forall\, \alpha \in \Sigma^+_{\rm red},\ \exists!\, \mathtt{t}(\alpha) \in \tilde{\Sigma}^+_{\rm red},\ \mathfrak{a}_{0,\alpha} = \tilde{\mathfrak{a}}_{0,\mathtt{t}(\alpha)}.$$

It is then clear that $\alpha \mapsto \mathtt{t}(\alpha) = \tilde{\alpha}$ gives the required bijection. Notice that $\operatorname{Ker} \alpha = \operatorname{Ker} \tilde{\alpha}$ (in $\mathfrak{a}_0$) implies that $\tilde{\alpha} = c_\alpha \alpha$ for some $c_\alpha \in \mathbb{R}^*$.

(2) Recall that W is generated by the reflections r_α, $\alpha \in \Sigma^+_{\rm red}$, and that the reflecting hyperplane of r_α is $\mathfrak{a}_{0,\alpha}$. Thus $r_\alpha = r_{\tilde{\alpha}}$ and it follows that $W = \tilde{W}$.

(3) We have $\operatorname{Ker} w.\alpha = w(\operatorname{Ker} \alpha)$, thus $w(\mathfrak{a}_{0,\alpha}) = w(\tilde{\mathfrak{a}}_{0,\tilde{\alpha}})$ is equivalent to $\operatorname{Ker} w.\alpha = \operatorname{Ker} w.\tilde{\alpha}$. Let $\epsilon = \pm 1$ such that $\epsilon w.\tilde{\alpha} \in \tilde{\Sigma}^+_{\rm red}$. Then $\operatorname{Ker} w.\tilde{\alpha} = \tilde{\mathfrak{a}}_{0,\epsilon w.\tilde{\alpha}} = \mathfrak{a}_{0,w.\alpha}$ and, by definition of $\mathtt{t}(w.\alpha)$, we obtain that $\mathtt{t}(w.\alpha) = \epsilon w.\tilde{\alpha}$.

(4) Let $\alpha, \beta \in \Sigma^+_{\rm red}$ having the same length and $w \in W$ be such that $\beta = w.\alpha$. By (3), $\tilde{\beta} = \pm w.\tilde{\alpha}$ and, therefore, $\tilde{\beta} = c_\beta \beta = \pm c_\alpha w.\alpha = \pm c_\alpha \beta$. Hence $c_\beta = \pm c_\alpha$. The assertion then follows easily (with the convention that all the roots are short when there is only one root length in Σ). □

Corollary 2.4. (1) *If* $\Sigma_{\rm red} \notin \{\mathsf{B}_p, \mathsf{C}_p\}$, *then* $\Sigma_{\rm red}$ *and* $\tilde{\Sigma}_{\rm red}$ *are of the same type.*

(2) *If* $\Sigma_{\rm red} \in \{\mathsf{B}_p, \mathsf{C}_p\}$, *then* $\tilde{\Sigma}_{\rm red} \in \{\mathsf{B}_p, \mathsf{C}_p\}$.

Proof. Recall that the Weyl group distinguishes irreducible root systems which are not of type B_p or C_p and that the Weyl groups of B_p and C_p are the same. The claims are therefore consequences of Proposition 2.3(2). $\square$

Observe that it could happen that $\Sigma_{\rm red} \cong \mathsf{B}_p$ and $\widetilde{\Sigma}_{\rm red} \cong \mathsf{C}_p$, the bijection $\mathtt{t}$ being given by $\mathtt{t}(\alpha) = 2\alpha$, α short, $\mathtt{t}(\alpha) = \alpha$, α long. (Similarly, $\Sigma_{\rm red} \cong \mathsf{C}_p$ and $\widetilde{\Sigma}_{\rm red} \cong \mathsf{B}_p$ could occur.) In case $\Sigma = \Sigma_{\rm red} \cong \mathsf{F}_4$ (resp. G_2) we must have $\widetilde{\Sigma} \cong \mathsf{F}_4$ (resp. G_2) but it possible that $\mathtt{t}$ interchanges the short and long roots. In summary, we have the following possibilities for the pair $(\Sigma_{\rm red}, \widetilde{\Sigma}_{\rm red})$:

- $(\mathsf{A}_p, \mathsf{A}_p)$, $(\mathsf{D}_p, \mathsf{D}_p)$, $(\mathsf{E}_p, \mathsf{E}_p)$;
- $(\mathsf{F}_4, \mathsf{F}_4)$, $(\mathsf{G}_2, \mathsf{G}_2)$;
- $(\mathsf{B}_p, \mathsf{B}_p)$, $(\mathsf{C}_p, \mathsf{C}_p)$, $(\mathsf{B}_p, \mathsf{C}_p)$, $(\mathsf{C}_p, \mathsf{B}_p)$.

For all $\lambda \in \Sigma^+_{\rm red}$ we set $\mathfrak{m}_\lambda = \mathrm{cent}_{\mathfrak{k}}(\mathfrak{a}_\lambda) = \{x \in \mathfrak{k} : [x, \mathfrak{a}_\lambda] = 0\}$. If, similarly, $\tilde{\mathfrak{m}}_{\tilde{\lambda}} = \mathrm{cent}_{\tilde{\mathfrak{k}}}(\tilde{\mathfrak{a}}_{\tilde{\lambda}})$ we obtain from $\mathfrak{a}_\lambda = \tilde{\mathfrak{a}}_{\tilde{\lambda}}$ that

$$\mathfrak{m}_\lambda = \tilde{\mathfrak{m}}_{\tilde{\lambda}} \cap \mathfrak{k}. \tag{2.1}$$

The Lie algebra $\mathfrak{m}_\lambda$ is described by the following well known lemma.

Lemma 2.5. *Let $\lambda \in \Sigma^+_{\rm red}$. Then, $\mathfrak{m}_\lambda = \mathfrak{m} \oplus \mathfrak{k}^\lambda \oplus \mathfrak{k}^{2\lambda}$ (with the convention that $\mathfrak{k}^{2\lambda} = 0$ if $2\lambda \notin \Sigma$).*

Proof. Let $\mathtt{X} \in \mathfrak{k}$ and set $\mathtt{X} = \mathtt{X}_0 + \sum_{\alpha\in\Sigma^+} \mathtt{X}_\alpha$, $\mathtt{X}_0 \in \mathfrak{m}$, $\mathtt{X}_\alpha \in \mathfrak{k}^\alpha$. Thus $\mathtt{X} \in \mathfrak{m}_\lambda$ if and only if $\sum_{\alpha\in\Sigma^+}[a, \mathtt{X}_\alpha] = 0$ for all $a \in \mathfrak{a}_\lambda$. But, since $[a, \mathtt{X}_\alpha] \in \mathfrak{p}^\alpha$, this is equivalent to $[a, \mathtt{X}_\alpha] = 0$ for all $\alpha \in \Sigma^+$ and $a \in \mathfrak{a}_\lambda$. Hence,

$$\begin{aligned} \mathtt{X} \in \mathfrak{m}_\lambda &\iff \forall \alpha \in \Sigma^+,\ \forall a \in \mathfrak{a}_\lambda,\ \mathtt{X}_\alpha \in \mathrm{Ker}\,\mathrm{ad}(a) = \mathrm{Ker}\,\mathrm{ad}(a)^2 \\ &\iff \forall \alpha \in \Sigma^+,\ \forall a \in \mathfrak{a}_\lambda,\ \alpha(a) = 0 \text{ or } \mathtt{X}_\alpha = 0. \end{aligned}$$

Therefore, if $\mathtt{X}_\alpha \neq 0$, $\mathfrak{a}_\lambda = \mathrm{Ker}\,\lambda \subset \mathrm{Ker}\,\alpha$; thus $\mathrm{Ker}\,\lambda = \mathrm{Ker}\,\alpha$ and $\alpha = \lambda$ or 2λ. Conversely, if $\mathtt{X} \in \mathfrak{k}^\lambda$ or $\mathfrak{k}^{2\lambda}$ we have $\mathtt{X} \in \mathrm{Ker}\,\mathrm{ad}(a)^2 = \mathrm{Ker}\,\mathrm{ad}(a)$ for all $a \in \mathfrak{a}_\lambda$. Hence $\mathtt{X} \in \mathrm{cent}_{\mathfrak{k}}(\mathfrak{a}_\lambda)$. $\square$

Let $\lambda \in \Sigma^+_{\rm red}$; set

$$\mathfrak{s}_\lambda = \mathfrak{k}^\lambda \oplus \mathfrak{k}^{2\lambda}, \quad s_\lambda = \dim \mathfrak{s}_\lambda = m_\lambda + m_{2\lambda}$$

(with $m_{2\lambda} = 0$ if $2\lambda \notin \Sigma$). Notice that $s_\lambda = \dim(\mathfrak{p}^\lambda \oplus \mathfrak{p}^{2\lambda})$.

Lemma 2.6. *One has $s_\lambda = \tilde{s}_{\tilde{\lambda}}$ for all $\lambda \in \Sigma^+_{\rm red}$.*

Proof. It follows from Lemma 2.5 and (2.1) that $\mathfrak{m} \oplus \mathfrak{s}_\lambda \subset \tilde{\mathfrak{m}} \oplus \tilde{\mathfrak{s}}_{\tilde{\lambda}}$. Let $\phi : \tilde{\mathfrak{m}}_{\tilde{\lambda}} \twoheadrightarrow \tilde{\mathfrak{s}}_{\tilde{\lambda}}$ be the projection afforded by the decomposition $\tilde{\mathfrak{m}}_{\tilde{\lambda}} = \tilde{\mathfrak{m}} \oplus \tilde{\mathfrak{s}}_{\tilde{\lambda}}$. By composing ϕ with the inclusions $\mathfrak{s}_\lambda \hookrightarrow \mathfrak{m}_\lambda \hookrightarrow \tilde{\mathfrak{m}}_{\tilde{\lambda}}$, we obtain a linear map $\varphi : \mathfrak{s}_\lambda \to \tilde{\mathfrak{s}}_{\tilde{\lambda}}$. Suppose that $\varphi(x) = 0$, then $x \in \tilde{\mathfrak{m}} \cap \mathfrak{s}_\lambda = \tilde{\mathfrak{m}} \cap \mathfrak{k} \cap \mathfrak{s}_\lambda = \mathfrak{m} \cap \mathfrak{s}_\lambda = 0$. Thus φ is injective and, consequently, $s_\lambda \leqslant \tilde{s}_{\tilde{\lambda}}$. Now, recall that

$$\mathfrak{p} = \tilde{\mathfrak{p}} = \mathfrak{a} \oplus \big(\oplus_{\lambda\in\Sigma^+_{\rm red}} \mathfrak{p}^\lambda \oplus \mathfrak{p}^{2\lambda}\big) = \mathfrak{a} \oplus \big(\oplus_{\tilde{\lambda}\in\widetilde{\Sigma}^+_{\rm red}} \tilde{\mathfrak{p}}^{\tilde{\lambda}} \oplus \tilde{\mathfrak{p}}^{2\tilde{\lambda}}\big).$$

Therefore $\sum_{\lambda\in\Sigma^+_{\rm red}} s_\lambda = \sum_{\tilde{\lambda}\in\widetilde{\Sigma}^+_{\rm red}} \tilde{s}_{\tilde{\lambda}}$ and, since $s_\lambda \leqslant \tilde{s}_{\tilde{\lambda}}$, $s_\lambda = \tilde{s}_{\tilde{\lambda}}$ for all $\lambda \in \Sigma^+_{\rm red}$. $\square$

Remark. One has $\mathfrak{p}^\lambda \oplus \mathfrak{p}^{2\lambda} = \tilde{\mathfrak{p}}^{\tilde{\lambda}} \oplus \tilde{\mathfrak{p}}^{2\tilde{\lambda}}$ for all $\lambda \in \Sigma^+_{\rm red}$. This can be shown as follows. Let $v \in \mathfrak{a}'$, then $\mathrm{ad}(v)$ induces an isomorphism $\mathfrak{k}^\alpha \xrightarrow{\sim} \mathfrak{p}^\alpha$ for all $\alpha \in \Sigma^+$.

Recall that if $\mathtt{X} \in \mathfrak{k}$, $[v, \mathtt{X}]^{\sim} = [v, \mathtt{X}]$. Thus $\mathrm{ad}_{\tilde{\mathfrak{g}}}(v)$ restricted to $\mathfrak{m}_\lambda$ coincides with $\mathrm{ad}(v)$. It follows that

$$\mathfrak{p}^\lambda \oplus \mathfrak{p}^{2\lambda} = \mathrm{ad}_{\tilde{\mathfrak{g}}}(v).\mathfrak{s}_\lambda \subset \mathrm{ad}_{\tilde{\mathfrak{g}}}(v).(\tilde{\mathfrak{m}} \oplus \tilde{\mathfrak{s}}_{\tilde{\lambda}}) = \tilde{\mathfrak{p}}^{\tilde{\lambda}} \oplus \tilde{\mathfrak{p}}^{2\tilde{\lambda}}.$$

Since $s_\lambda = \tilde{s}_{\tilde{\lambda}}$, we get that $\mathfrak{p}^\lambda \oplus \mathfrak{p}^{2\lambda} = \tilde{\mathfrak{p}}^{\tilde{\lambda}} \oplus \tilde{\mathfrak{p}}^{2\tilde{\lambda}}$.

We now set:

$$\begin{aligned} & s_1 = s_\lambda \ \text{ if } \lambda \in \Sigma^+_{\mathrm{red}} \text{ is short,} \\ & s_2 = s_\lambda \ \text{ if } \lambda \in \Sigma^+_{\mathrm{red}} \text{ is long,} \\ & s_2 = 0 \ \text{ if all } \lambda \in \Sigma^+ \text{ are short.} \end{aligned} \tag{2.2}$$

Hence, we can associate to the Lie algebra $\mathfrak{g}_0$ two ordered pairs (s_1, s_2) and (s_2, s_1). It is shown in Appendix A that these pairs almost determine $\mathfrak{g}_0$. A similar definition holds for the pair $\tilde{\mathfrak{g}}_0$ and gives the pairs $(\tilde{s}_1, \tilde{s}_2)$, $(\tilde{s}_2, \tilde{s}_1)$. We now compare the s_i and $\tilde{s}_j$.

Lemma 2.7. (1) *Assume that Σ is simply laced. Then, $(s_1, s_2) = (\tilde{s}_1, \tilde{s}_2)$.*

(2) *Assume that Σ has two root lengths. Then,*

$$(s_1, s_2) = \begin{cases} (\tilde{s}_1, \tilde{s}_2) \ \textit{or} \ (\tilde{s}_2, \tilde{s}_1) & \textit{if } (\Sigma_{\mathrm{red}}, \widetilde{\Sigma}_{\mathrm{red}}) = (\mathsf{F}_4, \mathsf{F}_4),\ (\mathsf{G}_2, \mathsf{G}_2),\ (\mathsf{B}_2, \mathsf{B}_2), \\ (\tilde{s}_1, \tilde{s}_2) & \textit{if } (\Sigma_{\mathrm{red}}, \widetilde{\Sigma}_{\mathrm{red}}) = (\mathsf{B}_p, \mathsf{B}_p),\ (\mathsf{C}_p, \mathsf{C}_p),\ p \geqslant 3, \\ (\tilde{s}_2, \tilde{s}_1) & \textit{if } (\Sigma_{\mathrm{red}}, \widetilde{\Sigma}_{\mathrm{red}}) = (\mathsf{B}_p, \mathsf{C}_p),\ (\mathsf{C}_p, \mathsf{B}_p),\ p \geqslant 3. \end{cases}$$

Proof. Observe first that $s_\alpha = s_\beta$ if α, β have the same length; then Lemma 2.6 yields $\tilde{s}_{\tilde{\alpha}} = \tilde{s}_{\tilde{\beta}} = \tilde{s}_1$ or $\tilde{s}_2$, depending on the length of $\tilde{\alpha}$.

(1) is clear.

(2) Recall that if Σ_{red} has two root lengths, then the number of short roots is equal to the number of long roots if, and only if, Σ_{red} is of type $\mathsf{B}_2 = \mathsf{C}_2$, F_4 or G_2. The assertion then follows from Lemma 2.6 and Proposition 2.3(4). □

Theorem 2.8. *Assume that $p \geqslant 2$. Then, $\mathfrak{g}_0 \cong \tilde{\mathfrak{g}}_0$ and, therefore, $\mathfrak{k}_0 = \tilde{\mathfrak{k}}_0$.*

Proof. By Corollary 2.4 and Lemma 2.7, the hypothesis (h.j), j = 1, …, 4, of Appendix A hold. Thus, by Theorem A.1, if $\mathfrak{g}_0 \ncong \tilde{\mathfrak{g}}_0$ we are in one of the following cases.

Case 1: Diagonal case with $\Sigma, \widetilde{\Sigma} \in \{\mathsf{B}_p, \mathsf{C}_p\}$. Then, $\dim \mathfrak{k}_0 = \dim \mathfrak{g}_0 = \dim \tilde{\mathfrak{g}}_0 = \dim \tilde{\mathfrak{k}}_0$ and $\mathfrak{k}_0 \subset \tilde{\mathfrak{k}}_0$ force $\mathfrak{k}_0 = \tilde{\mathfrak{k}}_0$ and, consequently, $\mathfrak{g}_0 \cong \tilde{\mathfrak{g}}_0$.

Case 2: $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$ are of type $\mathsf{BI}(p, p+1)$ or $\mathsf{CI}(p)$. This implies that $\mathfrak{k}_0$ and $\tilde{\mathfrak{k}}_0$ are isomorphic to $\mathfrak{so}(p) \times \mathfrak{so}(p+1)$ or $\mathfrak{u}(p)$, which are both of dimension p^2. Since $\mathfrak{k}_0 \subset \tilde{\mathfrak{k}}_0$, this implies $\mathfrak{k}_0 = \tilde{\mathfrak{k}}_0$. But $\mathfrak{so}(p) \times \mathfrak{so}(p+1) \cong \mathfrak{u}(p)$ only happens when $p = 2$ (see [4, p. 519]), in which case $\mathfrak{g}_0 \cong \tilde{\mathfrak{g}}_0 \cong \mathfrak{so}(2, 3)$. □

Proof of the Main Theorem. As noticed in Remark 1.2, we may assume that $(\mathfrak{g}, \mathfrak{k})$ is irreducible. Now, the assertion follows from Lemma 1.3 if $(\mathfrak{g}, \mathfrak{k})$ has rank one and from Theorem 2.8 if this rank is $\geqslant 2$. □

Appendix A

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. We adopt the notation of §§1 and 2. In particular, we fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and a Cartan subspace $\mathfrak{a}_0 \subset \mathfrak{p}_0$ of dimension p. Let $\tilde{\mathfrak{g}}_0$ be another semisimple Lie algebra with Cartan decomposition $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{p}}_0$. Any object x defined relatively to $\mathfrak{g}_0$ has an analogue for $\tilde{\mathfrak{g}}_0$ and it will be denoted by $\tilde{x}$.

We will assume that the pairs $(\mathfrak{g}_0, \mathfrak{k}_0)$ and $(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{k}}_0)$ are both irreducible and that the following hypothesis hold.

(h.1) $p \geqslant 2$.
(h.2) $\Sigma_{\rm red} \in \{\mathsf{B}_p, \mathsf{C}_p\}$ if, and only if, $\widetilde{\Sigma}_{\rm red} \in \{\mathsf{B}_p, \mathsf{C}_p\}$.
(h.3) $\Sigma_{\rm red} \cong \widetilde{\Sigma}_{\rm red}$ when $\Sigma_{\rm red}$ is not of type B_p or C_p.
(h.4) The pairs (s_1, s_2), $(\tilde{s}_1, \tilde{s}_2)$ being defined as in (2.2), one has

$$(s_1, s_2) = \begin{cases} (\tilde{s}_1, \tilde{s}_2) & \text{if } \Sigma \text{ is simply laced,} \\ (\tilde{s}_1, \tilde{s}_2) \text{ or } (\tilde{s}_2, \tilde{s}_1) & \text{if } (\Sigma_{\rm red}, \widetilde{\Sigma}_{\rm red}) = (\mathsf{F}_4, \mathsf{F}_4),\ (\mathsf{G}_2, \mathsf{G}_2),\ (\mathsf{B}_2, \mathsf{B}_2), \\ (\tilde{s}_1, \tilde{s}_2) & \text{if } (\Sigma_{\rm red}, \widetilde{\Sigma}_{\rm red}) = (\mathsf{B}_p, \mathsf{B}_p),\ (\mathsf{C}_p, \mathsf{C}_p),\ p \geqslant 3, \\ (\tilde{s}_2, \tilde{s}_1) & \text{if } (\Sigma_{\rm red}, \widetilde{\Sigma}_{\rm red}) = (\mathsf{B}_p, \mathsf{C}_p),\ (\mathsf{C}_p, \mathsf{B}_p),\ p \geqslant 3. \end{cases}$$

Observe that the hypothesis are symmetric in $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$.

The notation for the classification of irreducible symmetric pairs, i.e. of semisimple real Lie algebras, will be (almost) as in [4, X.6]; in particular, we adopt the notation of [4, pp. 532-534]. For instance, if $\mathfrak{g}_0 = \mathfrak{so}(p,q)$, $\mathfrak{k}_0 = \mathfrak{so}(p) \times \mathfrak{so}(q)$, $p \leqslant q$, $p+q$ even, we say that $\mathfrak{g}_0$ is of type $\mathsf{DI}(p,q)$.

Suppose that $\mathfrak{g}_0 = \mathfrak{g}_1^{\mathbb{R}}$ for some complex simple Lie algebra $\mathfrak{g}_1$. Define an involution ϑ on $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_1 \times \mathfrak{g}_1$ by $\vartheta(x,y) = (y,x)$. Then the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is isomorphic to $(\mathfrak{g}_1 \times \mathfrak{g}_1, \mathfrak{g}_1)$. This case will be called the *diagonal case* and $(\mathfrak{g}, \vartheta)$ is said to be of *diagonal type*.

Theorem A.1. *Up to symmetry between $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$, the following (exclusive) possibilities hold.*

(i) $\mathfrak{g}_0 \cong \tilde{\mathfrak{g}}_0$.
(ii) $(\mathfrak{g}, \vartheta)$ *and* $(\tilde{\mathfrak{g}}, \tilde{\vartheta})$ *are of diagonal type,* $\Sigma \cong \mathsf{B}_p$, $\widetilde{\Sigma} \cong \mathsf{C}_p$.
(iii) $\mathfrak{g}_0$ *is of type* $\mathsf{BI}(p, p{+}1)$, $\tilde{\mathfrak{g}}_0$ *is of type* $\mathsf{CI}(p)$, $p \geqslant 3$ *(thus* $\mathfrak{k}_0 \cong \mathfrak{so}(p) \times \mathfrak{so}(p{+}1)$, $\tilde{\mathfrak{k}}_0 \cong \mathfrak{u}(p)$*).*

Proof. The proof is a case by case analysis using [4, X, Table VI]: One computes the pairs (s_1, s_2) for each type of irreducible symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ and, then, one notes that the hypothesis (h.i), $\mathrm{i} = 1, \dots, 4$, yield the desired result. We will simply make a few remarks in order to explain the method and the appearance of cases (i), (ii), (iii).

If $(\mathfrak{g}, \vartheta)$ is of diagonal type with $\mathfrak{g} \cong \mathfrak{g}_1 \times \mathfrak{g}_1$, $\mathfrak{g}_1$ complex simple of type T_p $(\mathsf{T} = \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G})$, then $\Sigma \cong \mathsf{T}_p$ and $(s_1, s_2) = (2, 0)$ or $(2, 2)$. Then, the (h.i)'s show that only cases (i) or (ii) may occur.

If $\mathfrak{g}_0$ of type $\mathsf{AIII}(p,p)$, then $(s_1, s_2) = (2, 1)$, $\Sigma \cong \mathsf{C}_p$. The only possibility for $\tilde{\mathfrak{g}}_0$ and $(\tilde{s}_1, \tilde{s}_2) = (s_1, s_2)$ or (s_2, s_1) occurs when $\tilde{\mathfrak{g}}_0$ is of type $\mathsf{DI}(p, p+2)$. In this case $\widetilde{\Sigma} \cong \mathsf{B}_p$. When $p = 2$ we find the isomorphism $\mathsf{DI}(2, 2+2) \cong \mathsf{AIII}(2,2)$, see [4, p. 519]. When $p \geqslant 3$, the hypothesis (h.4) forces $(s_1, s_2) = (2, 1) = (\tilde{s}_2, \tilde{s}_1) = (1, 2)$, hence a contradiction.

If $\mathfrak{g}_0$ is of type $\mathsf{BI}(p, 2\ell + 1 - p)$, then $(s_1, s_2) = (2\ell - 2p + 1, 1)$, $\Sigma \cong \mathsf{B}_p$. From $s_2 = 1$ and s_1 odd, it follows that the only possibility for $\tilde{\mathfrak{g}}_0$ may occur in type $\mathsf{CI}(p)$, where $(\tilde{s}_2, \tilde{s}_1) = (1, 1)$, $\Sigma \cong \mathsf{C}_p$. But this forces $2\ell - 2p + 1 = 1$, i.e. $\ell = p$. Recalling that $\mathsf{BI}(2,3) \cong \mathsf{CI}(2)$, see [4, p. 519], this yields case (iii). $\square$

References

[1] J. Dadok, Polar coordinates induced by actions of compact Lie groups, *Trans. Amer. Math. Soc.*, **288** (1985), 125-137.
[2] J. Dadok and V. Kac, Polar representations, *J. Algebra*, **92** (1985), 504-524.
[3] J. Dixmier, Champs de vecteurs adjoints sur les groupes et algèbres de Lie semi-simples, *J. Reine Angew. Math.*, **309** (1979), 183-190.
[4] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic press, 1978.
[5] ———, *Groups and Geometric Analysis*, Academic press, 1984.
[6] B. Kostant, On invariant skew-tensors, *Proc. Nat. Acad. of Sci.*, **42** (1956), 148-151.
[7] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, *Amer. J. Math.*, **93** (1971), 753-809.
[8] A. Kowata, Spherical hyperfunctions on the tangent space of symmetric spaces, *Hiroshima Math. J.*, **21** (1991), 401-418.
[9] T. Levasseur and J. T. Stafford, Invariant differential operators on the tangent space of some symmetric spaces, in preparation, (1997).
[10] H. Ochiai, Invariant functions on the tangent space of a rank one semisimple symmetric space, *J. Fac. Sci. Univ. Tokyo*, **39** (1992), 17-31.
[11] A. L. Onischik and E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Verlag, Berlin/New York, 1990.
[12] C. Procesi and G. Schwarz, Inequalities defining orbit spaces, *Invent. Math.*, **81** (1985), 539-554.
[13] G. W. Schwarz, Lifting differential operators from orbit spaces, *Ann. Scient. Éc. Norm. Sup.*, **28** (1995), 253-306.

Département de Mathématiques, Université de Poitiers, 86022 Poitiers, France.
E-mail address, T. Levasseur: levasseu@mathlabo.univ-poitiers.fr
E-mail address, R. Ushirobira: rosane@mathlabo.univ-poitiers.fr