Introduction to G-Expectation, G-Brownian motion and G-Backward SDEs

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The aim of the talk

- *Introduction to G-stochastic analysis theory (S. Peng).*
- *Part of my work in this research field.*

1. Non linear expectations: a general framework

Setting

$$C_{l.Lip}(\mathbb{R}^n) := \{\varphi : \mathbb{R}^n \mapsto \mathbb{R} \mid \exists C_\varphi \in \mathbb{R}^+, m \in \mathbb{N}\ s.t.\ |\varphi(x) - \varphi(y)| \leq C_\varphi(1 + |x|^m + |y|^m)|x - y|\}$$

$$Lip_b(\mathbb{R}^n) := \{f : \mathbb{R}^n \mapsto \mathbb{R} \mid \exists M_f, C_f \in \mathbb{R}^+\ s.t.\ |f(x)| \leq M_f, |f(x) - f(y)| \leq C_f|x - y|\}$$

Let $\Omega = \mathbb{R}$ *and let* $\mathscr{H}$ *be a linear space of real functions s.t.*

$$X_1, \cdots, X_n \in \mathscr{H} \Rightarrow \varphi(X_1, \cdots, X_n) \in \mathscr{H}, \forall \varphi \in C_{l.Lip}(\mathbb{R}^n).$$

1. Non linear expectations: a general framework

Definition

A non linear expectation $\hat{\mathbb{E}}$ is a functional $\mathscr{H} \mapsto \mathbb{R}$ satisfying the following properties:

$$\bullet \textit{Monotonicity}: X, Y \in \mathscr{H} \text{ and } X \geq Y \Rightarrow \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y].$$
$$\bullet \textit{Preservation of constants}: \hat{\mathbb{E}}[c] = c, \ \forall c \in \mathbb{R}.$$

A sub linear expectation in addition satisfies:

$$\bullet \textit{Subadditivity}: \ \hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y], \forall X, Y \in \mathscr{H}.$$
$$\bullet \textit{Positive homogeneity}: \hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \forall \lambda \geq 0, X \in \mathscr{H}.$$

Remark

Norm on $\mathscr{H}$: $\|X\| := \hat{\mathbb{E}}[|X|]$, $X \in \mathscr{H}$.

$(\mathscr{H}, \|\cdot\|)$ is a normed space.

We denote its completion by $([\mathscr{H}], \|\cdot\|)$, or simply $[\mathscr{H}]$.

Let $0 \leq {\sigma_1}^2 \leq \sigma_2^2$, $X \in \mathscr{H}$; $G(a) = \frac{1}{2}(\sigma_2^2 a^+ - \sigma_1^2 a^-)$, $\forall a \in \mathbb{R}$.

Classical case

$\sigma^2 = \sigma_1^2 = \sigma_2^2$.

$X \sim \mathcal{N}(0, \sigma^2)$ *can be characterized by*

$u(t,x) = \mathbb{E}[\varphi(x + \sqrt{t}X)]$, *for* $\varphi \in C_{l.Lip}(\mathbb{R})$, *which satisfies*

$$\partial_t u(t,x) = \frac{1}{2}\sigma^2 \partial_{xx}^2 u(t,x),\ u(0,x) = \varphi(x).$$

G-case (Definition)

$X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$ *is characterized by*

$u(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, *for* $\varphi \in C_{l.Lip}(\mathbb{R})$,

which is the unique viscosity solution of

$$\partial_t u - G(\partial_{xx}^2 u) = 0,\ u(0,x) = \varphi(x).$$

Proposition

If $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$, *we have the following property:*

- *For each convex* $\varphi \in C_{l.Lip}(\mathbb{R})$,

$$u(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)] = \frac{1}{2\sqrt{\pi t}\sigma_2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_2{}^2 t}} \varphi(x) \mathrm{d}x;$$

- *For each concave* $\varphi \in C_{l.Lip}(\mathbb{R})$, *for* $\sigma_1 > 0$,

$$u(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)] = \frac{1}{2\sqrt{\pi t}\sigma_1} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_1{}^2 t}} \varphi(x) \mathrm{d}x.$$

Definition

A random vector $Y = (Y_1, \cdots, Y_n), Y_i \in \mathscr{H}$, is said to be *independent* of $X = (X_1, \cdots, X_m), X_i \in \mathscr{H}$, if for each $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$,

$$\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}].$$

Remark

Unlike the classical case, the independence under $\hat{\mathbb{E}}$ is not symmetric.

Example

Let $X,\ Y \in \mathscr{H}$ be s.t. $\sigma_2^2 := \hat{\mathbb{E}}[Y^2] > \sigma_1^2 := -\hat{\mathbb{E}}[-Y^2] > 0$, $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0$, and $\hat{\mathbb{E}}[|X|] > 0$. Then if Y is independent of X:

$$\begin{aligned} \hat{\mathbb{E}}[XY^2] &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[xY^2]_{x=X}] = \hat{\mathbb{E}}[(x^+\hat{\mathbb{E}}[Y^2] + x^-\hat{\mathbb{E}}[-Y^2])_{x=X}] \\ &= \hat{\mathbb{E}}[X^+\sigma_2^2 - X^-\sigma_1^2] = \hat{\mathbb{E}}[X^+](\sigma_2^2 - \sigma_1^2) > 0. \end{aligned}$$

But if X is independent of Y :

$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[Xy^2]_{y=Y}] = \hat{\mathbb{E}}[(y^2\hat{\mathbb{E}}[X])_{y=Y}] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[X]Y^2] = 0.$$

Definition

X, Y are said to be *identically distributed* ($X \sim Y$, or X is a copy of Y), if $\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)], \forall \varphi \in C_{l.Lip}(\mathbb{R})$.

Definition

A sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ is said to *converge in law* to $\eta \in \mathscr{H}$ under $\hat{\mathbb{E}}$ if

$$\lim_{i\to\infty} \hat{\mathbb{E}}[\varphi(\eta_i)] = \hat{\mathbb{E}}[\varphi(\eta)],$$

for each $\varphi \in Lip_b(\mathbb{R})$.

Central Limit Theorem (Peng)

Let $\{X_i\}_{i=1}^{\infty}$ *in* $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ *satisfy* $X_i \sim X_1$, *each* X_{i+1} *is independent of* $(X_1, \cdots, X_i)$,

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty, \text{ for some } \alpha > 0, \text{ and } \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

$S_n := X_1 + \cdots + X_n$. *Then* $S_n/\sqrt{n}$ *converges in law to* $\mathscr{N}(0; [\sigma_1^2, \sigma_2^2])$*:*

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(S_n/\sqrt{n})] = \hat{\mathbb{E}}[\varphi(X)], \forall \varphi \in Lip_b(\mathbb{R}),$$

where $X \sim \mathscr{N}(0, [\sigma_1^2, \sigma_2^2]), \sigma_1^2 = -\hat{\mathbb{E}}[-X_1^2], \sigma_2^2 = \hat{\mathbb{E}}[X_1^2]$.

Setting

Let $\Omega = C_0(\mathbb{R}^+)$ *the space of continuous paths* $(\omega_t)_{t\in\mathbb{R}^+}$ *with* $\omega_0 = 0$*; distance on* Ω*:*

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

Coordinate process on Ω*:* $B_t(\omega) = \omega_t$*,* $t \geq 0$*,* $\omega \in \Omega$*.*
For $t \in [0, +\infty)$*, we set*

$$\begin{aligned}\mathscr{H}_t : &= \{\varphi(\omega_{t_1}, \cdots, \omega_{t_n}) : \forall n \in \mathbb{N}, t_1, \cdots, t_n \in [0, t], \\ &\quad \forall\ \varphi \in C_{l.Lip}(\mathbb{R}^n)\};\end{aligned}$$

$$\mathscr{H} := \bigcup_{n=1}^{\infty} \mathscr{H}_n.$$

Definition

The coordinate process B on $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ is a *G-Brownian motion* if:

1. $B_0 = 0$;
2. $B_{t+s} - B_t$ is $\mathscr{N}(0; [\sigma_1^2 s, \sigma_2^2 s])$ - distributed, $\forall t, s \geq 0$;
3. For each $n \geq 2$, $0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent of $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$.

Theorem (Peng)

Let $B_t(\omega)_{t\geq 0}$ *be a process on* $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ *such that*

1. *For each* $0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ *is independent of* $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$;
2. B_t *has the same distribution as* $B_{t+s} - B_s$, $\forall t, s \geq 0$;
3. $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$,

then B *is a* G*-Brownian motion, for*

$$\sigma_1^2 = -\hat{\mathbb{E}}[-B_1^2],\ \sigma_2^2 = \hat{\mathbb{E}}[B_1^2].$$

For what follows: for simplicity, we choose $\sigma_2^2 = 1$.

Definition

The related *conditional expectation* of $X=\varphi(B_{t_1},B_{t_2}-B_{t_1},\cdots,B_{t_m}-B_{t_{m-1}})$ under $\mathscr{H}_{t_j}$ is defined by

$$\hat{\mathbb{E}}[X|\mathscr{H}_{t_j}]=\psi(B_{t_1},B_{t_2}-B_{t_1},\cdots,B_{t_j}-B_{t_{j-1}}),$$

where $\psi(x_1,\cdots,x_j)=\hat{\mathbb{E}}[\varphi(x_1,\cdots,x_j,B_{t_{j+1}}-B_{t_j},\cdots,B_{t_m}-B_{t_{m-1}})]$.

Definition

A process $(M_t)_{t\geq 0}$ is called a *G-martingale* **(resp.** *G-submartingale*, *G-supermartingale***)**, if $\forall t\in[0,+\infty)$, $M_t\in\mathscr{H}_t$ and for $s\in[0,t]$,

$$\hat{\mathbb{E}}[M_t|\mathscr{H}_s]=M_s\,(\text{resp. } \leq M_s,\ \geq M_s)\ a.s. \text{ in } \mathscr{H}.$$

Setting

$$L_G^2(\mathscr{H}_t) := \{\xi \in \mathscr{H}_t : \hat{\mathbb{E}}[|\xi|^2] < \infty\}.$$

$$L_G^{2,0}(0,T) := \{\eta(\omega) | \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t),$$

$$\text{where } \xi_j \in L_G^2(\mathscr{H}_{t_j}), 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

Definition

For each $(\eta_t)_{0 \le t \le T} \in L_G^{2,0}(0,T)$, we define its Itô's integral as

$$I(\eta) = \int_0^T \eta(s) \mathrm{d}B_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Lemma

We have

$$\hat{\mathbb{E}}[\int_0^T \eta(s)\mathrm{d}B_s] = 0,$$

and

$$\hat{\mathbb{E}}[(\int_0^T \eta(s)\mathrm{d}B_s)^2] \leq \int_0^T \hat{\mathbb{E}}[(\eta_s)^2]\mathrm{d}s.$$

We denote by $L_G^2(0,T)$ the completion of $L_G^{2,0}(0,T)$ under the norm

$$\|\eta\|_{L_G^2(0,T)} = \{\int_0^T \hat{\mathbb{E}}[|\eta_t|^2]\mathrm{d}t\}^{1/2}.$$

Hence the stochastic integral can be extended to $L_G^2(0,T)$.

4.Itô's integral of G-Brownian motion

We denote: $\langle B\rangle_t = B_t^2 - 2\int_0^t B_s \mathrm{d}B_s$, then

$$\hat{\mathbb{E}}[|\langle B\rangle_t - \sum_{k=0}^{N-1}(B_{t_{k+1}} - B_{t_k})^2|^2] \to 0,\ N \to +\infty.$$

$\langle B\rangle$ is an increasing process; it is called *the quadratic variation of B*.

Proposition

$$\hat{\mathbb{E}}[\langle B\rangle_t] = t,\ \textit{but}\ \hat{\mathbb{E}}[-\langle B\rangle_t] = -\sigma_1^2 t.$$

$$\hat{\mathbb{E}}[(\int_0^T h(s)\mathrm{d}B_s)^2] = \hat{\mathbb{E}}[\int_0^T h^2(s)\mathrm{d}\langle B\rangle_s], h \in L_G^2(0,T).$$

Consider

$$X_t = X_0 + \int_0^t \alpha_s \mathrm{d}s + \int_0^t \eta_s \mathrm{d}\langle B \rangle_s + \int_0^t \beta_s \mathrm{d}B_s.$$

Itô's formula (Peng)

Let α, β, η be bounded processes of $L_G^2(0,T)$. Then for each $t \geq s \geq 0$ and $\Phi(X_t) \in L_G^2(\mathscr{H}_t)$, we have

$$\begin{aligned}\Phi(X_t) = \Phi(X_s) \quad &+ \quad \int_s^t \Phi_x(X_u)\beta_u \mathrm{d}B_u + \int_s^t \Phi_x(X_u)\alpha_u \mathrm{d}u \\ &+ \quad \int_s^t [\Phi_x(X_u)\eta_u + \frac{1}{2}\Phi_{xx}(X_u)\beta_u^2]\mathrm{d}\langle B \rangle_u.\end{aligned}$$

Now we can consider the following G-SDE:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t h(X_s)\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(X_s)\mathrm{d}B_s, t > 0.$$

where $X_0 \in \mathbb{R}$, $b, h, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions.

Theorem (Peng)

There exists a unique solution $X \in L_G^2(0,T)$ of the above G-SDE.

5. SDE and BSDE driven by a G-Brownian motion

Recall

Classical BSDE (under a linear expectation):

$$\begin{cases} \mathrm{d}Y_t = -f(t, Y_t, Z_t)\mathrm{d}t + Z_t \mathrm{d}B_t, \\ Y_T = \xi \in L^2(\Omega, \mathscr{F}_T, P); \end{cases}$$

solution $(Y_t, Z_t)_{0 \le t \le T}$; existence and uniqueness for f Lipschitz in (y, z) and linear growth. For $f \equiv 0$: the martingale representation theorem.

Remark

No martingale representation theorem in G-theory up to now!

Unlike the classical case, we only consider the G-BSDE of the following type:

$$Y_t = \hat{\mathbb{E}}[\xi + \int_t^T f(s, Y_s)\mathrm{d}s | \mathscr{H}_t], \quad t \in [0, T],$$

where $\xi \in L_G^2(\mathscr{H}_T)$. $f(t,y) \in L_G^2(0,T), y \in \mathbb{R}$, is a given Lipschitz function with respect to y.

Theorem (Peng)

There exists a unique solution $(Y_t)_{t\in[0,T]} \in L_G^2(0,T)$.

$$Y_t^i = \hat{\mathbb{E}}[\xi^i + \int_t^T f_i(s, Y_s^i)\mathrm{d}s|\mathscr{H}_t], \quad t \in [0,T],$$

where $\xi^i \in L_G^2(\mathscr{H}_T), f_i(t,y) \in L_G^2(0,T), y \in \mathbb{R}$, for $i = 1,2$.

Comparison Theorem (Jing)

Suppose that $\xi^1 \geq \xi^2, f_1(t,y) \geq f_2(t,y), \forall (t,y), f_1 \uparrow$ or $f_2 \downarrow$ in y, both f_1 and f_2 are Lipschitz in y and bounded, then $Y_t^1 \geq Y_t^2$, $\forall t \in [0,T]$.

6.A property of G-martingales

Proposition

Let $x \in \mathbb{R}$, $Z \in L_G^2(0,T)$, *and* $\eta \in L_G^2(0,T)$, *then the process*

$$M_t = x + \int_0^t Z_s \mathrm{d}B_s + \int_0^t \eta_s \mathrm{d}\langle B \rangle_s - \int_0^t 2G(\eta_s)\mathrm{d}s$$

is a G*-martingale.*

Recall

If $\sigma_1 = 1$, $\int_0^t \eta_s \mathrm{d}\langle B \rangle_s - \int_0^t 2G(\eta_s)\mathrm{d}s = 0$.

Question

Is the stochastic integral w. r. t. this G*-martingale still a* G*-martingale?*

6.A property of G-martingales

Remark

$\{M_t\}$ being a G-martingale does not always imply that $\{-M_t\}$ is a G-martingale.

Example

$\{B_t\}$ and $\{-B_t\}$ are both G-martingales.
$\{\langle B\rangle_t - t\}$ is a G-martingale, but $\{-\langle B\rangle_t + t\}$ is not a G-martingale.

Theorem (Jing)

Let $0 \leq \sigma_1 < 1$ *and* $\xi \in L_G^2(0,T)$. *Put*

$$M_t = x + \int_0^t Z_s \mathrm{d}B_s + \int_0^t \eta_s \mathrm{d}\langle B\rangle_s - \int_0^t 2G(\eta_s)\mathrm{d}s,$$

and

$$N_t = \int_0^t \xi_s \mathrm{d}M_s, \quad t \geq 0.$$

Then $\{N_t\}$ *is a* G*-martingale w. r. t.* $\{\mathscr{H}_t\}$ *if and only if*

$$\begin{cases} \text{sgn}(\xi_s) \geq 0, & \text{on } \{s : \eta_s \neq 0\}; \\ \text{sgn}(\xi_s) \text{ can be arbitrary}, & \text{on } \{s : \eta_s = 0\}. \end{cases}$$

This talk is mainly based on

• Peng, S. (2006) *G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô's type*, Proceedings of 2005 Abel Symposium, Springer

• Peng, S. (2008) *Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation*, Stochastic Processes and their Applications, Vol. 118, Issue 12, Dec. 2008, 2223-2253

• Peng, S. (2008) *A New Central Limit Theorem under Sublinear Expectations*, arXiv:0803.2656v1 [math.PR] 18 Mar 2008

Thank you!

谢谢！