DIFFERENTIAL OPERATORS ON A REDUCTIVE LIE ALGEBRA

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1. Differential operators

Let $\mathcal{X}$ be an affine complex algebraic variety. Denote by $\mathcal{O}(\mathcal{X})$ the algebra of regular functions, and by $\mathcal{D}(\mathcal{X})$ the algebra of differential operators (on $\mathcal{X}$). Recall that $\mathcal{D}(\mathcal{X})$ is a filtered $\mathbb{C}$-algebra (by the order of differential operators): one defines, inductively,

$$\mathcal{D}_0(\mathcal{X}) = \mathcal{O}(\mathcal{X}), \quad \mathcal{D}_m(\mathcal{X}) = \{P \in \operatorname{End}_{\mathbb{C}}(\mathcal{O}(\mathcal{X})) \,:\, [P, \mathcal{O}(\mathcal{X})] \subset \mathcal{D}_{m-1}(\mathcal{X})\}.$$

Then $\mathcal{D}(\mathcal{X}) = \bigcup_m \mathcal{D}_m(\mathcal{X})$ and we denote by

$$\operatorname{gr} \mathcal{D}(\mathcal{X}) = \bigoplus\nolimits_m \mathcal{D}_m(\mathcal{X})/\mathcal{D}_{m-1}(\mathcal{X})$$

the associated graded algebra. The principal symbol of an element $P \in \mathcal{D}(\mathcal{X})$ is denoted by $\operatorname{gr}(P)$.

Assume that $\mathcal{X}$ is smooth. Then, $\mathcal{D}(\mathcal{X})$ is generated by $\mathcal{O}(\mathcal{X})$ and $\operatorname{Der} \mathcal{O}(\mathcal{X})$ (the module of $\mathbb{C}$-linear derivations on $\mathcal{O}(\mathcal{X})$). Furthermore, $\operatorname{gr} \mathcal{D}(\mathcal{X}) = \mathrm{S}_{\mathcal{O}(\mathcal{X})}(\operatorname{Der} \mathcal{O}(\mathcal{X}))$. Here $\mathrm{S}_{\mathcal{O}(\mathcal{X})}(\operatorname{Der} \mathcal{O}(\mathcal{X}))$ is the symmetric algebra of the module $\operatorname{Der} \mathcal{O}(\mathcal{X})$, that we identify with $\mathcal{O}(T^*\mathcal{X})$, the ring of regular functions on the cotangent bundle of $\mathcal{X}$.

For any affine algebraic subvariety $\mathcal{X} \subset \mathbb{C}^n$, let $\mathcal{A}(\mathcal{X})$ the radical ideal defining $\mathcal{X}$. Conversely if $E \subset \mathcal{O}(\mathbb{C}^n)$ is a subset, let $\mathcal{V}(E) \subseteq \mathbb{C}^n$ be the variety of zeroes of E. In particular, for any subset E of $\mathcal{D}(\mathfrak{g})$, $\mathcal{V}(\operatorname{gr} E)$ is an affine subvariety of $T^*\mathcal{X}$.

Let $\mathcal{Y}$ be a smooth affine algebraic variety, and $\varphi : \mathcal{X} \to \mathcal{Y}$ be a morphism. Recall that φ is étale at $x \in \mathcal{X}$, if φ yields an isomorphism $d_x\varphi : T_x\mathcal{X} \xrightarrow{\sim} T_{\varphi(x)}\mathcal{Y}$. The following result is classical.

Proposition 1.1. *Assume that $\varphi : \mathcal{X} \to \mathcal{Y}$ is étale. Then, for all $m \in \mathbb{N}$, one has natural identifications*

$$\mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}(\mathcal{Y})} \mathrm{S}^m(\operatorname{Der} \mathcal{O}(\mathcal{Y})) \xrightarrow{\sim} \mathrm{S}^m(\operatorname{Der} \mathcal{O}(\mathcal{X})), \quad \mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}(\mathcal{Y})} \mathcal{D}_m(\mathcal{Y}) \xrightarrow{\sim} \mathcal{D}_m(\mathcal{X}).$$

Remark . Assume that $\mathcal{X} = V$ is an n-dimensional complex vector space. Then $\mathcal{D}(V)$ is a Weyl algebra on $2n$ generators. We have $\mathcal{O}(V) = \mathrm{S}(V^*)$ and we will identify $\mathrm{S}(V)$ with the algebra of constant coefficient differential operators. If we fix a coordinate basis $\{x_i, \partial_i; 1 \leq i \leq n\}$, we then have

$$\mathrm{S}(V) = \mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial(v)\,;\, v \in V],$$

where $\partial(v)$ is the derivation given by $\partial(v)(f)(x) = \frac{d}{dt}_{|t=0} f(x + tv)$. Note that $\mathcal{D}(V) = \mathrm{S}(V^*) \otimes_{\mathbb{C}} \mathrm{S}(V)$ as an $\mathcal{O}(V)$-module.

Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Assume that $\mathcal{X}$ is a G-variety[1]. We denote by $\mathcal{X}/G$ the affine variety whose ring of regular functions is the ring of invariants $\mathcal{O}(\mathcal{X})^G$. Recall that $\mathcal{X}/G$ can be identified with the variety of closed orbits in $\mathcal{X}$ and that we have a natural surjective morphism $p : \mathcal{X} \twoheadrightarrow \mathcal{X}/G$. For $x \in \mathcal{X}$ we denote by G^x its stabilizer in G and we set $\mathfrak{g}^x = \operatorname{Lie}(G^x)$. Recall (Matsushima's theorem) that if $G.x$ is closed, then G^x is reductive.

[1] G acts rationally on $\mathcal{X}$.

The action of G induces a morphism of Lie algebras $\tau_{\mathcal{X}} : \mathfrak{g} \to \operatorname{Der} \mathcal{O}(\mathcal{X})$, given by $\tau_{\mathcal{X}}(\xi)(f) = \frac{d}{dt}_{|t=0}(\exp(t\xi).f)$.

Example . Consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. Set (for simplicity) $\tau_{\mathfrak{g}} = \tau$ in this case. Since $\mathfrak{g}$ is reductive, we can fix a nondegenerate invariant bilinear symmetric form κ on $\mathfrak{g}$. Then $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified through κ by $x \mapsto \kappa_x = \kappa(\ ,x)$. It follows easily that $\tau(\xi)(\kappa_x) = \kappa_{[\xi,x]}$, for all $\xi \in \mathfrak{g}$. The elements of $\mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$ will be called "adjoint vector fields" on $\mathfrak{g}$. An easy computation also shows that the principal symbol of $\tau(\xi)$, denoted by $\sigma(\xi)$, is the function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \equiv \mathfrak{g} \times \mathfrak{g}$, given by $\sigma(\xi)(a,b) = \kappa([b,a],\xi)$ for all $a, b \in \mathfrak{g}$.

In this situation an orbit $G.x$ is closed if and only if G^x is reductive, if and only if x is semisimple.

Return now to the general situation. The group G acts on $\mathcal{D}(\mathcal{X})$ by $(g.P)(f) = g.(P(g^{-1}.f))$ for all $g \in G$, $P \in \mathcal{D}(\mathcal{X})$ and $f \in \mathcal{O}(\mathcal{X})$. It is not difficult to see that this G-action is rational and that $G.\mathcal{D}_m(\mathcal{X}) \subseteq \mathcal{D}_m(\mathcal{X})$ for all m. Denote by $\mathcal{D}(\mathcal{X})^G$ the ring of invariant differential operators, that we filter by the $\mathcal{D}_m(\mathcal{X})^G$. Since G is reductive, it follows that

$$\operatorname{gr}[\mathcal{D}(\mathcal{X})^G] = [\operatorname{gr} \mathcal{D}(\mathcal{X})]^G = \mathcal{O}(T^*\mathcal{X})^G = \mathcal{O}(T^*\mathcal{X}/G).$$

By restriction we obtain a morphism

$$\psi : \mathcal{D}(\mathcal{X})^G \to \mathcal{D}(\mathcal{X}/G), \quad \psi(P)(f) = P(f) \text{ for all } f \in \mathcal{O}(\mathcal{X}/G).$$

It is clear that $\psi(\mathcal{D}_m(\mathcal{X})^G) \subseteq \mathcal{D}_m(\mathcal{X}/G)$. Note that $\mathcal{O}(\mathcal{X})^G \subseteq \{f \in \mathcal{O}(\mathcal{X}) : \tau_{\mathcal{X}}(\mathfrak{g})(f) = 0\}$, with equality when G is connected. Moreover the differential of the action of G on $\mathcal{D}(\mathcal{X})$ is given by: $\xi.P = [\tau_{\mathcal{X}}(\xi), P]$ for all $\xi \in \mathfrak{g}$, $P \in \mathcal{D}(\mathcal{X})$. Set

$$\mathcal{J}(\mathcal{X}) = \{D \in \mathcal{D}(\mathcal{X}) : D(\mathcal{O}(\mathcal{X})^G) = 0\}, \quad \mathcal{I}(\mathcal{X}) = \mathcal{J}(\mathcal{X}) \cap \mathcal{D}(\mathcal{X})^G.$$

Clearly $\operatorname{Ker} \psi = \mathcal{I}(\mathcal{X})$ and $\mathcal{J}(\mathcal{X}) \supseteq \mathcal{D}(\mathcal{X})\tau_{\mathcal{X}}(\mathfrak{g})$.

Assume now that $G = W$ is a finite sugroup of $\mathrm{GL}(V)$, where V is a complex vector space of dimension ℓ. Then, the morphism $p : V \twoheadrightarrow V/W$ is finite and every orbit is closed. Define a W-stable open subset of V by

$$V' := \{v \in V \mid p \text{ étale at } v\}.$$

Hence, $V' = \{v \in V \mid \operatorname{rk}_v p = \ell \text{ and } p(v) \text{ is a smooth point}\}$.

Note that if the action of W is not faithful, we may decompose $V = V_W \oplus V^W$ so that $V/W = (V_W/W) \oplus V^W$ and $(V_W)^W = 0$. Therefore the analysis of the situation always reduces to the case of a faithful action of W on V. In this case, it is a classical result that $V' = \{v \in V \mid W^v = \{1\}\}$.

Recall that $\mathcal{D}(V)$ is a simple ring, and, since W is finite, $\mathcal{D}(V)^W$ is also simple [10]. Hence $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is an embedding. The following result is well known[2].

Theorem 1.2. *The following are equivalent*

(1) *ψ is a (filtered) isomorphism;*
(2) $\operatorname{codim}(V \setminus V') \geq 2$*;*
(3) *W does not contain any pseudoreflection ($\neq 1$).*

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[2]We shall not use this result.

Recall that V/W is smooth if and only if W is generated by pseudoreflections. Therefore, if $W \neq \{1\}$ and V/W is smooth, $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is not surjective. Actually, if W acts faithfully on V and $\mathrm{S}(V^*)^W = \mathbb{C}[p_1, \dots, p_\ell]$ is a polynomial ring, it is not difficult to see that there does not exist any $d \in \mathcal{D}(V)^W$ such that $\psi(d) = \frac{\partial}{\partial p_i}$.

Example . The following case is obvious, but will prove useful in the sequel. Assume that $\dim V = 1$ and set

$$\mathrm{S}(V^*) = \mathbb{C}[z], \qquad \mathrm{S}(V) = \mathbb{C}[\partial_z].$$

Let $W = \{\pm 1\}$ act on V by multiplication. Then

$$\mathrm{S}(V^*)^W = \mathbb{C}[z^2], \quad \mathrm{S}(V) = \mathbb{C}[\partial_z^2], \quad \mathcal{D}(V)^W = \mathbb{C}[z^2, z\partial_z, \partial_z^2]^3.$$

Set $t = z^2$. Then $\mathcal{D}(V/W) = \mathbb{C}[t, \partial_t]$ and the morphism $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is given by

$$\psi(z^2) = t, \quad \psi(z\partial_z) = 2t\partial_t, \quad \psi(\partial_z^2) = 4t\partial_t^2 + 2\partial_t.$$

Note that $\partial_t \notin \operatorname{Im} \psi$. We have $V' = V \setminus \{0\}$, and if we localize at the invariant function $t = z^2$, we obtain

$$\psi : \mathcal{D}(V)^W_{z^2} = \mathbb{C}[z^{\pm 2}, z^{-1}\partial_z] \xrightarrow{\sim} \mathcal{D}(V/W)_t = \mathbb{C}[t^{\pm 1}, \partial_t],$$

since $\psi(\frac{1}{2}z^{-1}\partial_z) = \partial_t$. Thus $\mathcal{D}(V')^W \xrightarrow{\sim} \mathcal{D}(V'/W)$.

2. The map δ: definition

Let G be a *connected* reductive algebraic group with maximal torus H. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and denote by $W = W(\mathfrak{g}, \mathfrak{h})$ the associated Weyl group. Let R be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Fix a basis B of R and let R^+ be the set of positive roots. We set $\mathfrak{n}^\pm = \oplus_{\{\pm\alpha \in R^+\}} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm\alpha} = \mathbb{C}X_{\pm\alpha}$. If $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$, we have

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}, \quad \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{z}, \quad \mathfrak{s} = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{t}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{s}$. We set $n = \dim \mathfrak{g}$, $\ell = \dim \mathfrak{h}$ and $k = \dim \mathfrak{t}$. As in §1, we denote by κ an invariant symmetric form on $\mathfrak{g}$. Recall that the discriminant of $\mathfrak{g}$ is the invariant function d_ℓ defined by

$$\det(t\mathrm{Id} - \operatorname{ad} x) = t^n + \cdots + (-1)^\ell d_\ell(x) t^\ell.$$

The set of generic[4] elements is $\mathfrak{g}' = \{x \in \mathfrak{g} \mid d_\ell(x) \neq 0\}$. Then $\mathfrak{g}'$ is the set of points where the morphism $p : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/G$ is smooth.

Recall the fundamental result of Chevalley:

Theorem 2.1. *There is a natural isomorphism $\mathfrak{h}/W \xrightarrow{\sim} \mathfrak{g}/G$: the restriction of functions from $\mathfrak{g}$ to $\mathfrak{h}$ yields an isomorphism of algebras,*

$$\phi : \mathrm{S}(\mathfrak{g}^*)^G \xrightarrow{\sim} \mathrm{S}(\mathfrak{h}^*)^W, \quad \phi(f) = f_{|\mathfrak{h}}.$$

Similarly, there exists an isomorphism $\phi : \mathrm{S}(\mathfrak{g})^G \to \mathrm{S}(\mathfrak{h})^W$, induced by the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ given by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{n}^+ \oplus \mathfrak{n}^-)$.

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[3]Observe that $[z^2, \partial_z^2] = 4z\partial_z + 2$, and thus $\mathcal{D}(V)^W = \mathbb{C}[z^2, \partial_z^2]$.

[4]An element x is called *generic* if it is semisimple and $\dim \mathfrak{g}^x = \ell$.

For sake of simplicity, all the isomorphisms related to the previous Chevalley isomorphisms will be denoted by the same symbol, ϕ.

Note that we may write $\mathrm{S}(\mathfrak{g}^*)^G = \mathbb{C}[u_1, \dots, u_k, u_{k+1}, \dots, u_\ell]$, where $u_i \in \mathrm{S}(\mathfrak{s}^*)^{\mathfrak{s}}$ for $i = 1, \dots, k$ and $u_j \in \mathfrak{z}^*$ for $i = k+1, \dots, \ell$ (hence $\mathrm{S}(\mathfrak{z}^*) = \mathbb{C}[u_{k+1}, \dots, u_\ell]$). We set $p_j = u_{j|\mathfrak{h}}$ and we denote by $p : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/W$ the associated morphism. Then $\mathrm{S}(\mathfrak{h}^*)^W = \mathrm{S}(\mathfrak{t}^*)^W \otimes \mathrm{S}(\mathfrak{z}^*) = \mathbb{C}[p_1, \dots, p_\ell]$. Define an element of $\mathrm{S}(\mathfrak{h}^*)$ by

$$\pi = \prod_{\alpha \in R^+} \alpha.$$

The following are well known, see [3, Proposition 3.13]:

- Let $\epsilon(w)$ be the signature of $w \in W$, then,
$$\mathrm{S}(\mathfrak{h}^*)^W \pi = \{f \in \mathrm{S}(\mathfrak{h}^*) \mid \forall w \in W,\, w.f = \epsilon(w) f\};$$
- $\phi(d_\ell) = (\pm)\pi^2 \in \mathrm{S}(\mathfrak{h}^*)^W$;
- up to a nonzero constant, $\pi(x) = \det \mathrm{Jac}(p)(x)$ and p is étale at $h \in \mathfrak{h}$ if, and only if, $h \in \mathfrak{h}' = \{x \in \mathfrak{h} \,:\, \pi(x) \neq 0\}$.

Recall [16, Corollary 3.11] that if $x \in \mathfrak{g}$ is semisimple, then G^x is a connected reductive subgroup of G. One can conjugate x and assume that $x \in \mathfrak{h}$. If we set $\Gamma = \{\alpha \in B \,:\, \alpha(x) = 0\}$, then: $\mathfrak{g}^x = \mathfrak{h} \oplus \big(\sum_{\{\beta \in \mathbb{Z}\Gamma \cap R\}} \mathfrak{g}_\beta\big)$, $[x, \mathfrak{g}] = \oplus_{\{\beta \notin \mathbb{Z}\Gamma\}} \mathfrak{g}_\beta$.

The Chevalley isomorphism ϕ induces an isomorphism

$$\phi : \mathcal{D}(\mathfrak{g}/G) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}/W), \quad \phi(P)(f) = \phi(P(\phi^{-1}(f)))$$

for all $P \in \mathcal{D}(\mathfrak{g}/G)$, $f \in \mathcal{O}(\mathfrak{h}/W) = \mathrm{S}(\mathfrak{h}^*)^W$. By composing with the natural morphism $\psi : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{g}/G)$, we obtain the morphism

$$r = \psi \circ \phi : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W), \quad r(P)(f) = \phi(P(\phi^{-1}(f))).$$

The element $r(P)$ is called the *radial component* of P. It is clear that

$$\operatorname{Ker} r = \mathcal{I} = \{P \in \mathcal{D}(\mathfrak{g})^G \,:\, P\big(\mathrm{S}(\mathfrak{g}^*)^G\big) = 0\}.$$

Since the morphism $p : \mathfrak{h}' \twoheadrightarrow \mathfrak{h}'/W$ is étale, it follows from Proposition 1.1 that we can identify $\mathcal{D}(\mathfrak{h}')^W$ with $\mathcal{D}(\mathfrak{h}'/W)$ (observe that $\mathcal{D}(\mathfrak{h}') = \mathcal{O}(\mathfrak{h}') \otimes_{\mathcal{O}(\mathfrak{h}'/W)} \mathcal{D}(\mathfrak{h}'/W)$ and take the W-invariants). Therefore

$$\operatorname{Im} r \subset \mathcal{D}(\mathfrak{h}/W) \subset \mathcal{D}(\mathfrak{h}'/W) \equiv \mathcal{D}(\mathfrak{h}')^W \subset \mathcal{D}(\mathfrak{h}').$$

Inside $\mathcal{D}(\mathfrak{h}')$ we can consider the inner automorphism

$$\iota : D \mapsto \pi \circ D \circ \pi^{-1}, \text{ i.e. } \iota(D)(f) = \pi D(\pi^{-1} f) \text{ for all } f \in \mathcal{O}(\mathfrak{h}').$$

From $w.\iota(D) = \pi \circ w.D \circ \pi^{-1}$, we get that $\iota(\mathcal{D}(\mathfrak{h}')^W) = \mathcal{D}(\mathfrak{h}')^W$.

Definition 2.2. *The Harish-Chandra map $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}')^W$ is defined to be $\delta = \iota \circ r$, i.e.*

$$\forall\, D \in \mathcal{D}(\mathfrak{g})^G, \forall\, f \in \mathcal{O}(\mathfrak{h})^W, \quad \delta(D)(f) = \pi r(D)(\pi^{-1} f).$$

In the next two sections we will sketch a proof of the following result of Harish-Chandra.

Theorem 2.3. (1) $\operatorname{Im} \delta \subseteq \mathcal{D}(\mathfrak{h})^W$.

(2) δ coincides with the Chevalley isomorphisms on $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$.

We end this section by the following slight generalization of the definition of δ. Let $U \subseteq \mathfrak{g}$ be a G-stable open subset. Set $\tilde{\mathfrak{h}} = U \cap \mathfrak{h}$ and $\tilde{\mathfrak{h}}' = U \cap \mathfrak{h}'$. Then the Chevalley isomorphism yields $U/G \xrightarrow{\sim} \tilde{\mathfrak{h}}/W$, and we can define in a similar way the "radial component" of elements of $\mathcal{D}(U)^G$. We then have a morphism

$$r : \mathcal{D}(U)^G \to \mathcal{D}(\tilde{\mathfrak{h}}/W) \hookrightarrow \mathcal{D}(\tilde{\mathfrak{h}}'/W) \equiv \mathcal{D}(\tilde{\mathfrak{h}}')^W.$$

After composition with ι (i.e. conjugation by the restriction of π on $\tilde{\mathfrak{h}}'$), we obtain a morphism

$$\delta = \iota \circ r : \mathcal{D}(U)^G \to \mathcal{D}(\tilde{\mathfrak{h}}')^W$$

which extends the previously defined δ.

3. The map δ in the $\mathfrak{sl}(2)$-case

In this section we assume that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$, where as usual $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $\mathfrak{h} = \mathbb{C}h$, $R = \{\pm\alpha\}$ where $\alpha(h) = 2$. We choose $\kappa(a,b) = \operatorname{tr}(ab)$, hence $\kappa(e,f) = 1$, $\kappa(h,h) = 2$. Let $\{x,y,z\}$ be the dual basis of $\{e,f,h\}$, thus $x = \kappa_f$, $y = \kappa_e$ and $z = \frac{1}{2}\kappa_h$. Furthermore $\partial(e) = \partial_y$, $\partial(f) = \partial_x$ and $\partial(h) = \partial_z$. Then

$$\mathrm{S}(\mathfrak{g}^*)^G = \mathbb{C}[z^2 + xy], \qquad \mathrm{S}(\mathfrak{g})^G = \mathbb{C}[\partial_z^2 + 4\partial_x\partial_y].$$

We set

$$\zeta = z^2 + xy, \;\; \omega = \partial_z^2 + 4\partial_x\partial_y, \;\; \varepsilon_{\mathfrak{g}} = x\partial_x + y\partial_y + z\partial_z, \;\; \varepsilon_{\mathfrak{h}} = z\partial_z.$$

Observe that $E_{\mathfrak{g}} := \varepsilon_{\mathfrak{g}} + 3/2 = [-\frac{1}{4}\zeta, \omega]$.

Recall that $W = \{1, s\}$, where $s : h \mapsto -h$. Therefore we are in the situation of the example $W = \{\pm 1\}$ given in §1. Hence, if $t = z^2$,

$$\psi : \mathcal{D}(\mathfrak{h})^W = \mathbb{C}[z^2, \partial_z^2] \hookrightarrow \mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[t, \partial_t]$$

is given by $\psi(z^2) = t$, $\psi(\partial_z^2) = 4t\partial_t^2 + 2\partial_t$. The Chevalley isomorphisms are determined by $\phi(\zeta) = z^2 = t$, $\phi(\omega) = \partial_z^2$. Recall that $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W)$.

Lemma 3.1. *We have:*

1. $\mathcal{D}(\mathfrak{g})^G = \mathbb{C}[\zeta, \omega] \cong U(\mathfrak{sl}(2))$;
2. $r(\zeta) = t$, $r(\omega) = 4t\partial_t^2 + 6\partial_t$.

Proof. (1) By an usual argument of associated graded ring, we will obtain generators of $\mathcal{D}(\mathfrak{g})^G$ by computing

$$[\operatorname{gr}\mathcal{D}(\mathfrak{g})]^G = \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g})^G \equiv \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)^G.$$

Here, G acts diagonally on $\mathfrak{g}^* \times \mathfrak{g}^*$ by $g.(a,b) = (g.a, g.b)$ and we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through κ. Under this identification, $\partial_z \leftrightarrow 2z$, $\partial_x \leftrightarrow y$ and $\partial_y \leftrightarrow x$. Therefore $\operatorname{gr}\mathcal{D}(\mathfrak{g}) \equiv \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*) = \mathbb{C}[U,V]$, where U and V are the generic matrices $U = \left[\begin{smallmatrix} z & x \\ y & -z \end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix} \frac{1}{2}\partial_z & \partial_y \\ \partial_x & -\frac{1}{2}\partial_z \end{smallmatrix}\right]$. Then, classical invariant theory gives that $\mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)^G$ is generated by

$$\operatorname{tr}(U^2) = \zeta, \quad \operatorname{tr}(UV) = \varepsilon_{\mathfrak{g}}, \quad \operatorname{tr}(V^2) = \omega/4.$$

Thus $\mathcal{D}(\mathfrak{g})^G = \mathbb{C}[\zeta, \omega, E_{\mathfrak{g}}] = \mathbb{C}[\zeta, \omega]$. Now observe that

$$[E_{\mathfrak{g}}, -\zeta/4] = 2\zeta, \quad [E_{\mathfrak{g}}, \omega] = -2\omega, \quad [-\zeta/4, \omega] = E_{\mathfrak{g}}.$$

Therefore, there exists a surjective morphism $\nu : U(\mathfrak{sl}(2)) \twoheadrightarrow \mathcal{D}(\mathfrak{g})^G$, such that $\nu(e) = -\frac{1}{4}\zeta$, $\nu(f) = \omega$ and $\nu(h) = E_{\mathfrak{g}}$. To prove that ν is injective[5], one can either show that

[5] We leave the details to the reader.

GKdim $\mathcal{D}(\mathfrak{g})^G = \mathrm{GKdim}\,\mathrm{gr}\,\mathcal{D}(\mathfrak{g})^G = \mathrm{GKdim}\,U(\mathfrak{sl}(2)) = 3$, see Corollary 5.8 (note that the maximal dimension of a G-orbit in $\mathfrak{g}\times\mathfrak{g}$ is 3), or prove that, if Ω is the Casimir element of $U(\mathfrak{sl}(2))$, then $\nu(\Omega - c) \neq 0$ for all $c \in \mathbb{C}$.

(2) The equality $r(\zeta) = t$ is clear. It is easily seen that

$$r(\omega)(1) = 0,\ \ r(\omega)(t) = 6,\ \ r(\omega)(t^2) = 20t.$$

Hence, $r(\omega) = 4t\partial_t^2 + 6\partial_t$ as desired. $\square$

Remark . Observe that $r(\omega) = \partial_z^2 + 4\partial_t \notin \mathcal{D}(\mathfrak{h})^W$, since $\partial_t \notin \mathcal{D}(\mathfrak{h})^W$ (see §1). Thus $\mathrm{Im}\, r \not\subset \mathcal{D}(\mathfrak{h})^W$.

Lemma 3.2. *$\delta(\omega) = \partial_z^2$ and $\delta(\zeta) = z^2$.*

Proof. In the notation of §2, we have $\pi = \alpha = 2z$ and $\mathfrak{h}' = \mathfrak{h}\setminus\{0\}$. Recall that we can identify $\mathcal{D}(\mathfrak{h}'/W) = \mathbb{C}[t^{\pm1}, \partial_t]$ with $\mathcal{D}(\mathfrak{h}')^W = \mathbb{C}[z^{\pm2}, \frac{1}{2}z^{-1}\partial_z]$. Now, since $z\partial_z z^{-1} = \partial_z - z^{-1}$ and $r(\omega) = 4t\partial_t^2 + 6\partial_t = \partial_z^2 + 2z^{-1}\partial_z$, we obtain

$$\delta(\omega) = \iota(r(\omega)) = (\partial_z - z^{-1})^2 + 2z^{-1}(\partial_z - z^{-1}) = \partial_z^2.$$

The second equality is obvious. $\square$

Proposition 3.3. (1) $\delta(\mathcal{D}(\mathfrak{g})^G) = \mathcal{D}(\mathfrak{h})^W$.

(2) δ coincides with the Chevalley isomorphisms on $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$.

Proof. The claims follow from Lemma 3.1 and Lemma 3.2. $\square$

Remark . From $\mathcal{D}(\mathfrak{g})^G \cong U(\mathfrak{sl}(2))$ we get that δ induces isomorphisms

$$\mathcal{D}(\mathfrak{h})^W \cong \mathcal{D}(\mathfrak{g})^G/\mathcal{I} \cong U(\mathfrak{sl}(2))/(\Omega + \lambda),$$

where $\lambda \in \mathbb{C}$ and Ω is the Casimir element. It is not difficult to see that $\lambda = 3/4$.

4. The map δ in the general case

In this section we sketch the proof of Theorem 2.3 given by G. Schwarz [14]. We continue with the notation of §2[6].

Fix a coordinate basis $\{z_1, \dots, z_\ell\}$ of $\mathfrak{h}^*$ and set $\partial_i = \frac{\partial}{\partial z_i}$. Let $P \in \mathcal{D}(\mathfrak{g})^G$. We have, with the usual conventions,

$$\delta(P) = \sum_m c_m(z)\partial^m,\ \ c_m \in \mathcal{O}(\mathfrak{h}')\ \text{ for all } m \in \mathbb{N}^\ell.$$

We want to show that $a_m \in \mathcal{O}(\mathfrak{h})$. Since $\mathcal{O}(\mathfrak{h}') = \mathcal{O}(\mathfrak{h})_\pi$, this is equivalent to showing that the a_m have no pole along the reflecting hyperplanes $\mathcal{H}_\gamma = \{h \in \mathfrak{h} : \gamma(h) = 0\}$ for $\gamma \in R^+$.

Fix $\gamma \in R^+$. Choose $b \in \mathcal{H}_\gamma$, $b \notin \mathcal{H}_\beta$ for $\beta \in R^+\setminus\{\gamma\}$. The idea is to prove that $\delta(P)$ is smooth in a neighborhood of b; this will be done by a "Luna's slice type argument". We have

$$\mathfrak{g}^b = \mathfrak{sl}(2)_\gamma \oplus \mathcal{H}_\gamma,\ \text{ where } \mathfrak{sl}(2)_\gamma = \mathbb{C}H_\gamma + \mathbb{C}X_\gamma + \mathbb{C}X_{-\gamma}.$$

The group G^b is reductive and we have a G^b-decomposition $\mathfrak{g} = \mathfrak{g}^b \oplus [b, \mathfrak{g}]$. Recall that, since $G.b \equiv G/G^b$ via the adjoint action, $T_b(G.b) = \mathfrak{g}/\mathfrak{g}^b \cong [\mathfrak{g}, b]$ is generated by the tangent vectors $\tau(\xi)_b = [b, \xi]$. Note also that $W(\mathfrak{g}^b, \mathfrak{h}) = W^b = \{1, s = s_\gamma\}$, $R(\mathfrak{g}^b, \mathfrak{h}) = \{\pm\gamma\}$.

Set $p = \dim G.b$ and define

$$U = \{u \in \mathfrak{g} \,:\, \exists\, X_1, \dots, X_p \in \mathfrak{g},\ \mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_u, \dots, \tau(X_p)_u\rangle_{\mathbb{C}}\}.$$

[6] Note that we may, if necessary, assume that $\mathfrak{g}$ is simple and that $G \subset \mathrm{GL}(\mathfrak{g})$ is the adjoint group.

(a) U is an open neighbourhood of b. Indeed: Let $u \in U$ and let $X_1, \dots, X_p$ be such that $\mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_u, \dots, \tau(X_p)_u \rangle_{\mathbb{C}}$, then

$$U' = \{u' \in \mathfrak{g} \,:\, \mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_{u'}, \dots, \tau(X_p)_{u'} \rangle_{\mathbb{C}}\}$$

is an affine open neighbourhood of u and $U' \subseteq U$.

(b) U is G^b-stable. Let $u \in U$. Note first that, for all $g \in G$,

$$g.\tau(X_i)_u = g.[u, X_i] = [g.u, g.X_i] = \tau(g.X_i)_{g.u}.$$

When $g \in G^b$, we also have $g.\mathfrak{g}^b = \mathfrak{g}^b$. Hence

$$\mathfrak{g} = g.\mathfrak{g} = \mathfrak{g}^b \oplus g.\langle \tau(X_1)_u, \dots, \tau(X_p)_u \rangle_{\mathbb{C}} = \mathfrak{g}^b \oplus \langle \tau(g.X_1)_{g.u}, \dots, \tau(g.X_p)_{g.u} \rangle_{\mathbb{C}}.$$

This shows that $g.u \in U$.

(c) Let $t_1, \dots, t_{\ell-1}$ be coordinate functions on $\mathcal{H}_\gamma$, and let $\{x, y, z\}$ be the dual basis of $\{X_\gamma, X_{-\gamma}, H_\gamma\}$. It follows from (a) and (b) that, on the open subset U,

$$\mathcal{D}(U) = \sum_{i,j,k \in \mathbb{N}, \mu \in \mathbb{N}^{\ell-1}} \mathcal{O}(U) \partial_x^i \partial_y^j \partial_z^k \partial_t^\mu + \mathcal{D}(U)\tau(\mathfrak{g}).$$

Therefore we can write $P = \tilde{P} + Q$ (on U), with $\tilde{P} \in \sum \mathcal{O}(U)\partial_x^i \partial_y^j \partial_z^k \partial_t^\mu$ and $Q \in \mathcal{D}(U)\tau(\mathfrak{g})$. Since $P \in \mathcal{D}(\mathfrak{g})^G \subset \mathcal{D}(U)^{G^b}$, and since G^b is reductive, we may as well assume that $\tilde{P}$ and Q are G^b-invariant.

Set $\tilde{U} = U \cap \mathfrak{g}^b$, $\tilde{\mathfrak{h}} = U \cap \mathfrak{h}$ and $\tilde{\mathfrak{h}}' = U \cap \{h \in \mathfrak{h} \,:\, \gamma(h) \neq 0\}$. Denote by $\tilde{r}$ and $\tilde{\delta} = \gamma \circ \tilde{r} \circ \gamma^{-1}$ the morphisms from $\mathcal{D}(\tilde{U})^{G^b}$ to $\mathcal{D}(\tilde{\mathfrak{h}}')^{W^b}$. From the $\mathfrak{sl}(2)$-case we can deduce that $\operatorname{Im} \tilde{\delta} \subseteq \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$. Therefore $\tilde{\delta}(\tilde{P}) = \gamma \circ \tilde{r} \circ \gamma^{-1} \in \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$.

Note that, since $\tau(\mathfrak{g})$ kills the G-invariant functions, $P(f) = \tilde{P}(f)$ for all $f \in \mathcal{O}(U)^G$. In particular, since $\mathcal{O}(\tilde{\mathfrak{h}})^W \subset \mathcal{O}(\tilde{\mathfrak{h}})^{W^b}$, we have that $r(P) = \tilde{r}(\tilde{P})$ on $A := \mathcal{O}(\tilde{\mathfrak{h}})^W$. Set $\tilde{\pi} = \prod_{\{\gamma \neq \alpha \in R^+\}} \alpha$; then $\pi = \tilde{\pi}\gamma$ and $\tilde{\pi}^{\pm 1}$ is smooth on a neighbourhood of b. Now, write $\delta(P) = \tilde{\pi}\gamma r(P)\gamma^{-1}\tilde{\pi}^{-1}$. From the above we know that, on A, $\delta(P) = \tilde{\pi}(\gamma \tilde{r}(\tilde{P})\gamma^{-1})\tilde{\pi}^{-1}$. But, we have seen that $\tilde{\delta}(\tilde{P}) = \gamma \tilde{r} \gamma^{-1} \in \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$ and $\tilde{\pi}^{\pm 1}$ are smooth on a neighbourhood of b. Hence, the same is true of $\delta(P)$.

(d) To complete the proof of Theorem 2.3, it remains to show that δ coincide with the Chevalley isomorphisms. Recall that this is obvious, by construction, for δ on $\mathrm{S}(\mathfrak{g}^*)^G$. We thus have to show that $\delta = \phi$ on $\mathrm{S}(\mathfrak{g})^G$; this will be done by "Fourier transform". Without loss of generality we can reduce to the case when $\mathfrak{g}$ is simple.

Choose coordinates on $\mathfrak{g}$ such that $\kappa = -\frac{1}{2}\sum_{i=1}^n x_i^2$ and set

$$\omega = \tfrac{1}{2}\textstyle\sum_{i=1}^n \partial_{x_i}^2, \quad \varepsilon_{\mathfrak{g}} = \sum_{i=1}^n x_i \partial_{x_i}.$$

Then, as in the $\mathfrak{sl}(2)$-case, one checks that

$$[\kappa, \omega] = E_{\mathfrak{g}} := \varepsilon_{\mathfrak{g}} + n/2, \quad [E_{\mathfrak{g}}, \kappa] = 2\kappa, \quad [E_{\mathfrak{g}}, \omega] = -2\omega.$$

Hence, $\mathfrak{k} = \mathbb{C}\kappa + \mathbb{C}\omega + \mathbb{C}E_{\mathfrak{g}} \cong \mathfrak{sl}(2) = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$. Recall that $\operatorname{gr} \mathcal{D}(\mathfrak{g}) = \mathcal{O}(T^*\mathfrak{g}) \equiv \mathcal{O}(\mathfrak{g} \times \mathfrak{g})$. Since $\mathfrak{g} \times \mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}^2$, there is a natural action of $\mathrm{SL}(2)$ on $\mathfrak{g} \times \mathfrak{g}$, and therefore on $\operatorname{gr} \mathcal{D}(\mathfrak{g}) = \mathcal{O}(T^*\mathfrak{g})$. This action lifts to an $\mathrm{SL}(2)$-action on $\mathcal{D}(\mathfrak{g})$. Tracing the identifications, one sees that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)$ acts on $\mathcal{D}(\mathfrak{g})$ in the following way

$$g.x_i = ax_i + c\partial_{x_i}, \qquad g.\partial_{x_j} = bx_j + d\partial_{x_j}.$$

Observe now that $[E_{\mathfrak{g}}, x_i] = x_i$, $[E_{\mathfrak{g}}, \partial_{x_i}] = -\partial_{x_i}$, $[\omega, x_i] = \partial_{x_i}$, $[\omega, \partial_{x_i}] = 0$, $[\kappa, x_i] = 0$, $[\kappa, \partial_{x_i}] = x_i$. It follows that, inside $\mathcal{D}(\mathfrak{g})$,

$$\exp(te) = \exp(t \operatorname{ad} \kappa), \quad \exp(tf) = \exp(t \operatorname{ad} \omega), \quad \exp(th) = \exp(t \operatorname{ad} E_{\mathfrak{g}}).$$

Hence, the adjoint action of $\mathfrak{k}$ integrates to the SL(2)-action that we just described. Observe that, since $\kappa, \omega, E_{\mathfrak{g}}$ are G-invariant, the SL(2)-action commutes with the G-action. Consider now the "Weyl group element" $w = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2)$ (here $i = \sqrt{-1} \in \mathbb{C}$). It acts on $\mathcal{D}(\mathfrak{g})$ by $w.x_j = i\partial_{x_j}$, $w.\partial_{x_j} = ix_j$ for all $j = 1, \dots, n$.

Let $\kappa_{\mathfrak{h}}$, $\omega_{\mathfrak{h}}$ and $\varepsilon_{\mathfrak{h}}$ be the analogous elements of $\mathcal{D}(\mathfrak{h})^W$. We have

$$\delta(\kappa) = \kappa_{\mathfrak{h}}, \quad [\kappa_{\mathfrak{h}}, \omega_{\mathfrak{h}}] = E_{\mathfrak{h}} := \varepsilon_{\mathfrak{h}} + \ell/2.$$

Let $f \in \mathrm{S}^p(\mathfrak{g}^*)^G$. Then, $\delta([\varepsilon_{\mathfrak{g}}, f]) = [\delta(\varepsilon_{\mathfrak{g}}), \phi(f)] = \delta(pf) = p\phi(f)$. This implies that $\delta(\varepsilon_{\mathfrak{g}}) = \varepsilon_{\mathfrak{h}} - c$ for some $c \in \mathbb{C}$. We know that $\delta(\omega) \in \mathcal{D}_2(\mathfrak{h})^W$. Note that

$$\delta([E_{\mathfrak{g}}, \omega]) = [\delta(E_{\mathfrak{g}}), \delta(\omega)] = [\epsilon_{\mathfrak{h}}, \delta(\omega)] = -2\delta(\omega).$$

In the appropriate coordinate basis of $\mathfrak{h}$, this forces

$$\delta(\omega) = \sum_{\{|\mu|-|\nu|=-2, |\nu|\leq 2\}} a_{\mu,\nu} x^\mu \partial_x^\nu, \quad a_{\mu,\nu} \in \mathbb{C},$$

and it follows that

$$\delta(\omega) = \sum_\nu a_\nu \partial_x^\nu \in \mathrm{S}^2(\mathfrak{h})^W = \mathbb{C}\omega_{\mathfrak{h}}.$$

Thus $\delta(\omega) = a\omega_{\mathfrak{h}}$ for some $a \in \mathbb{C}$. Then, $\delta([\kappa, \omega]) = [\kappa_{\mathfrak{h}}, a\omega_{\mathfrak{h}}] = \varepsilon_{\mathfrak{h}} - c + n/2$ implies that $a = 1$ and $c = \frac{1}{2}(n - \ell)$. Hence, we have shown

$$\delta(\kappa) = \kappa_{\mathfrak{h}}, \quad \delta(\omega) = \omega_{\mathfrak{h}}, \quad \delta(E_{\mathfrak{g}}) = E_{\mathfrak{h}}.$$

Recall that $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{h})$ have natural SL(2)-actions, which integrate the adjoint actions of $\mathbb{C}\kappa + \mathbb{C}\omega + \mathbb{C}E_{\mathfrak{g}}$ and $\mathbb{C}\kappa_{\mathfrak{h}} + \mathbb{C}\omega_{\mathfrak{h}} + \mathbb{C}E_{\mathfrak{h}}$ respectively. The above formulas prove that the map δ is SL(2)-equivariant. Let $P \in \mathrm{S}^m(\mathfrak{g})^G$. By definition of w, and the fact that the SL(2)-action commutes with the G-action, we obtain that $w.P \in \mathrm{S}^m(\mathfrak{g}^*)^G$. Therefore

$$w.\delta(P) = \delta(w.P) = (w.P)_{|\mathfrak{h}}$$

implies that

$$\delta(P) = w^{-1}.\delta(w.P) = w^{-1}(w.P)_{|\mathfrak{h}}.$$

The definition of w then shows that $w^{-1}(w.P)_{|\mathfrak{h}}$ is the projection of P onto $\mathrm{S}^m(\mathfrak{h})^W$, as required. $\square$

5. Surjectivity of δ

We have shown that there exists a homomorphism

$$\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$$

with kernel

$$\mathcal{I} = \{P \in \mathcal{D}(\mathfrak{g})^G \mid P\big(\mathcal{O}(\mathfrak{g})^G\big) = 0\}.$$

Evidently, $\operatorname{Im}\delta$ contains the images of $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$ which, by Theorem 2.3 coincide with $\mathrm{S}(\mathfrak{h}^*)^W$ and $\mathrm{S}(\mathfrak{h})^W$. Denote by B the subalgebra of $\mathcal{D}(\mathfrak{h})^W$ generated by $\mathrm{S}(\mathfrak{h}^*)^W$ and $\mathrm{S}(\mathfrak{h})^W$. Two questions naturally arise.

(†) Is δ surjective?

Recall that δ is a filtered morphism. The second question is more precise: Is it true that $\delta(\mathcal{D}_m(\mathfrak{g})^G) = \mathcal{D}_m(\mathfrak{h})^W$ for all $m \in \mathbb{N}$? Equivalently:

(††) Is $\operatorname{gr}(\delta) : \operatorname{gr}\mathcal{D}(\mathfrak{g})^G \to \operatorname{gr}\mathcal{D}(\mathfrak{h})^W$ surjective?

If this is true, we shall say that δ is *graded-surjective.*

N. Wallach has shown [17] that (††) has a positive answer when $\mathfrak{g}$ has no factor of type E_p, $p = 6, 7, 8$. In [7] it is shown that (†) is true in all cases. This follows from a general result about differential operators invariant under a finite group action:

Theorem 5.1. [7] *Let V be a finite dimensional $\mathbb{C}$-vector space and W be a finite subgroup of $\mathrm{GL}(V)$. Then $\mathcal{D}(V)^W$ is generated by $\mathrm{S}(V)^W$ and $\mathrm{S}(V^*)^W$.*

The proof of Theorem 5.1 is not difficult. In this section we shall give a proof in the case we are presently interested: $(V, W) = (\mathfrak{h}, W = \text{Weyl group})$. The idea of the proof is exactly the same, but, in this particular case, we will bring a little bit more of information.

We fix a coordinate basis $\{x_1, \dots, x_\ell; \partial_1, \dots, \partial_\ell\}$ of $\mathfrak{h}^* \times \mathfrak{h}$[7]. In this situation we may also suppose that $\{\partial_1, \dots, \partial_\ell\}$ is an orthonormal basis, with respect to κ, on a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$. Then, each $w \in W$ acts on $\mathfrak{h}$ via an orthogonal matrix: $w.\partial_j = \sum_{i=1}^{\ell} w_{ij}\partial_i$.

Recall that $\pi^2 \in B$ and that, up to a nonzero scalar (that we ignore), we have $\pi = \det \mathrm{Jac}(p)$, where $\mathrm{Jac}(p) = [\frac{\partial p_i}{\partial x_j}] \in \mathrm{M}_\ell(\mathrm{S}(\mathfrak{h}^*))$. Moreover $\mathfrak{h}' = \{h : \pi(h) \neq 0\}$ is the set of points where $p : \mathfrak{h} \to \mathfrak{h}/W$ is étale. Define, as usual, the gradient vector field associated to the invariant function p_j by

$$\nabla(p_j) = \sum_{i=1}^{\ell} \partial_i(p_j)\partial_i, \quad j = 1, \dots, \ell.$$

Lemma 5.2. *The following assertions hold:*

(1) $\nabla(p_j) \in [\mathrm{Der}\, \mathcal{O}(\mathfrak{h})]^W \cap B$;

(2) $\mathrm{Der}\, \mathcal{O}(\mathfrak{h}') = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')\nabla(p_j)$;

(3) $[\mathrm{Der}\, \mathcal{O}(\mathfrak{h}')]^W = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')^W \nabla(p_j)$, and

$$[\mathrm{Der}\, \mathcal{O}(\mathfrak{h})]^W = \textstyle\bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h})^W \nabla(p_j)$$

is a free $\mathcal{O}(\mathfrak{h})^W$-module.

Proof. (1) Note first that

$$w.\partial_j(p_k) = (w.\partial_j)(w.p_k) = (w.\partial_j)(p_k) = \textstyle\sum_i w_{ij}\partial_i(p_k).$$

Therefore

$$\begin{aligned} w.\nabla(p_k) &= \sum_j w.\partial_j(p_k)\, w.\partial_j = \sum_{i,j,s} w_{ij}\partial_i(p_k)\, w_{sj}\partial_s \\ &= \sum_{i,s}(\sum_j w_{ij}w_{sj})\partial_i(p_k)\partial_s = \sum_{i,s} \delta_{is}\partial_i(p_k)\, \partial_s \\ &= \nabla(p_k). \end{aligned}$$

Hence, $\nabla(p_k)$ is W-invariant. Recall that $\omega_\mathfrak{h} = \frac{1}{2}\sum_i \partial_i^2 \in \mathrm{S}^2(\mathfrak{h})^W$. Note that

$$[\omega_\mathfrak{h}, p_j] = \frac{1}{2}\sum_i [\partial_i^2, p_j] = \nabla(p_j) + \frac{1}{2}\omega_\mathfrak{h}(p_j).$$

Thus, $\nabla(p_j) = [\omega_\mathfrak{h}, p_j] - \frac{1}{2}\omega_\mathfrak{h}(p_j) \in B$.

[7] The elements of $\mathfrak{h}$ are identified with $\mathbb{C}$-linear derivations with constant coefficients on $\mathrm{S}(\mathfrak{h}^*)$, hence $\partial_i = \frac{\partial}{\partial x_i}$.

(2) Denote by $[a_{ij}] \in \mathrm{M}_\ell(\mathcal{O}(\mathfrak{h})_\pi)$ the inverse matrix of $\mathrm{Jac}(p)$. Then, $\pi[a_{ij}] \in \mathrm{M}_\ell(\mathcal{O}(\mathfrak{h}))$ and

$$\sum_m a_{mk}\nabla(p_m) = \sum_i (\sum_m a_{mk}\partial_i(p_m))\partial_i = \sum_i \delta_{ik}\partial_i = \partial_k.$$

Hence, $\mathrm{Der}\,\mathcal{O}(\mathfrak{h}') = \bigoplus_k \mathcal{O}(\mathfrak{h}')\partial_k = \bigoplus_k \mathcal{O}(\mathfrak{h}')\nabla(p_k)$. Observe that we have also shown that

$$\pi\,\mathrm{Der}\,\mathcal{O}(\mathfrak{h}) = \bigoplus_m \mathcal{O}(\mathfrak{h})\nabla(p_m). \tag{5.1}$$

(3) The first claim is consequence of (2) by taking W-invariants. Let $d \in \mathrm{Der}\,\mathcal{O}(\mathfrak{h})^W$. From (5.1), we get that $\pi d = \sum_m \varphi_m \nabla(p_m)$ for some $\varphi_m \in \mathcal{O}(\mathfrak{h})$. Thus, for all $w \in W$,

$$w.(\pi d) = w.\pi\, w.d = \epsilon(w)\pi d = \sum_m w.\varphi_m\nabla(p_m).$$

It follows that $w.\varphi_m = \epsilon(w)\varphi_m$, and therefore $\varphi_m = \pi\gamma_m$ for some $\gamma_m \in \mathcal{O}(\mathfrak{h})^W$. Hence, $d = \sum_j \gamma_j\nabla(p_j) \in \bigoplus_j \mathcal{O}(\mathfrak{h})^W\nabla(p_j)$, as required. □

Recall that, since the elements of $\mathcal{O}(\mathfrak{h})$ act locally nilpotently on $\mathcal{D}(\mathfrak{h})$, we can localize at any Öre subset of $\mathcal{O}(\mathfrak{h})$.

Proposition 5.3. *We have:* $B_{\pi^2} = \mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)]$.

Proof. Recall that

$$\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{D}(\mathfrak{h}'/W) = [\mathcal{D}(\mathfrak{h})_\pi]^W = \mathcal{O}(\mathfrak{h}')^W[\mathrm{Der}\,\mathcal{O}(\mathfrak{h}'/W)].$$

But, since $p : \mathfrak{h}' \twoheadrightarrow \mathfrak{h}'/W$ is étale, we obtain from Lemma 5.2(3) that

$$\mathrm{Der}\,\mathcal{O}(\mathfrak{h}'/W) = [\mathrm{Der}\,\mathcal{O}(\mathfrak{h}')]^W = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')^W\nabla(p_j).$$

Hence, using Lemma 5.2(1),

$$\mathcal{D}(\mathfrak{h})^W_{\pi^2} \subseteq \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)] \subseteq B_{\pi^2}.$$

The other inclusion being obvious, we have the desired equalities. □

We filter $\mathcal{D}(\mathfrak{h})$ and its subspaces by the order of differential operators. In particular, if $B_m = \mathcal{D}_m(\mathfrak{h}) \cap B$, we obtain

$$\mathrm{gr}\,B = \bigoplus B_m/B_{m-1} \hookrightarrow \mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W = \mathcal{O}(\mathfrak{h} \times \mathfrak{h}^*)^W = \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^W \subset \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})$$

where the group W acts diagonally.

Lemma 5.4. *The ring B is a noetherian domain, and $\mathcal{D}(\mathfrak{h})^W$ is a finitely generated (left and right) B-module.*

Proof. Clearly, $B \supseteq \mathrm{S}(\mathfrak{h}^*)^W \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{h})^W = \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^{W\times W}$. It is well known, since the group $W \times W$ is finite, that $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})$ is a finite module over the finitely generated algebra $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^{W\times W}$. It follows easily that $\mathrm{gr}\,B$ is a finitely generated $\mathbb{C}$-algebra and that $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^W$ is a finitely generated $(\mathrm{gr}\,B)$-module. A routine argument then yields the claim. □

Lemma 5.5. *Let $B \subseteq A$ be two noetherian domains. Assume that A is simple and finitely generated as a left or right B-module. Then, if A and B have the same fraction field, we have $A = B$.*

Proof. Set $L = \{b \in B \mid bA \subseteq B\}$. Since A is a finitely generated right B-module, and $\mathrm{Frac}(A) = \mathrm{Frac}(B)$, L is nonzero. Similarly, $L' = \{b \in B \mid Ab \subseteq B\} \neq 0$. Since L' and L are, respectively, left and right ideals of A, $L'L$ is a two-sided ideal of A. But A being a domain, $L'L \neq 0$. Therefore $A = L'L \subseteq B$, and $A = B$ as required. $\square$

Theorem 5.6. *The homomorphism* $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ *is surjective.*

Proof. We apply Lemma 5.5 to $B = \operatorname{Im}\delta \subseteq A = \mathcal{D}(\mathfrak{h})^W$. Recall [10] that A is simple. The theorem then follows from Proposition 5.3 and Lemma 5.4. $\square$

The previous theorem shows that (†) has a positive answer, but does not give the graded surjectivity of δ. In the next sections we will see that question (††) is closely related to geometric questions about the commuting variety of $\mathfrak{g}$. Before going into this interpretation, we have to remark that the graded surjectivity of δ is easy once we have localized at the discriminant[8]. Indeed:

Proposition 5.7. *The map* $\delta : \mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \mathcal{D}(\mathfrak{h})^W_{\pi^2}$ *is graded-surjective.*

Proof. Fix an orthonormal basis of $\mathfrak{g}$ with respect to κ and denote the associated coordinate system on $\mathfrak{g}^* \times \mathfrak{g}$ by $\{x_1, \dots, x_n; \partial_1, \dots, \partial_n\}$. Assume that the numbering is chosen such that $\{x_1, \dots, x_\ell; \partial_1, \dots, \partial_\ell\}$ is the previous coordinate system on $\mathfrak{h}^* \times \mathfrak{h}$.

Define the gradient vector field of $u_j \in \mathcal{O}(\mathfrak{g})^G$, by $\nabla(u_j) = \sum_{k=1}^n \partial(u_k)\partial_k$. Recall that $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W)$. It is easily checked that

$$r(\nabla(u_j)) = \nabla(p_j), \quad j = 1, \dots, \ell.$$

We have seen in Proposition 5.3 that $\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)]$, hence

$$\operatorname{gr}\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \operatorname{gr}\mathcal{D}(\mathfrak{h}')^W = \mathbb{C}[p_1, \dots, p_\ell, \pi^{-2}, \operatorname{gr}(\nabla(p_1)), \dots, \operatorname{gr}(\nabla(p_\ell))].$$

Therefore, with obvious notation,

$$\operatorname{gr}_m \mathcal{D}(\mathfrak{h})^W_{\pi^2} = \textstyle\sum_{|k|=m} \mathbb{C}[p_1, \dots, p_\ell, \pi^{-2}]\nabla(p)^k.$$

Recall that $\delta(P) = \pi r(P)\pi^{-1}$; it follows that $\operatorname{gr}(\delta) = \operatorname{gr}(r)$. Since $\phi(u_j) = p_j$, $\phi(d_\ell) = \pi^2$ and $\operatorname{gr}(\delta)(\nabla(u_j)) = \operatorname{gr}(\nabla(u_j))$, we obtain from the above description of $\operatorname{gr}_m \mathcal{D}(\mathfrak{h})^W_{\pi^2}$ that $\operatorname{gr}(\delta) : \operatorname{gr}\mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \operatorname{gr}\mathcal{D}(\mathfrak{h})^W_{\pi^2}$ is surjective. $\square$

Set $A = \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$. Recall that we can identify $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})^G$ with $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})$ and $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$ respectively. Let $\mathbf{q}$ be the kernel of the graded morphism

$$\operatorname{gr}(\delta) : \operatorname{gr}\mathcal{D}(\mathfrak{g})^G \to \operatorname{gr}\mathcal{D}(\mathfrak{h})^W.$$

Hence, $\operatorname{gr}\mathcal{I} \subseteq \mathbf{q}$ and $\mathbf{q}$ is prime. Since $\operatorname{gr}\mathcal{I}$ and $\mathbf{q}$ are contained in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$, they define affine subvarieties $\mathcal{V}(\mathbf{q}) \subseteq \mathcal{V}(\operatorname{gr}\mathcal{I}) \subseteq (\mathfrak{g} \times \mathfrak{g})/G$.

Corollary 5.8. *One has*[9]*:*

(1) $\operatorname{GKdim}\mathcal{D}(\mathfrak{g})^G = \dim(\mathfrak{g} \times \mathfrak{g})/G = n + \ell - k$;
(2) $\operatorname{GKdim} A = \operatorname{GKdim}\operatorname{gr} A = \operatorname{GKdim}\mathcal{D}(\mathfrak{h})^W = 2\ell$;
(3) $\operatorname{height}(\operatorname{gr}\mathcal{I}) = \operatorname{height}(\mathbf{q}) = n - \ell - k$.

[8] In the rest of this section we do not assume that the surjectivity of δ has been proved.

[9] Recall that $\dim\mathfrak{g} = n$, $\ell = \operatorname{rk}\mathfrak{g} = \dim\mathfrak{h}$, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{t}$ and $k = \dim\mathfrak{t}$ (hence $\dim\mathfrak{z} = \ell - k$). The heights of the ideals in (3) are computed in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$.

Proof. (1) Clearly, if S is the connected semisimple subgroup of G such that $\mathrm{Lie}(S) = \mathfrak{s}$, we have

$$(\mathfrak{g} \times \mathfrak{g})/G \cong ((\mathfrak{s} \times \mathfrak{s})/S) \times (\mathfrak{z} \times \mathfrak{z}).$$

The maximal dimension of an S-orbit in $\mathfrak{s} \times \mathfrak{s}$ is $n - \ell + k$: pick $(x, y) \in \mathfrak{s} \times \mathfrak{s}$, with x generic and y regular nilpotent; then $\mathfrak{s}^x$ is a Cartan subalgebra of $\mathfrak{s}$ and $\mathfrak{s}^y$ is contained in the nilpotent cone of $\mathfrak{s}$. Hence, $\mathfrak{s}^x \cap \mathfrak{s}^y = 0$ and[10] $\dim S.(x, y) = \dim \mathfrak{s} = n - \ell + k$. Therefore, $\dim(\mathfrak{g} \times \mathfrak{g})/G = n - \ell + k + 2(\ell - k) = n + \ell - k$.

(2) From Proposition 5.7, we deduce that there is a filtered isomorphism $A_{d_\ell} \cong \mathcal{D}(\mathfrak{h})^W_{\pi^2}$. The localization at d_ℓ commutes with gr, hence

$$\mathrm{gr}\big(\mathcal{D}(\mathfrak{g})^G_{d_\ell}/\mathcal{J}_{d_\ell}\big) = \mathrm{gr}\, A_{d_\ell} = (\mathrm{gr}\, A)_{d_\ell} \xrightarrow{\sim} \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2}.$$

From $\mathrm{Ker}\big(\mathrm{gr}(\delta) : \mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2}\big) = \mathbf{q}_{d_\ell}$, it follows that $\mathbf{q}_{d_\ell} = (\mathrm{gr}\, \mathcal{J})_{d_\ell}$ and $\big(\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q}\big)_{d_\ell} \cong \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W_{\pi^2}$. Observe that, since $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q}$ is a domain,

$$\mathrm{GKdim}\, \mathrm{gr}\, A_{d_\ell} = \mathrm{GKdim}\big(\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q}\big)_{d_\ell} = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} = 2\ell.$$

Note that d_ℓ is a nonzero divisor in A: $\delta(d_\ell) = \pi^2$ is a nonzero element of the domain $\mathcal{D}(\mathfrak{h})^W$, and $\delta : A \to \mathcal{D}(\mathfrak{h})^W$ is injective by definition of $\mathcal{J}$. Moreover, d_ℓ acts locally ad-nilpotently on A. Therefore, by [6, Lemma 4.7, page 49], $\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_{d_\ell}$. Hence,

$$\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_{d_\ell} = \mathrm{GKdim}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathrm{GKdim}\, \mathcal{D}(\mathfrak{h})^W = 2\ell.$$

Now, by [6, Lemma 6.5, page 75] and the previous remarks,

$$2\ell = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} \le \mathrm{GKdim}\, \mathrm{gr}\, A \le \mathrm{GKdim}\, A = 2\ell.$$

Thus $\mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} = \mathrm{GKdim}\, \mathrm{gr}\, A = \mathrm{GKdim}\, A = 2\ell$.

(3) Since $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$ is a finitely generated domain,

$$\mathrm{height}(\mathrm{gr}\, \mathcal{J}) = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G - \mathrm{GKdim}\, \mathrm{gr}\, A = n + \ell - k - 2\ell = n - \ell - k.$$

Similarly,

$$\mathrm{height}(\mathbf{q}) = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G - \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q} = n - \ell - k$$

as desired. □

Remark . Corollary 5.8(3) shows that $\mathbf{q}$ is a minimal prime ideal over $\mathrm{gr}\, \mathcal{J}$, and that $\dim \mathcal{V}(\mathrm{gr}\, \mathcal{J}) = \dim \mathcal{V}(\mathbf{q}) = 2\ell$.

Corollary 5.9. *The following are equivalent:*

(a) δ is graded-surjective;

(b) δ is surjective and $\mathrm{gr}\, \mathcal{J}$ is a prime ideal.

Proof. (a) $\Rightarrow$ (b) The hypothesis says that $\mathrm{gr}(\delta) : \mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G \twoheadrightarrow \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W$ is surjective. Thus δ is surjective. We have to show that $\mathrm{Ker}\, \mathrm{gr}(\delta) = \mathrm{gr}\, \mathcal{J}$. Let $a \in \mathcal{D}_m(\mathfrak{g})^G$ be such that $0 = \mathrm{gr}(\delta(a)) \in \mathrm{gr}_m\, \mathcal{D}(\mathfrak{h})^W$, i.e. $\delta(a) \in \mathcal{D}_{m-1}(\mathfrak{h})^W$. Since $\mathcal{D}_{m-1}(\mathfrak{h})^W = \delta(\mathcal{D}_{m-1}(\mathfrak{g})^G)$, we obtain $a \in \mathcal{D}_{m-1}(\mathfrak{g})^G + \mathcal{J}$. Hence, $\mathrm{gr}(a) \in \mathrm{gr}\, \mathcal{J}$ as required.

(b) $\Rightarrow$ (a) Since $\mathrm{gr}\, \mathcal{J} = \mathbf{q}$, $\mathrm{gr}(\delta)$ yields an injection: $\mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G/\mathrm{gr}\, \mathcal{J} \hookrightarrow \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W$. Let $b \in \mathcal{D}_m(\mathfrak{h})^W$. Then, $b = \delta(a)$ for some $a \in \mathcal{D}_p(\mathfrak{g})^G$. If $p \le m$ we are done; otherwise, $\mathrm{gr}(\delta)(\mathrm{gr}_p(a)) = \mathrm{gr}_p(b) = 0$. Hence, $\mathrm{gr}_p(a) \in \mathrm{gr}\, \mathcal{J}$ and therefore $a \in \mathcal{J} + \mathcal{D}_{p-1}(\mathfrak{g})^G$. By induction we get that $b = \delta(a')$ for some $a' \in \mathcal{D}_m(\mathfrak{g})^G$, proving the graded surjectivity of δ. □

[10]Since S is semisimple, $\dim(\mathfrak{s} \times \mathfrak{s})/S = 2 \dim \mathfrak{s} - \max\{\dim S.(x, y)\,;\, x, y \in \mathfrak{s}\}$.

6. The commuting variety of $\mathfrak{g}$

The *commuting variety* of $\mathfrak{g}$ is the closed subvariety of $\mathfrak{g} \times \mathfrak{g}$ defined by

$$\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}.$$

Note that $\mathcal{C}(\mathfrak{g})$ is a G-subvariety of $\mathfrak{g} \times \mathfrak{g}$ under the diagonal (adjoint) action of G.

Remark . In general, i.e. for an arbitrary Lie algebra, $\mathcal{C}(\mathfrak{g})$ is not irreducible. Take, for example, the 3-dimensional solvable Lie algebra $\mathfrak{g} = \mathbb{C}u + \mathbb{C}v + \mathbb{C}w$, where the nonzero brackets are

$$[u, v] = v, \qquad [u, w] = w.$$

Let $\{x, y, z\}$ be the dual basis of $\{u, v, w\}$ and set $\mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathbb{C}[x, y, z] \otimes_{\mathbb{C}} \mathbb{C}[x', y', z']$. Then,

$$\mathcal{C}(\mathfrak{g}) = \mathcal{V}(xy' - x'y, xz' - x'z)$$

is 4-dimensional and has two irreducible components

$$\mathcal{V}(x, x') = (\mathbb{C}v + \mathbb{C}w) \times (\mathbb{C}v + \mathbb{C}w), \quad \mathcal{V}(xy' - x'y, xz' - x'z, y'z - yz').$$

But, when $\mathfrak{g}$ is reductive, we have the following result.

Theorem 6.1. [12] *The variety $\mathcal{C}(\mathfrak{g})$ is irreducible. Indeed,*

$$\mathcal{C}(\mathfrak{g}) = \overline{G.(\mathfrak{h} \times \mathfrak{h})}.$$

Remark . The study of $\mathcal{C}(\mathfrak{g})$ reduces easily to the case when $\mathfrak{g}$ is semisimple: Write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{z}$ is the centre, then

$$\mathcal{C}(\mathfrak{g}) = \mathcal{C}(\mathfrak{s}) \times (\mathfrak{z} \times \mathfrak{z}),$$

where we have identified $\mathfrak{g} \times \mathfrak{g}$ with $(\mathfrak{s} \times \mathfrak{s}) \times (\mathfrak{z} \times \mathfrak{z})$. Therefore we shall assume in this section that $\mathfrak{g}$ is semisimple, and that G is the adjoint group.

Denote by $\mathbf{p}$ the prime ideal of $\mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)$ such that $\mathcal{C}(\mathfrak{g}) = \mathcal{V}(\mathbf{p})$. Clearly, $(x, y) \in \mathcal{C}(\mathfrak{g})$ is and only if $\kappa(a, [x, y]) = 0$ for all $a \in \mathfrak{g}$. Let $\sigma_a \in \mathcal{O}(\mathfrak{g} \times \mathfrak{g})$ be the function $(x, y) \mapsto \kappa(a, [x, y])$, and define the ideal

$$\mathbf{a} = (\sigma_a \,;\, a \in \mathfrak{g}).$$

Thus, $\sqrt{\mathbf{a}} = \mathbf{p}$ and $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G) = \mathcal{V}(\mathbf{a}^G)$. The main questions concerning $\mathcal{C}(\mathfrak{g})$ are the following:

- Is $\mathbf{a} = \mathbf{p}$? If true, this would imply that $\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$[11].
- Is $\mathcal{C}(\mathfrak{g})$ normal? Cohen-Macaulay?
- Is $\mathcal{C}(\mathfrak{g})/G$ normal? Cohen-Macaulay? We shall relate the normality of $\mathcal{C}(\mathfrak{g})/G$ to the graded-surjectivity of δ in the next section.

We need to know the dimension of $\mathcal{C}(\mathfrak{g})$; this computation is implicit in [12], for sake of completeness we indicate a proof.

Lemma 6.2. $\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \operatorname{rk} \mathfrak{g}$.

[11] Actually, $\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ has been proved [8]. The equality $\mathbf{a} = \mathbf{p}$ would imply a stronger result: $\operatorname{gr} \mathcal{J} = \operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.

Proof. Let $\eta : \mathfrak{g} \times \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ be the first projection. Since $\eta(\mathcal{C}(\mathfrak{g})) = \mathfrak{g}$, we have a surjective morphism: $\eta : \mathcal{C}(\mathfrak{g}) \twoheadrightarrow \mathfrak{g}$. Note that, for all $u = (x, x') \in \mathcal{C}(\mathfrak{g})$,

$$\eta^{-1}(\eta(u)) = \{(x, y) \,:\, y \in \mathfrak{g}^x\} \cong \mathfrak{g}^x$$

is an irreducible variety.

By a standard result, see [15, Theorem 4.1.6], there exists a non-empty open subset $U \subseteq \mathcal{C}(\mathfrak{g})$ such that, for all $u \in U$,

$$\dim U = \dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \dim \eta^{-1}(\eta(u)).$$

Since $(\mathfrak{g}' \times \mathfrak{g}) \cap \mathcal{C}(\mathfrak{g})$ is a non-empty open subset of $\mathcal{C}(\mathfrak{g})$, we can pick $u = (x, y) \in U$ with $x \in \mathfrak{g}'$. Then $\mathfrak{g}^x$ is a Cartan subalgebra of $\mathfrak{g}$. Hence $\dim \mathcal{C}(\mathfrak{g}) = n + \ell$. □

Again, the situation is easy after localization at the discriminant $d_\ell \in \mathcal{O}(\mathfrak{g}) \equiv \mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{C}} 1 \subset \mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{g})$.

Lemma 6.3. $\mathbf{a}_{d_\ell} = \mathbf{p}_{d_\ell}$.

Proof. Let $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ and $v \in \mathfrak{g}$. The differential of σ_v at (x, y), that we denote by $d\sigma_v(x, y) \in T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g})$, is given by

$$\forall\, (a, b) \in \mathfrak{g} \times \mathfrak{g}, \quad d\sigma_v(x, y)(a, b) = \frac{d}{dt}_{|t=0} \sigma_v(x + ta, y + tb) = \kappa(v, [x, b] + [a, y]).$$

It follows that $d\sigma_v(x, y) = 0$ if, and only if, $v \in ([x, \mathfrak{g}] + [y, \mathfrak{g}])^\perp = \mathfrak{g}^x \cap \mathfrak{g}^y$, where $\perp$ denotes the orthogonal with respect to κ. Therefore, the linear map

$$\vartheta : \mathfrak{g} \to T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g}), \quad v \mapsto d\sigma_v(x, y),$$

has rank $n - \dim(\mathfrak{g}^x \cap \mathfrak{g}^y)$.

Now, suppose that $(x, y) \in \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g})$. Then $y \in \mathfrak{g}^x$ and $\mathfrak{g}^x$ is a Cartan subalgebra of $\mathfrak{g}$. Thus, $\mathfrak{g}^y \supseteq \mathfrak{g}^x$ and $\operatorname{rk} \vartheta = n - \ell$. Let $v_1, \ldots, v_{n-\ell} \in \mathfrak{g}$ be such that $d\sigma_{v_1}(x, y), \ldots, d\sigma_{v_{n-\ell}}(x, y)$ are linearly independent. Denote by $(A, \mathbf{m})$ the local ring of $\mathfrak{g} \times \mathfrak{g}$ at the point (x, y); recall that $T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g}) \equiv \mathbf{m}/\mathbf{m}^2$. Since $(A, \mathbf{m})$ is a regular local ring, the functions $\sigma_{v_1}, \ldots, \sigma_{v_{n-\ell}} \in \mathbf{m}$ can be included in a regular system of parameters. In particular, they generate an ideal of height $n - \ell$ in A. Note that they also belong to $\mathbf{a}_{(x,y)} \subseteq \mathbf{p}_{(x,y)}$, and that $\operatorname{height}(\mathbf{p}_{(x,y)}) = \operatorname{height}(\mathbf{p}) = \operatorname{codim} \mathcal{C}(\mathfrak{g}) = n - \ell$. Hence,

$$(\sigma_{v_1}, \ldots, \sigma_{v_{n-\ell}}) = \mathbf{a}_{(x,y)} = \mathbf{p}_{(x,y)}.$$

Since $(x, y) \in \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g}) = \mathcal{C}(\mathfrak{g})_{d_\ell}$ was arbitrary, we obtain that $\mathbf{a}_{d_\ell} = \mathbf{p}_{d_\ell}$. □

Remark . The proof of Lemma 6.3 shows that, if $(x, y) \in \mathcal{C}(\mathfrak{g})$ and $\dim(\mathfrak{g}^x \cap \mathfrak{g}^y) = \operatorname{rk} \mathfrak{g}$, then (x, y) is a smooth point of $\mathcal{C}(\mathfrak{g})$. Hence, $\mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g})$ is a smooth open subset of $\mathcal{C}(\mathfrak{g})$.

Recall the following theorem:

Theorem 6.4. [13, Theorem 3.2] *Let $(x, y) \in \mathfrak{g} \times \mathfrak{g}$. Then, the orbit $G.(x, y)$ is closed if and only if the algebraic hull of the Lie subalgebra of $\mathfrak{g}$ generated by x and y is reductive in $\mathfrak{g}$.*

Since we are, here, interested in orbits in $\mathcal{C}(\mathfrak{g})$, we will give a proof of Theorem 6.4 in this particular case.

Lemma 6.5. *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$. Then, $G.(x, y)$ is closed if and only if x and y are semisimple.*

Proof. Recall the Jordan-Chevalley decomposition of $x \in \mathfrak{g}$: $x = x_s + x_n$, x_s semisimple, x_n nilpotent, $[x_s, x_n] = 0$. Note that $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ are polynomials in $\operatorname{ad} x$. Thus, $[x, y] = 0$ if and only if $[x_s, y_s] = [x_s, y_n] = [x_n, y_s] = [x_n, y_n] = 0$. Since x_s, y_s are commuting semisimple elements, we may assume (after conjugacy) that $x_s, y_s \in \mathfrak{h}$. Observe that $\mathfrak{k} = \mathfrak{g}^{x_s} \cap \mathfrak{g}^{y_s}$ is a reductive Lie algebra in $\mathfrak{g}$, see §2. Denote by $K \subseteq G$ the adjoint group of $\mathfrak{k}$. Furthermore, $x_n, y_n \in [\mathfrak{k}, \mathfrak{k}]$ are nilpotent, see [4, §3 Remark 9]; since they commute, there exists a maximal nilpotent subalgebra $\mathfrak{u}$ of $\mathfrak{k}$ containing x_n and y_n. Then, it easy to show that there is a one-parameter subgroup, $\lambda : \mathbb{C}^* \to K$, such that $\lim_{t\to 0} \lambda(t).z = 0$ for all $z \in \mathfrak{u}$.

Now assume that $G.(x, y)$ is closed. Then $\lim_{t\to 0} \lambda(t).(x, y) = (x_s, y_s)$, and therefore $(x_s, y_s) \in G.(x, y)$. This shows that x, y are semisimple.

Conversely, assume that $x, y \in \mathfrak{g}$ are commuting semisimple elements. We may suppose (after conjugacy) that $x, y \in \mathfrak{h}$. Thus the stabilizer $G^{(x,y)} = G^x \cap G^y$ contains H. Then [5, III.2.5, Folgerung 3] gives that $G.(x, y)$ is closed. (The proof goes as follows. Let $B = NH$ be a Borel subgroup. Since N is unipotent, $Z = B.(x, y) = N.(x, y)$ is closed. Recall now the well known fact: Let P be a parabolic subgroup of G and Z be a P-stable closed subset of some G-variety V, then, $G.Z$ is closed. (Set $\varphi : G \times V \xrightarrow{\sim} G \times V$, $\varphi((g, v)) = (g, g.v)$, $\eta : G \times V \twoheadrightarrow G/P \times V$, $\eta((g, v)) = (\bar{g}, v)$, and $\varpi : G \times V \twoheadrightarrow V$, $\varpi((g, v)) = v$. Since $\varphi(G \times Z)$ is closed, $\eta(\varphi(G \times Z))$ is closed if and only if $\varphi(G \times Z) = \eta^{-1}(\eta(\varphi(G \times Z)))$, which is clear. Then, since G/P is complete, $G.Z = \varpi(\varphi(G \times Z))$ is closed.)) □

Set $N = N_G(H)$, so that $W = N/H$. We have a natural surjective morphism

$$\mu : \mathcal{X} = G \times_N (\mathfrak{h} \times \mathfrak{h}) \to G.(\mathfrak{h} \times \mathfrak{h}), \quad [g, (h_1, h_2)] \mapsto (g.h_1, g.h_2).$$

By Theorem 6.1, μ induces a dominant morphism from $\mathcal{X}$ to $\mathcal{C}(\mathfrak{g})$. Furthermore, $\dim \mathcal{X} = \dim G + 2 \dim \mathfrak{h} - \dim N = n + \ell$.

Theorem 6.6. *Set $\mathcal{X}' = G \times_N (\mathfrak{h}' \times \mathfrak{h})$ and $\mathcal{S} = \{(x, y) \in \mathcal{C}(\mathfrak{g}) \mid x \in \mathfrak{g}'\}$. Then,*
(1) $\mu : \mathcal{X}' \to \mathcal{S}$ is an isomorphism;
(2) μ is a birational morphism from $\mathcal{X}$ to $\mathcal{C}(\mathfrak{g})$.

Proof. (1) If $x \in \mathfrak{g}'$, x is conjugate to an element of $\mathfrak{h}'$, say $x = g.x_1$. Let $(x, y) \in \mathcal{S}$ and set $y = g.y_1$. Then $[x, y] = [x_1, y_1] = 0$, hence $y_1 \in \mathfrak{g}^{x_1} = \mathfrak{h}$. It follows that $(x, y) = g.(x_1, y_1) \in \mu(\mathcal{X}')$. Hence, $\mu : \mathcal{X}' \to \mathcal{S}$ is surjective. Suppose that $[g, (h_1, h_2)], [g', (h_1', h_2')] \in \mathcal{X}'$ and satisfy $g.h_i = g.h_i'$ for $i = 1, 2$. Then $h_i = g^{-1}g'.h_i'$; in particular, the two generic elements h_1, h_1' are G-conjugate. This implies that h_1 and h_1' are W-conjugate. Indeed, there exists $n \in N$ such that $h_1' = n.h_1$. Therefore $h_1 = g^{-1}g'n.h_1$, forcing $t := g^{-1}g'n \in H$. We obtain that $g^{-1}g' = tn^{-1} \in N$ and

$$[g', (h_1', h_2')] = [gtn^{-1}, (h_1', h_2')] = [g, tn^{-1}(h_1', h_2')] = [g, (h_1, h_2)].$$

This proves that μ restricted to $\mathcal{X}'$ is bijective. We know that $\mathcal{S}$ is contained in the set of smooth points of $\mathcal{C}(\mathfrak{g})$ (see the remark after Lemma 6.3). Therefore $\mu_{|\mathcal{X}'}$ is an isomorphism.

(2) Since $\mathcal{X}'$ and $\mathcal{S}$ are non-empty open subsets of the irreducible varieties $\mathcal{X}$ and $\mathcal{C}(\mathfrak{g})$ respectively, the result follows from (1). □

The previous theorem says that $\mathfrak{h} \times \mathfrak{h}$ is a *rational section* of the action of G on $\mathcal{C}(\mathfrak{g})$, see [11, II.2.5, II.2.8].

The group G acts on $\mathcal{X}$ by left translation and we have a natural isomorphism

$$\mathcal{X}/G \cong (\mathfrak{h} \times \mathfrak{h})/W.$$

The G-equivariant morphism μ then induces $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$. This morphism μ will be called the *Chevalley restriction map*; it is easily seen that μ is given by restriction of functions:

$$\mu : (\mathfrak{h} \times \mathfrak{h})/W \longrightarrow \mathcal{C}(\mathfrak{g})/G; \quad \mu : W.(x, y) \mapsto G.(x, y).$$

The comorphism of μ is

$$\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W, \quad \mu^\sharp(f) = f_{|\mathfrak{h}\times\mathfrak{h}}.$$

Since $\mathcal{C}(\mathfrak{g}) = \overline{G.(\mathfrak{h} \times \mathfrak{h})}$, it is clear that a function f on $\mathcal{C}(\mathfrak{g})$ is determined by its values on $G.(\mathfrak{h} \times \mathfrak{h})$. If f is G-invariant, it is therefore determined by $f_{|\mathfrak{h}\times\mathfrak{h}}$. Hence, $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is injective, i.e. μ is dominant.

The open question is to show that $\mathfrak{h} \times \mathfrak{h}$ is a *Chevalley section* [11, II.3.8], i.e. $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$. The next result shows that μ is at least bijective.

Theorem 6.7. *The morphism $\mu : (\mathfrak{h} \times \mathfrak{h})/W \longrightarrow \mathcal{C}(\mathfrak{g})/G$ is bijective and is the normalization of $\mathcal{C}(\mathfrak{g})/G$.*

Proof. 1. *μ is surjective.* Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ be such that $G.(x, y)$ is closed in $\mathcal{C}(\mathfrak{g})$ (hence in $\mathfrak{g} \times \mathfrak{g}$). Then, by Lemma 6.5, x and y are commuting semisimple elements. Therefore they are contained in a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}$. By conjugacy of the Cartan subalgebras, we can find $g \in G$ such that $g.\mathfrak{h}_1 = \mathfrak{h}$. Thus, $g.(x, y) = (g.x, g.y) \in \mathfrak{h} \times \mathfrak{h}$. This proves the surjectivity of μ.
2. *μ is injective.* Recall the following well-known facts, cf. [4] for example.

(1) If $x \in \mathfrak{g}$ is semisimple, then G^x is a connected reductive subgroup of G.
(2) If $y \in \mathfrak{h}$, then $G.y \cap \mathfrak{h} = W.y$.

We shall denote by $\dot{w} \in N = N_G(H)$ a representative element of $w \in W$. We have to show that: if $x, x', y, y' \in \mathfrak{h}$ are such that $g.x = x', g.y = y'$ for some $g \in G$, then there exists $\dot{u} \in N$ such that $x' = \dot{u}.x, y' = \dot{u}.y$. Since $x' \in G.x \cap \mathfrak{h}$ and $y' \in G.y \cap \mathfrak{h}$, we know from (2) that $x' = \dot{w}_1.x, y' = \dot{w}_2.y$ for some $w_1, w_2 \in W$. Set $y'' = \dot{w}_1^{-1}.y', g' = \dot{w}_1^{-1}g$. We have $g'.x = x, g'.y = y''$; thus, $G.(x, y) = (x, y'')$. Therefore, it is enough to show that there exists $w \in W^x$ such that $y'' = w.y$. Indeed, $y'' = w.y$ implies $y' = \dot{w}_1\dot{w}.y$ and we have $x' = \dot{w}_1\dot{w}.x$. Thus, the result follows by setting $\dot{u} = \dot{w}_1\dot{w}$.

Therefore we may, and we do, assume that $x = x'$, $g \in G^x$, $y' = g.y \in \mathfrak{h}$. The proof of the injectivity of μ reduces then to show that

$$x,\, y \in \mathfrak{h} \implies G^x.y \cap \mathfrak{h} = W^x.y.$$

Observe that $H \subset G^x$. Therefore $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{g}^x$. Futhermore, cf. (1), G^x is a connected reductive subgroup of G. Since $H \subseteq N \cap G^x = N_{G^x}(H)$, the Weyl group of G^x is W^x (with respect to the choice of the Cartan $\mathfrak{h}$). Now, use (2) in the connected reductive group G^x to get $G^x.y \cap \mathfrak{h} = W^x.y$.

By [2, Theorem 4.6], μ is then birational. Since $(\mathfrak{h} \times \mathfrak{h})/W$ is a normal variety, the result follows. □

Remark . The fact that μ is the normalization of $\mathcal{C}(\mathfrak{g})$ is a corollary of [9, Proposition 2.2][12]. Recall [9, Lemme 1.8] that, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a surjective birational morphism between affine irreducible varieties, and if $\mathcal{Y}$ is normal, then φ is an isomorphism. Therefore, the open problem of whether $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism, is equivalent to showing that $\mathcal{C}(\mathfrak{g})/G$ is normal, cf. Corollary 7.2.

[12] Apply this proposition to $M = \mathfrak{g} \times \mathfrak{g}$.

7. Graded-surjectivity of δ

We begin with a preliminary remark. Recall that the map $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ is equal to $\iota \circ r$, where $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}')^W$ is the "radial component" map. We noticed that, when we restrict to the generic elements,

$$\operatorname{gr}(\delta) = \operatorname{gr}(r) : \mathcal{O}(\mathfrak{g}' \times \mathfrak{g})^G = \mathcal{O}((\mathfrak{g}' \times \mathfrak{g})/G) \longrightarrow \mathcal{O}(\mathfrak{h}' \times \mathfrak{h})^W = \mathcal{O}((\mathfrak{h}' \times \mathfrak{h})/W).$$

From the definition of r, it is immediate that $\operatorname{gr}(r)$ is induced by restriction of functions: $\operatorname{gr}(r)(f) = f_{|\mathfrak{h}' \times \mathfrak{h}}$. Since $(\mathfrak{g}' \times \mathfrak{g})/G$ is open and dense in $(\mathfrak{g} \times \mathfrak{g})/G$, it follows that $\operatorname{gr}(\delta)$ is also given by restriction of functions.

Proposition 7.1. *With the notation of §6, we have*

(1) $\mathbf{q} = \mathbf{p}^G$ and $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\operatorname{gr} \mathcal{J})$;

(2) $\mathbf{p}^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$ and $\mathcal{J}_{d_\ell} = \big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G_{d_\ell}$.

Proof. (1) Note first that, since $\mathcal{V}(\mathbf{a}) = \mathcal{C}(\mathfrak{g})$, $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G) = \mathcal{V}(\mathbf{a}^G)$. Moreover, $\mathbf{a} \subseteq \operatorname{gr} \mathcal{J}$ implies $\mathbf{a}^G \subset (\operatorname{gr} \mathcal{J})^G = \operatorname{gr} \mathcal{J}$. Therefore,

$$\mathcal{V}(\mathbf{q}) \subseteq \mathcal{V}(\operatorname{gr} \mathcal{J}) \subseteq \mathcal{V}(\mathbf{a}^G) = \mathcal{C}(\mathfrak{g})/G.$$

By Corollary 5.8 and Theorem 6.7, $\dim \mathcal{C}(\mathfrak{g})/G = \dim \mathcal{V}(\mathbf{q}) = 2\ell$. Hence,

$$\mathcal{V}(\mathbf{q}) = \mathcal{V}(\operatorname{gr} \mathcal{J}) = \mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G).$$

This proves the claims.

(2) We have seen (cf. the proof of Corollary 5.8) that $\mathbf{q}_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Thus, the first assertion follows from (1). By Lemma 6.3, $\mathbf{a}^G_{d_\ell} = \mathbf{p}^G_{d_\ell}$ (recall that d_ℓ is invariant) and therefore, $\mathbf{a}^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Since $\mathbf{a}^G \subseteq \big(\operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G \subseteq \operatorname{gr} \mathcal{J}$, we obtain the equality $\big(\operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Hence $\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G_{d_\ell} = \mathcal{J}_{d_\ell}$. □

Corollary 7.2. *The following are equivalent:*

(a) δ is graded-surjective;

(b) $\mathcal{C}(\mathfrak{g})/G$ is a normal variety;

(c) the Chevalley restriction map $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism, i.e. $\mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$.

Proof. By Proposition 7.1 and the preliminary remark, the comorphism of $\operatorname{gr}(\delta) : \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G / \operatorname{gr} \mathcal{J} \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is the map, $(\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{V}(\operatorname{gr} \mathcal{J}) = \mathcal{C}(\mathfrak{g})/G$, induced by restriction of functions.

(b) $\Leftrightarrow$ (c) is consequence of Theorem 6.7.

(a) $\Rightarrow$ (c) If δ is graded-surjective, then $\operatorname{gr} \mathcal{J} = \mathbf{q} = \mathbf{p}^G$ by Corollary 5.9. Hence, $\operatorname{gr}(\delta) : \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{p}^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ and (c) follows.

(c) $\Rightarrow$ (a) If the Chevalley restriction map is an isomorphism, we deduce that $\operatorname{gr}(\delta)$ gives an isomorphism $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{p}^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$. In particular, $\operatorname{gr}(\delta)$ is surjective. Hence the result. □

The (equivalent) conditions of Corollary 7.2 hold when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. As explained in [1], this follows from the fact that, in this case, $\mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is well understood. Recall that when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, one can choose $\mathfrak{h} = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathrm{M}_n(\mathbb{C})\}$ as a Cartan subalgebra. Then, the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ identifies with the symmetric group $\mathfrak{S}_n$ acting on $\mathfrak{h}$ by permutation of the entries:

$$w.\operatorname{diag}(\lambda_1, \dots, \lambda_n) = \operatorname{diag}(\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)}).$$

Set $\mathcal{O}(\mathfrak{h}\times\mathfrak{h}) = \mathbb{C}[X_1,\dots,X_n]\otimes_{\mathbb{C}}\mathbb{C}[Y_1,\dots,Y_n]$. Thus W acts on $\mathcal{O}(\mathfrak{h}\times\mathfrak{h})$ by $w.X_j = X_{w(j)}$, $w.Y_j = Y_{w(j)}$.

For every $r,s\in\mathbb{N}$, define the "polarized power sums" $p_{r,s}\in\mathcal{O}(\mathfrak{h}\times\mathfrak{h})$ by

$$p_{r,s} = \sum_{i=1}^{n} X_i^r Y_i^s.$$

Clearly, $p_{r,s}$ is W-invariant. One has the following result, due to H. Weyl:

Theorem 7.3. [18] *$\mathcal{O}(\mathfrak{h}\times\mathfrak{h})^W$ is generated by the polynomials $p_{r,s}$.*

Corollary 7.4. *Assume that $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Then, the Chevalley restriction map $\mu : (\mathfrak{h}\times\mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism.*

Proof. We have already noticed in §6 that $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h}\times\mathfrak{h})^W$ is injective. It remains to show that $\mu^\sharp$ is surjective. By Theorem 7.3, this is equivalent to showing that $p_{r,s}\in\operatorname{Im}\mu^\sharp$. Consider the polynomial function $u_{r,s}$ on $\mathfrak{g}\times\mathfrak{g}$ defined by

$$u_{r,s}(x,y) = \operatorname{tr}(x^r y^s).$$

Then, $u_{r,s}$ is G-invariant and induces a function $u_{r,s}\in\mathcal{O}(\mathcal{C}(\mathfrak{g}))^G$. Obviously, $u_{r,s|\mathfrak{h}\times\mathfrak{h}} = p_{r,s}$; hence the result. □

Remark . When $\mathfrak{g}$ is of type B_n or G_2, then Theorem 7.3 has an analog and the same proof yields Corollary 7.4. For $\mathfrak{g}$ of type D_n and F_4, Theorem 7.3 fails, but Wallach [17] has shown that Corollary 7.4 is true. Therefore, it remains to investigate the types $\mathsf{E}_6,\mathsf{E}_7$ and E_8.

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