THE POTENTIAL POINT OF VIEW FOR RENORMALIZATION

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Abstract. We study the renormalization for potentials defined by

$$\mathcal{R}(V) := V \circ T \circ H + V \circ H,$$

where $T : X \circlearrowleft$ is the dynamics, $H : X \to X$ is one-to-one and $V : X \to \mathbb{R}$ is a potential. We explain how this operator is obtained from the usual renormalization operator for maps and why it has a fixed point.

For the Manneville-Pomeau map, $f : [0,1] \circlearrowleft$, close to the fixed and indifferent point 0 we have, $H(x) = \frac{x}{2}$ and V*:=log f' is a fixed point for $\mathcal{R}$. We are interested in characterizing potentials V such that $\mathcal{R}^n(V)$ converges to $\log f'$. We recover here the importance of the germ close to the fixed indifferent point.

For the shift σ in $\Sigma = \{0,1\}^{\mathbb{N}}$ we prove that under mild assumptions there exists a unique kind of H. Consequently, there is a unique kind of fixed potentials for $\mathcal{R}$. These are the" Hofbauer-like" potentials.

In the last part, we construct a two-parameters family of potentials defined on Σ related to this renormalization procedure. We show they are less regular than the class $R(X)$ introduced in [35]. We study the thermodynamic formalism for these potentials and exhibit phase transitions.

1. Introduction

1.1. **General presentation.** Renormalization can have in general different meanings in Mathematics or in Physics. From the mathematical point of view, it is usually associated to the period doubling renormalization operator as introduced by M. Feigenbaum and by P. Coullet and C. Tresser (see [6] [10] [11] [30] [31] [14]). We recall that for $f : [0,1] \circlearrowleft$, the renormalization of f is defined by

$$\widetilde{\mathcal{R}}(f)(x) = h^{-1} \circ f^2 \circ h(x), \tag{1}$$

where h is an affine map defined on $[0,1]$. This defines an operator $\widetilde{\mathcal{R}}$ acting on dynamical transformations f and the point is to study the hyperbolicity of $\widetilde{\mathcal{R}}$ at fixed points. As far as we know, renormalization from the mathematical point of view essentially studies class of one-dimensional dynamical systems which are associated to critical points (see [17] [21] [13] [38] [25] [24]). Nevertheless, we point out here that, what is usually called renormalization in the mathematical setting is mathematical rigorous (of course), but not related— in our point of view— to the real physical meaning of the term.

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The Renormalization in the physical world is associated to phase transitions and polynomial decay of correlation (see [12], [16] [18],[36], [37]). As far as we know, using the mathematical vocabulary, this is associated to "indifferent fixed points" and not to "critical points". Moreover, it acts on potentials and not on maps (see for instance [12], [8] or [15]). Renormalization in Physics is also sometimes presented as a way to rescale the action of a potential.

We want here to explore this "potential" point of view (and associated to indifferent fixed points) of the renormalization. Our main and first motivation is chapter 5 of the book [26], where the renormalization is associated to transformations with a weakly expanding fixed point. There, the reasoning of several of the results which are presented are difficult to be understood from the pure mathematical point of view.

In a first step (and following [26]) we define the renormalization operator $\mathcal{R}$ for potentials and for a given Manneville-Pomeau-like map f. By construction, if f satisfies (1), the potential $\log f'$ is a fixed point for $\mathcal{R}$. Then, we are interested in studying the "stable set" of this fixed point: what are the potentials V such that $\mathcal{R}^n(V)$ converges to $\log f'$.

Our first result (Theorem A) recovers the importance of the germ of these dynamics close to the indifferent fixed points: only the germ of the potential determines if $\mathcal{R}^n(V)$ converges or not to the fixed point.

The Mannevile-Pomeau-like maps are (semi)-conjugated to the full 2-shift $\Sigma := \{0,1\}^{\mathbb{N}}$. We are thus naturally led to study the renormalization operator for potentials in Σ. This case is closer to the problems which arise in Statistical Mechanics of the one-dimensional lattice (see [12] [36]): one can see 0 as + (positive spin) and 1 as − (negative spin).

Moreover, we point out that, if we may consider several dynamics (see *e.g.* the f_t's with different t below) in the interval $[0,1]$, this is not the case in Σ. This means that in Σ, the potential point of view has an even more important place.

Indeed, our second result (Theorem B) explains why the potential of Hofbauer in the shift is so important for Statistical Mechanics (see [13][21]).

In the last part, we are interested in thermodynamic formalism in Σ for "less regular" potentials. Sufficient conditions yielding existence and uniqueness of *equilibrium state* in Σ are well-known: following notations from [35], it is sufficient that the potential is Hölder continue, or satisfies the Walters condition (hence belongs to $W(X,T)$) or the Bowen's condition (hence belongs to $Bow(X,T)$) to ensure existence and uniqueness of the equilibrium state (see [33, 34]). Nevertheless, these conditions are only sufficient conditions, and for several reasons people are now interested in studying the thermodynamic formalism for less regular potentials. One of these reasons is to exhibit phase transition.

In [35], P. Walters defined a new class of potentials for which mains properties of the thermodynamic formalism holds. This class contains the Hofbauer potential, and more generally all the Hofbauer-like potentials that arise from our Theorem B. It is known that they have phase transition. Nevertheless, potentials defined in [35] must satisfy a quite strong condition regarding the dynamics and the renormalization. We explain better this point later.

With this background, our last result (Theorem C) deals with thermodynamic formalism for potentials which do not satisfy the strong condition on [35].

1.2. **Statement of results.** Consider for $t > 0$,

$$\begin{cases} f_t(x) = \dfrac{x}{(1-x^t)^{1/t}} = (\dfrac{x^t}{1-x^t})^{1/t}, \text{ if } 0 \le x \le \dfrac{1}{2^{1/t}}, \\ f_t(x) = (2 - \dfrac{1}{x^t})^{1/t}, \text{ if } \dfrac{1}{2^{1/t}} < x \le 1, \end{cases}$$

For this map the fixed points are $x = 0$ and $x = 1$. In these points the derivative is equal to one. For any $x \neq 0, 1$, $f_t'(x) > 1$. These maps are thus weakly expanding.

Definition 1.1. *The renormalization operator $\mathcal{R}_t$ acts on the set of potentials V by means of*

$$\mathcal{R}_t(V)\,(x) = V(\,f_t(\frac{x}{2^{1/t}})\,) + V(\frac{x}{2^{1/t}}).$$

Our definition of the renormalization operator $\mathcal{R}_t$ is based on the following observation. Let us set $h(x) = \frac{x}{2^{1/t}}$. Then, taking derivative in (1), and keeping in mind that h is affine, we get

$$f_t'(f_t \circ h(x)) f_t' \circ h(x) = f_t'(x).$$

Then, taking the logarithm in this last equation and setting $V_t^*(x) := \log f_t'(x)$, we finally get

$$V_t^*(f(h(x))) + V_t^*(h(x)) = V_t^*(x). \tag{2}$$

Definition 1.2. *For a given value $t > 0$ we denote $\mathcal{F}_t$ the set of non-negative continuous functions $V : [0,1] \to \mathbb{R}$ such that $V(x) = (1 + \dfrac{1}{t})x^t + o(x^t)$.*

For this setting, we have the following result on the action of the renormalization operator:

Theorem A. *For any $V \in \mathcal{F}_t$ and for every x in $[0, \frac{1}{2^{1/t}}[$ we have*

$$\lim_{n\to\infty} {\mathcal{R}_t}^n(V)(x) = V_t^*(x).$$

Remark 1. Note that $\mathcal{R}_t$ is linear, hence for every λ, λV_t^* is also a fixed point. Moreover, if $V(x) = \lambda x^t + o(x)$ then $\mathcal{R}_t^n(V)(x)$ converges to $\frac{\lambda}{1+\frac{1}{t}} V_t^*(x)$ as n goes to $+\infty$.

The proof of Theorem A is presented in the proof of Theorem 2.1. We remind that we are doing renormalization close to the fixed point 0. The same could be done close to the other fixed indifferent point 1.

It is well-known that for each value t, the nature of the germ of the dynamics close to the fixed and weakly hyperbolic point (see [38, 23, 27, 28, 29]) determines, on one hand, the existence or non-existence of an absolutely continuous invariant measure with respect to Lebesgue measure and, on the other hand, the different power laws of decay of correlation (when the SRB-measure does exist [38]). We remind that this SRB-measure is the measure associated to the potential $-\log f_t'$. We claim that Theorem A recovers the importance of the germ of these dynamics close to the fixed indifferent points.

Owing to the dynamics we are studying, we are naturally led to study the lifted equation of (2) in Σ. We shall (in a first step) set

$$H((\underbrace{0,\ldots,0}_{c_1},\underbrace{1,\ldots,1}_{c_2}\underbrace{0,\ldots,0}_{c_3},1,\ldots)) = (\underbrace{0,\ldots,0}_{2c_1},\underbrace{1,\ldots,1}_{c_2}\underbrace{0,\ldots,0}_{c_3},1,\ldots),$$

and then consider:

$$V(\sigma(H(x))) + V(H(x)) = V(x). \tag{3}$$

In Section 2 we state and prove results similar to Theorem A but for the one-sided or two-sided shift (see Subsections 2.2 and 2.3). There, we show that the fixed potential for the renormalization is the Hofbauer potential defined by

$$V(x) = \log\frac{n+1}{n}$$

if x belong to $[\underbrace{000\ldots00}_{n}1]$ (and $n > 0$).

Note that H is defined above only on the cylinder [0]. This corresponds to the fact that $x \mapsto \frac{x}{2}$ has $[0, 1\frac{1}{2}]$ for image (ad not the hole interval [0, 1]). Our definition, for the case of the Bernoulli space, is very much similar to the one described for the two dimensional lattice in Statistical Mechanics (see for instance [8]), where one takes a square box, and then consider a new renormalized box such that each side is scaled by a factor of 2. The old potential is also rescaled in the new box. This is a local procedure and we believe this is a mathematical way to understand why physicists consider renormalization on potential as a way integrate (or rescale) it.

In the shift Σ the dynamics is fixed and one can ask if other kind of renormalization operators (with a different H) could be considered (giving similar results). We address this question now.

Theorem B. *Let H be an increasing function on the shift Σ (for the lexicographic order), such that*

1. *for every $\underline{x} = (1, x_2, x_3, \ldots)$, $H(\underline{x}) = (\underbrace{0,\ldots,0}_{a\ terms}, 1, x_2, x_3, \ldots)$, where $a \geq 1$;*
2. *$H^{-1} \circ \sigma^2 \circ H = \sigma$,*
3. *$H(\underline{0}^\infty) = \underline{0}^\infty$*

Then, for every $\underline{x} = (\underbrace{0,\ldots,0}_{n_0\ terms}, 1, x_{n_0+2}\ldots)$, we have $H(x) = (\underbrace{0,\ldots,0}_{2n_0+a\ terms}, 1, x_{n_0+2}, \ldots)$.

In other words, Theorem B shows that there exists a unique type of maps $H : \Sigma \circlearrowleft$, and (as a consequence) a unique type of "good" potential V which satisfy the fixed point equation (3). There appears the special importance of the Hofbauer potential.

We want here to emphasize that the assumptions on H are very natural if we consider the rescaling procedure described above. The lexicographic order is a good way (and may be the unique one) to consider blocks at different scales. The assumption "$H([1]) = [0^a 1]$" is a good way to send blocks on blocks.

The last part of the paper deals with thermodynamic formalism. We recall that, given a function ϕ, a probability measure μ is said to be ϕ-conformal if there exists a positive real number λ_ϕ such that every Borel[1] set A satisfying that $\sigma : A \to \sigma(A)$ is an homeomorphism, then

$$\mu(\sigma(A)) = \lambda_\phi \int_A e^{-\phi}\, d\mu.$$

If ϕ is continuous, there necessarily exists a ϕ-conformal measure. Indeed the Transfer Operator

$$\mathcal{P}(\psi)(x) := \sum_{y,\ \sigma(y)=x} e^{\phi(y)}\psi(y)$$

acts on continuous functions, hence its adjoint acts on measures. We then use the Schauder-Tychonoff theorem to get some eigen-measure. This measure is a ϕ-conformal measure. The question is then to study existence (and uniqueness) of a σ-invariant probability measure equivalent to the ϕ-conformal measure. Such a measure is said to be *ϕ-quasi-conformal*. We shall simply say quasi-conformal when the function ϕ is clearly understandable.

We denote by h_μ the Kolmogorov entropy of the invariant probability μ. We recall that given a function $\phi : \Sigma \to \mathbb{R}$, an invariant *probability* measure μ is called an equilibrium state for the *potential* ϕ if it satisfies

$$h_\mu + \int \phi\, d\mu = \sup_\nu \left\{ h_\nu + \int \phi\, d\nu \right\}.$$

In "good" cases, given a potential ϕ, there exists a unique ϕ-quasi-conformal invariant probability which is also the unique equilibrium state for ϕ. This however does not hold in any case, in particular the intermittent maps furnish counter-examples. Our last theorem studies this question for a special two-parameters family of potentials $\gamma.\phi_\beta$.

Potentials in the class $R(X)$ defined in [35] are "good cases". Nevertheless they are constant on cylinders of the form $[0^n 1]$ or $[10^n 1]$ or $[1^n 0]$ or $[01^n 0]$. This means they only takes account one fixed indifferent point and then do not distinguish points in function of the time their orbit spend in the second laminar regime.

More precisely, a typical orbit is an infinite alternation of sequence of 0's and sequences of 1's. We would like to study the thermodynamic formalism for potentials which take account all these alternations and not only the first string of 0's or of 1's. These potentials cannot be constant on the cylinders of the form $[0^n 1]$ or $[10^n 1]$ or $[1^n 0]$ or $[01^n 0]$.

[1]The set Σ is a compact and metric space with $d((x_n),(y_n)) = 2^{-\min(x_n \neq y_n)}$

The family of potentials we are considering is defined as follows. We consider real numbers, α in $[1,+\infty[$, β in $]0,1]$, and a natural number $a \geq 0$. We assume that these parameters satisfy

$$\frac{1}{\left(\frac{3}{2}\right)^{\beta}-1}-\frac{1}{2^{\beta}-1} = (1+\frac{1}{a+1})^{\alpha}-1. \tag{4a}$$

$$\frac{1}{\alpha} = 2^{\beta}-1. \tag{4b}$$

This system of conditions is referred to as (4). We shall prove in Lemma 4.4 that for each choice of one parameter, (4) gives a unique value for the two others parameters (except that a may not be integer). Hence, for each positive integer value of a we have the corresponding values α_a and β_a. In this way, several renormalization operators, with different values $a \in \mathbb{N}$, can be considered as in Theorem B. In the following, we however prefer to keep β as parameter.

Given $\bar{x} = (\underbrace{0,\dots,0}_{n_0},\underbrace{1,\dots,1}_{n_1},\underbrace{0,\dots,0}_{n_2},1,\dots) \in \Sigma = \{0,1\}^{\mathbb{N}}$ we define a real number in the following way:

$$\theta_\beta(\bar{x}) = \cfrac{1}{\cfrac{(n_0+1)^\beta}{(n_0+2)^\beta-(n_0+1)^\beta}+\cfrac{1}{\cfrac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha-(n_1+a)^\alpha}+\cfrac{1}{\cfrac{n_2^\beta}{(n_2+1)^\beta-n_2^\beta}+\cfrac{1}{\cfrac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha-(n_3+a)^\alpha}+\dots}}}}$$

With these notations, the potential ϕ_β is defined by:

$$\phi_\beta(x) = -2\log\left(\frac{\theta_\beta\circ\sigma(x)}{\theta_\beta(x)}\right) \text{ if } x \in [0],$$

$$\phi_\beta(x) = -2\log\left(\frac{2^\beta-1-\theta_\beta\circ\sigma(x)}{2^\beta-1-\theta_\beta(x)}\right) \text{ if } x \in [1].$$

Heuristically speaking, the potential ϕ_β should be seen as what one should expect to be the $-\log$ of the derivative of a "global" Manneville-Pomeau map $\widehat{f}_\beta$ defined for the Bernoulli space after the "change of coordinates" θ_β. We are studying existence of γ-conformal measures for our virtual Manneville-Pomeau maps:

Theorem C. *For any $\gamma \in]0,\frac{1}{2}]$ and for any β there exist a unique $\gamma\phi_\beta$-conformal measure and an unique quasi-conformal and σ-invariant probability measure.*
For $\gamma \in]\frac{1}{2},1]$, there exists a critical value $\beta_c := \beta_c(\gamma) > 0$, which is maximal with this property, such that for any $\beta < \beta_c$ there exist an unique $\gamma\phi_\beta$-conformal measure and an unique equivalent quasi-conformal σ-invariant probability measure.
In both cases the invariant quasi-conformal probability is the unique equilibrium state associated to the potential $\gamma\phi_\beta$.

The definition of the family of potentials results from the next series of observations: for $\gamma = 1$ and $\beta = \alpha = 1$ (and $a = 0$), θ_1 is the usual continuous fraction and is also the conjugacy of Σ with the Manneville-Pomeau map f_1 (see [21]):

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \theta_1 \downarrow & \circlearrowleft & \downarrow \theta_1(0^{n_0}1^{n_1}0^{n_2}\ldots) = \frac{1}{1+n_0+\frac{1}{n_1+\frac{1}{n_2+\ldots}}} \\ [0,1] & \xrightarrow{f} & [0,1]. \end{array}$$

In other words the virtual Manneville-Pomeau maps $\widehat{f_1}$ is, in that case, the true Manneville-Pomeau map f_1 and the potential is $-\log f_1'$.

It is noteworthy that θ_1 has the same germ than the Hofbauer potential close to 0^∞. Moreover, it takes account the two competitive laminar regimes of the map f; θ_1 seems to be a way to consider for $x \in \Sigma$ how all the whole orbit approaches the two fixed and indifferent points.

Let us set $V_0(x) = \log \dfrac{n_0+1}{n_0}$ if $x = 0^{n_0}\ldots$ with $n_0 > 0$ and $V_1(y) = \log \dfrac{n_1+1}{n_1}$ if $y = 1^{n_1}\ldots$ with $n_1 > 0$. These are the two Hofbauer potentials associated to the two fixed and indifferent points 0^∞ and 1^∞. Note that

$$n = \frac{n}{(n+1)-n},$$

and that for any β and α, βV_0 and αV_1 are also fixed point for the renormalization operators associated to 0^∞ and 1^∞.

Then, the function θ_β is just an extension of θ_1, considering, in the one hand, close to 0^∞ a potential of the form βV_0 and a renormalization associated to $a = 1$ (in Theorem B), and in the other hand, close to 1^∞ a renormalization associated to $a = a$ and a potential "multiplied by α". And finally, the relations (4) ensure that θ_β is onto the interval $[0, 2^\beta - 1]$(this proof is left to the reader and uses properties of the function g_γ defined in Subsection 4.1).

It is well known that f_1 has no absolutely continuous invariant probability (with respect to the Lebesgue measure). Then, the fact that for $\gamma = 1$ and for sufficiently small β we again get a finite quasi-conformal measure is thus non-obvious and non-intuitive. This proves existence of a phase transition when β increases.

Regarding to this problem of phase transition, several questions are still unsolved. This work is a first step to study phase transitions on our setting.

The case $\beta = \gamma = 1$ should indicate that for $\gamma > \frac{1}{2}$, there exists another critical value $\bar{\beta}_c = \bar{\beta}_c(\gamma)$ such that for $\beta > \bar{\beta}_c$ there exists no $\gamma\phi_\beta$-quasi-conformal probability.

Similarly and probably consequently, it is expected that for fixed β, the one family of potentials $\gamma.\phi_\beta$ presents a phase transition: for γ sufficiently big, the pressure of $\gamma.\phi_\beta$ is affine.

1.3. **Structure of the paper.** This paper can be separated in three parts.

The first part is the entire Section 2. There, we study the renormalization for the Manneville-Pomeau maps, and also for the full 2-shift. Both studies are restricted to the basin of backward-attraction of a weakly expanding fixed point. This is what we call the local point of view.

In Subsection 2.1 we show the fixed point property for the renormalization operator associated to Manneville-Pomeau transformations and also Theorem A. In Subsection 2.2 we consider the one-sided shift and we define there the natural renormalization operator with respect to the class of dynamics we are considering. As a by-product we extend the operator to the 2-side case in Subsection 2.3, and then consider a kind of two dimensional bijective Baker Manneville-Pomeau map in Subsection 2.4.

In the second part of the paper (Section 3), we prove Theorem B , that is, there exits a unique renormalization operator (up to an integer positive parameter a) for the shift which respects the class of dynamics we are considering (two coupled laminar regimes with two fixed and weakly repelling points).

The third part is the proof of Theorem C. In section 4, we prove properties on θ_β, discuss the motivation for the relation of the parameters. Section 5 is devoted to the strict proof of Theorem C, namely to the existence of Gibbs measures associated to the potential in function of the values of the parameters.

This part contains very long and perhaps unpleasant computations. They are however simple and necessary to get complete mathematical proofs. Nevertheless we indicate to the reader that almost all the computations from the proofs of Section 4 should be omitted in a first reading. In Section 5, the reader is supposed to be familiar with some basic knowledge of the transfer operator theory, even if we tried to make the paper as self-contained as possible. We refer the reader to [1] to basic notions on this theory.

2. The local renormalization operator

2.1. **The Manneville-Pomeau model.** We set

$$\begin{cases} f(x) = \frac{x}{1-x}, \text{ if } , 0 \leq x \leq \frac{1}{2}, \\ f(x) = 2 - \frac{1}{x}, \text{ if } , \frac{1}{2} < x \leq 1, \end{cases}$$

Note that one branch above is obtained from the other by the change of coordinate $x \to (1-x)$.

We remind that we get for $t \geq 0$,

$$\begin{cases} f_t(x) = \dfrac{x}{(1-x^t)^{1/t}} = (\dfrac{x^t}{1-x^t})^{1/t}, \text{ if } 0 \leq x \leq \dfrac{1}{2^{1/t}}, \\ f_t(x) = (2 - \dfrac{1}{x^t})^{1/t}, \text{ if } \dfrac{1}{2^{1/t}} < x \leq 1, \end{cases}$$

Let us set $h_t(x) = x^t$. Then, for every x in $[0, \frac{1}{2^{1/t}}[$ we have $f_t = h_t^{-1} \circ f \circ h_t$. Using $x \to 1-x$ we get the same kind of result for $x \geq \frac{1}{2^{1/t}}$. Therefore, in all this section we shall only state and prove results for the map f.

Note that $f_1(x) = \frac{x}{1-x}$ can be considered as a translation by -1 in the variable $s = 1/x$. Seeing it as a shift helps to understand the partition and other things that come later.

In this way we have a natural partition by fundamental domains for the branch of f in $(0,(1/n))$ by $(\frac{1}{3},\frac{1}{2}),...,(\frac{1}{k},\frac{1}{(k+1)}),...$ (see also page 153 in [21]).

For a given y , the two inverse branches by f are $x_1(y)=\dfrac{y}{(1+y)}$ and $x_2(y)=(\frac{1}{2-y})$.

The image of x_1 is in $[0\,,\,(0.5)]$ and the image of x_2 is $[(0.5)\,,\,1]$.

Note that $f'(x)=\frac{1}{(1-x)^2}$ for $x\in(0,0.5)$ and $f'(x)=\frac{1}{x^2}$ for $x\in(0.5,1)$

We point out the main property of f:

$$f^2(\frac{x}{2}) \;=\; (f\circ f)\,(\frac{x}{2})=\frac{1}{2}f(x). \tag{5}$$

One can see by induction that

$$f^j(x)=\frac{1}{(\frac{1}{x}-j)}. \tag{6}$$

We remind that we set $\mathcal{R}(V)(x):=V(f(\frac{x}{2}))+V(\frac{x}{2})$. And we consider V in $\mathcal{F}_1$, *i.e.*

$$V(x)=x+o(x).$$

This is meaningful only close to 0. We also set $V^*(x)=-\log(1-x)$. Taking derivative of both sides of (5) one can see that V^* is a fixed point for $\mathcal{R}$.

By recurrence and using (5) one can easily see that

$$\mathcal{R}^n(V)\,(x) \;=\; [S_{2^n}(V)]\,(\frac{x}{2^n})=\sum_{j=0}^{2^n}V(f^j(\frac{x}{2^n})).$$

From (6) this yields

$$\mathcal{R}^n(V)\,(x) \;=\; \sum_{j=0}^{2^n}V(\frac{1}{(\frac{2^n}{x}-j)}). \tag{7}$$

Our main interest is on universality type properties for the renormalization operator.

Theorem 2.1. *For any $V\in\mathcal{F}_1$ and for very x in $[0,\frac{1}{2}[$ we have*

$$\lim_{n\to\infty}\mathcal{R}^n(V)(x)=V^*(x).$$

Proof. Let x be in $\left[\dfrac{1}{(m+1)},\dfrac{1}{m}\right]$, with $m\geq 2$.

Then, $\dfrac{x}{2}$ belongs to $\left[\dfrac{1}{2^n(m+1)},\dfrac{1}{2^n m}\right]$.

Hence, the smallest value for $(\frac{2^n}{x}-j)$, $j=0,1,\ldots,2^n$ is obtained when $j=2^n$, and is larger than $2^n(2-1)$. Therefore each term $\dfrac{1}{\frac{2^n}{x}-j}$ is very close to 0, and it makes sense to approximate $V\left(f^j(\frac{x}{2^n})\right)$. Hence we have

$$\mathcal{R}^n(V)(x) = \sum_{j=0}^{2^n} V\left(\frac{1}{(\frac{2^n}{x}-j)}\right) \tag{8}$$

$$= \sum_{j=0}^{2^n} \frac{1}{(\frac{2^n}{x}-j)} + o(\frac{1}{(\frac{2^n}{x}-j)})$$

$$= \frac{1}{2^n}\sum_{j=0}^{2^n} \frac{1}{(\frac{1}{x}-\frac{j}{2^n})} + o\left(\frac{1}{2^n}\sum_{j=0}^{2^n} \frac{1}{(\frac{1}{x}-\frac{j}{2^n})}\right). \tag{9}$$

For a fixed x, $\frac{1}{2^n}\sum_{j=0}^{2^n} \frac{1}{(\frac{1}{x}-\frac{j}{2^n})}$ converges to

$$\int_0^1 \frac{1}{(\frac{1}{x}-r)}\,dr = -[\log(\frac{1}{x^t}-r)]_0^1 = -\log(1-x) = V^*(x),$$

as n goes to $+\infty$. Hence, $o(\frac{1}{2^n}\sum_{j=0}^{2^n}\frac{1}{(\frac{1}{x}-\frac{j}{2^n})})$ is in $o(1)$ and then converges to 0 as n goes to $+\infty$.

This finishes the proof of the theorem. □

Note that if $V(x) = c.x^t + o(x^t)$ with $t \neq 1$, then, assuming $c > 0$, the same kind of commutation than above yields that

$$\mathcal{R}^n(V) \to +\infty \text{ if } t < 1 \text{ or } \mathcal{R}^n(V) \to 0 \text{ if } t > 1.$$

Therefore, only potentials V in $\mathcal{F}_1$ can converge to the fixed point V^*.

Remark 2. Let us now assume that V belongs to $\mathcal{F}_t$. Let us set $g : [0,1] \hookleftarrow$ such that $V = \log g'$. Then $\mathcal{R}^n V \to V^*$ is the expression that says that g belongs to the "stable set" of f for the action of $\widetilde{\mathcal{R}}$ (this expression, "stable set", has a clear meaning but we do not have here the ambition to say something rigorous about the general set of maps g, satifying $\widetilde{\mathcal{R}}^n(g) \to f$). As we said, this exactly means that g has the same germ than f.

The bottom line is: the results described in [26] (for maps) are now obtained for the potential point of view (and, in a rigorous way). The fixed point for $\mathcal{R}$ is $V_1^*(x) = \log f'$.

2.2. **The one-side shift Σ.** We consider here the Bernoulli space $\Sigma = \{0,1\}^{\mathbb{N}}$ and the shift acting on Σ.

We denote by $M_n \subset \Sigma$, for $n \geq 1$, the cylinder set $[\underbrace{000\ldots00}_{n}1]$ and by M_0 the cylinder set [1]. The ordered collection $(M_n)_{n=0}^{\infty}$ is a partition of Σ.

Definition 2.1. *Consider $\mathcal{F}$ the set of non-negative continuous functions $V : \Sigma \to \mathbb{R}$ which are constant in the set M_n, for all $n \geq 1$. We denote by a_n the value of V on each M_n. We further assume that $a_n = \frac{1}{n} + o(\frac{1}{n})$.*

Definition 2.2. *We define the renormalization operator in the following way:*
For $x := (\underbrace{0,\dots,0}_{c_1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots)$ *we set*

$$\mathcal{R}(V)(x) = V((\underbrace{0,\dots,0}_{2c_1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots))+$$
$$V((\underbrace{0,\dots,0}_{2c_1+1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots)).$$

Note that the potential V^*, with value $\log\frac{k+1}{k}$ in M_k, is invariant by $\mathcal{R}$. Indeed we have

$$\log\frac{k+1}{k} = \log\frac{2k+1}{2k} + \log\frac{2k+1+1}{2k+1}.$$

Theorem 2.2. *Each $V\in\mathcal{F}$ and for every x in Σ, $\mathcal{R}^n(V)(x)$ goes to $V^*(x)$ as n goes to $+\infty$.*

Proof. An easy computation, by induction, gives the formula

$$\mathcal{R}^n(V)(x) = S_{2^n}(V)(x_n) \tag{10}$$

where $x_n = (\underbrace{0,\dots,0}_{2^nc_1+2^n-1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots)$ and $S_k(V)$ is the Birkhoff sum $V(.)+V\circ\sigma(.)+\dots+V\circ\sigma^{k-1}(.)$.

Equation (10) yields for $x\in M_{c_1}$

$$\begin{aligned}\mathcal{R}(V)(x) = \sum_{j=0}^{2^n-1} a_{2^nc_1+j} &= \sum_{j=0}^{2^n-1}\frac{1}{(2^nc_1+j)} + o\left(\frac{1}{(2^nc_1+j)}\right)\\ &= \frac{1}{2^n}\sum_{j=0}^{2^n-1}\frac{1}{(c_1+\frac{j}{2^n})} + o(\frac{1}{2^n}\sum_{j=0}^{2^n-1}\frac{1}{(c_1+\frac{j}{2^n})})\end{aligned}$$

The first term in the right hand side is a Riemann sum, and converges, as $n\to\infty$, to $\displaystyle\int_0^1\frac{1}{(c_1+r)}dr$. Again the second term goes to zero.

Note that the integral $\displaystyle\int_0^1\frac{1}{(c_1+r)}dr$ is the same as $-\log\frac{c_1+1}{c_1}$. Thus, and in the same way as before , if the potential V satisfies the condition

$$a_k = \frac{1}{k} + o(\frac{1}{k}),$$

we have convergence of $\mathcal{R}^n(V)(x)$ to $V^*(x)$ when n goes to $+\infty$.

□

2.3. **The two-sided shift $\hat\Sigma$.** We denote $\hat\Sigma=\{0,1\}^{\mathbb{Z}}$ and also denote each point in this set by $<y|\,x> = <\dots y_2,y_1|\,x_0,x_1,x_2..>$ where x is future and y is past. The shift $\hat\sigma$ is defined by

$$\hat\sigma(<\dots y_2,y_1|\,x_0,x_1,x_2..>) = <\dots y_2,y_1,x_0|\,x_1,x_2..>.$$

Definition 2.3. *Consider $\mathcal{F}$ the set of non-negative continuous functions $V:\Sigma\to\mathbb{R}$, which are constant in the sets of the form*

$$M_m|M_n=\{<y,x>, x\in M_n, y\in M_m\},$$

for each pair $m,n\geq 1$. We denote by $a_{m,n}=V(m,n)$ the value of V on each $M_m\times M_n$. We further assume that $a_{m,n}=\dfrac{m+n}{(m-1)\,n}+o\left(\dfrac{1}{m}\right)+o\left(\dfrac{1}{n}\right)$.

Definition 2.4. *We define the renormalization operator in the following way:*
For

$$z:=(\underbrace{0,\dots,0}_{d_3},\underbrace{1,\dots,1}_{d_2}\underbrace{0,\dots,0}_{\zeta})\,|\,(\underbrace{0,\dots,0}_{c_1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots)\in M_\zeta\times M_{c_1},$$

we set

$$\mathcal{R}(V)(z)=V(\underbrace{0,\dots,0}_{d_3},\underbrace{1,\dots,1}_{d_2}\underbrace{0,\dots,0}_{2\,\zeta-1})\,|\,(\underbrace{0,\dots,0}_{2\,c_1+1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots))+$$
$$V(\underbrace{0,\dots,0}_{d_3},\underbrace{1,\dots,1}_{d_2}\underbrace{0,\dots,0}_{2\,\zeta})\,|\,(\underbrace{0,\dots,0}_{2\,c_1},\underbrace{1,\dots,1}_{c_2}\underbrace{0,\dots,0}_{c_3},1,\dots)).$$

In order to simplify the notation we write

$$\mathcal{R}(V)(z)=V(2\zeta-1,2c_1+1)+V(2\,\zeta,2c_1).$$

One can show that for $V\in\mathcal{F}$, and $z\in M_m|M_n$, we have that

$$\mathcal{R}^n(V)(z)=\sum_{k=0}^{2^n-1}V(2^n\zeta-2^n+1+k,2^nc_1+2^n-1-k).$$

It is easy to see that the potential given by: for each $z\in M_j|M_k$

$$V^*(z)=\log\frac{j\,(k+1)}{(j-1)\,k},$$

defines a fixed point potential for $\mathcal{R}$.

Theorem 2.3. *Each $V\in\mathcal{F}$ is attracted, in the pointwise sense, by the renormalization operator $\mathcal{R}$ to the fixed point V^*.*

Proof. Given $V\in\mathcal{F}$, we have

$$\begin{aligned}\mathcal{R}^n(V)(z) &= \sum_{k=0}^{2^n-1}V(2^n\zeta-2^n+1+k,2^nc_1+2^n-1-k)\\ &= \sum_{k=0}^{2^n-1}\left(\frac{2^n(c_1+\zeta)}{(2^n\zeta-2^n+k)\,(2^nc_1+2^n-1-k)}+o\left(\frac{1}{(2^n\zeta-2^n+k+1)}\right)\right.\\ &\qquad\left.+\,o\left(\frac{1}{(2^nc_1+2^n-1-k)}\right)\right)\\ &= \frac{1}{2^n}\sum_{k=0}^{2^n-1}\frac{(c_1+\zeta)}{((\zeta-1)+\frac{k}{2^n})\,(\,(c_1+1)-\frac{k+1}{2^n})}+o(1),\end{aligned}$$

where we use $\frac{1}{(2^n\zeta - 2^n + k + 1)} \to_{n\to+\infty} \int_0^1 \frac{1}{\zeta - 1 + x} dx$ and $\frac{1}{(2^n c_1 + 2^n - 1 - k)} \to_{n\to+\infty} \int_0^1 \frac{1}{c_1 + -x} dx$.

Taking n large we get

$$\lim_{n\to+\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{(c_1 + \zeta)}{((\zeta - 1) + \frac{k}{2^n})\,((c_1 + 1) - \frac{k+1}{2^n})} = \int_0^1 \frac{(c_1 + \zeta)}{((\zeta - 1) + x)\,((c_1 + 1) - x)} dx =$$

$$(c_1 + \zeta) \int_0^1 \frac{1}{(\zeta - 1 + x)} + \frac{1}{(c_1 + 1 - x)} dx = \left[\log \frac{\zeta - 1 + x}{c_1 + 1 - x} \right]_0^1 =$$

$$\log\left(\frac{\zeta\,(c_1 + 1)}{(\zeta - 1)\, c_1} \right) = V^*(z).$$

□

2.4. **The Baker Manneville-Pomeau bijective transformation.** Using the notation of the first section, for a fixed value of t, consider

$$F_t : [0,1] \times [0,1] \to [0,1] \times [0,1],$$

a bijective transformation such that satisfies for each x and y

$$F_t(x, f_t(y)) = (f_t(x), y).$$

In order to simplify the notation we consider here only the case $t = 1$. Similar results will be true for the general case $t > 0$. We use the notation $\mathcal{F}_1 = \mathcal{F}$.

Definition 2.5. *We denote $\mathcal{F}$ the set of non-negative continuous functions $V : [0,1] \times [0,1] \to \mathbb{R}$ such that $V(x,y) = \log(\frac{1+x}{1-y}) + o(x) + o(y)$ when (x,y) is close to $(0,0)$.*

Definition 2.6. *The renormalization operator $\mathcal{R}$ acts on the set of functions V on $\mathcal{F}$ by means of*

$$\mathcal{R}(V)\,(x,y) = V(\frac{2}{2+x}, \frac{y}{2-y}) + V(\frac{x}{2}, \frac{y}{2})$$

In the same way as before one can show that

$$V^*(x,y) = \log(\frac{1+x}{1-y}),$$

is a fixed point for $\mathcal{R}$.

We leave for the reader the proof of the theorem:

Theorem 2.4. *Each $V \in \mathcal{F}$ is attracted, in the pointwise sense, by the renormalization operator $\mathcal{R}$ to the fixed point V^*.*

3. Proof of Theorem B

Bellow we consider X_1, X_2 metric spaces, $T_1 : X_1 \to X_1$, $T_2 : X_2 \to X_2$, $H_1 : X_1 \to X_1$, $H_2 : X_2 \to X_2$, are continuous transformations, and $V_1 : X_1 \to \mathbb{R}$, $V_2 : X_1 \to \mathbb{R}$ are also continuous. We first state a simple lemma:

Lemma 3.1. *Let (X_1, T_1) and (X_2, T_2) be two conjugated dynamical systems. Let $\theta : X_1 \to X_2$ be the continuous conjugacy. If H_1 satisfies $H_1^{-1} \circ T_1^2 \circ H_1 = T_1$, then $H_2 := \theta \circ H_1 \circ \theta^{-1}$ satisfies*

$$H_2^{-1} \circ T_2^2 \circ H_2 = T_2.$$

Moreover if V_1 satisfies $V_1(T_1(H_1(x))) + V_1(H_1(x)) = V_1(x)$, then $V_2 := V_1 \circ \theta^{-1}$ satisfies

$$V_2(T_2(H_2(x))) + V_2(H_2(x)) = V_2(x).$$

The proof follows just by taking the compositions.

In view of those results, it's meaningful to study maps H on the shift which satisfy the property $H^{-1} \circ \sigma^2 \circ H = \sigma$. More assumptions are necessary, if one wants to respect some other properties of the map $x \mapsto x/2$ in the interval. If $\bar{0}^\infty$ in the shift represents the 0 of the interval, then $H(\underline{0}^\infty) = \underline{0}^\infty$ needs to hold. Moreover the map H has to "increase", which can be translated into "H respect the lexicographic order in Σ".

We can now prove Theorem B.

Let H be an increasing function on the shift Σ_2 (for the lexicographic order), such that

(1) for every $\underline{x} = (1, x_2, x_3, \ldots)$, $H(\underline{x}) = (\underbrace{0, \ldots, 0}_{a\ terms}, 1, x_2, x_3, \ldots)$, where $a \geq 1$;
(2) $H^{-1} \circ \sigma^2 \circ H = \sigma$,
(3) $H(\underline{0}^\infty) = \underline{0}^\infty$

We want to prove that for every $\underline{x} = (\underbrace{0, \ldots, 0}_{n_0\ terms}, 1, x_{n_0+2} \ldots)$, we have

$$H(x) = (\underbrace{0, \ldots, 0}_{2n_0+a\ terms}, 1, x_{n_0+2}, \ldots).$$

Note that by assumption, this is already proved for every $\underline{x}$ on the form $(1, \ldots)$.

We first consider the case where $a \geq 2$. Note that we took $a = 1$ in a previous section where we considered the shift.

Let us pick some x, which necessarily has to be of the form $\underline{x} = (\underbrace{0, \ldots, 0}_{n_0\ terms}, 1, x_{n_0+2} \ldots)$.

We assume $n_0 > 1$. We point out that $\sigma(\underline{x}) \geq \underline{x}$, because a "1" appears sooner in $\sigma(\underline{x})$ than in $\underline{x}$. Therefore we must have

$$H(\sigma(\underline{x})) > H(\underline{x}), \text{ if } x \neq \underline{0}^\infty, \underline{1}^\infty. \tag{11}$$

Now, $\sigma^{n_0}(\underline{x})$ belongs to the cylinder [1], hence $H(\sigma^{n_0}(\underline{x})) = [\underline{a}\underline{x}]$, where $\underline{a}$ is the finite word $\underbrace{0, \ldots, 0}_{a\ terms}$, and [ ] is the concatenation of words in the shift. As we said before, in the moment we are considering such $a \geq 2$. The constraint $H^{-1} \circ \sigma^2 \circ H = \sigma$, yields $\sigma^{2n_0} \circ H = H \circ \sigma^{n_0}$. Therefore

$$H(\underline{x}) = (\underbrace{?, \ldots, ?}_{2n_0 \ terms}, \underbrace{0, \ldots, 0}_{a \ terms}, 1, x_{n_0+2}, \ldots), \tag{12}$$

where the first $2n_0$ digits are unknown.

As H has the increasing property, its image is in the cylinder [0], and the first digit in (12) is 0. The property $H^{-1} \circ \sigma^2 \circ H = \sigma$, also means $\sigma^2 \circ H = H \circ \sigma$. Therefore, each odd unknown digit in (12) is 0.

Now, we prove that no even unknown digit can be 1. Let us assume that the second digit is 1. Doing the same work for $\sigma(\underline{x})$ (here we use $n_0 > 1$), we have

$$H \circ \sigma(\underline{x}) = (\underbrace{0, ?, \ldots, 0, ?}_{2n_0-2 \ terms}, \underbrace{0, \ldots, 0}_{a \ terms}, 1, x_{n_0+2}, \ldots), \tag{13}$$

where each unknown digit at position $2p$ is the same digit than the digit in position $2p + 2$ in (12). To get these equalities, we again used $\sigma^2 \circ H = H \circ \sigma$.

If the second digit in (12) is a "1", then to respect (11), the second digit in (13) must be a "1" too. Therefore, the cascade rule yields that each even unknown digit must be 1, in (12) and in (13). In that case, and as we assume $a \geq 2$, there will be a "1" in $H(\underline{x})$ in position $2n_0$, and a "0" for $H \circ \sigma(\underline{x})$, and the two words coincide before that position. Hence, $H(\sigma(\underline{x})) < H(\underline{x})$, which is impossible by (11). This proves that the assumption is false, and the second unknown digit in (12) must be a "0".

Note that this also holds if $n_0 = 1$. Indeed, in that case we completely know $H \circ \sigma(\underline{x})$, by assumption *(1)* in the proposition. Therefore the above discussion means that for every $\underline{\xi} = (0, \ldots)$, $H(\underline{\xi})$ starts with 3 "0". Here again, the cascade rule between (12) and (13) yields that every even unknown digit is "0".

To complete the proof of Theorem B, we have to deal with the case $a = 1$. In that case, the assumption "the second unknown digit in (12) in 1" yields to

$$\begin{aligned} H(\underline{x}) &= (\underbrace{0, 1, \ldots, 0, 1, 0, 1}_{2n_0 \ terms}, \overset{\downarrow a}{0}, 1, x_{n_0+2}, \ldots), \\ H \circ \sigma(\underline{x}) &= (\underbrace{0, 1, \ldots, 0, 1}_{2n_0-2 \ terms}, \overset{\downarrow a}{0}, 1, x_{n_0+2}, \ldots). \end{aligned}$$

Hence, the unique possibility to respect the increasing property for H would be to alternate "0" and "1" for the tail of $\underline{x}$. But even in that case, this will be in contradiction with (11). This finishes the proof.

The conclusion is that each renormalization operator has to be of the form: take a fixed $a \in \mathbb{N}$, then given $V : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, for any $x := (\underbrace{0, ..., 0}_{c_1}, \underbrace{1, ..., 1}_{c_2} \underbrace{0, ..., 0}_{c_3}, 1, ...)$, we set

$$\mathcal{R}(V)(x) = V((\underbrace{0, ..., 0}_{2c_1+a}, \underbrace{1, ..., 1}_{c_2} \underbrace{0, ..., 0}_{c_3}, 1, ...)) + V((\underbrace{0, ..., 0}_{2c_1+a-1}, \underbrace{1, ..., 1}_{c_2} \underbrace{0, ..., 0}_{c_3}, 1, ...)).$$

In that case, the potential defined by $\log \dfrac{k+a}{k+a-1}$ on M_k ($k \geq 1$) is invariant by $\mathcal{R}$. It is a "Hofbauer-like" potential.

4. Properties for θ_β, parameters and virtual Manneville-Pomeau maps

In this section, we state and prove the main properties for the function θ_β. We explain where the condition (4) comes from: it yields the compatibility of the two regimes associated to different local renormalizations.

4.1. Main properties for θ_β.

4.1.1. Convergence of the continued fraction expansion defined by θ_β. Here, we define a generalization of the continued fraction expansion. We consider real numbers, α in $[1, +\infty[$, β in $]0, 1]$, and the natural number $a \geq 0$. These parameters are not supposed to satisfy (4)

Lemma 4.1. *Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that $a_0 = 0$, each a_{2k+1} is larger than 1, and all the even terms a_{2k}, $k > 0$, are positive and uniformly bounded away from zero. Then, the sequence of real numbers (r_k) defined by*

$$r_k = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}},$$

converges to a real number denoted by $[0, a_1, a_2, a_3, \ldots]$, and we have

$$[0, a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \cfrac{1}{\ddots}}}}}.$$

Proof. Let $(a_k)_{k\in\mathbb{N}}$ be as in the assumptions. We define two new sequences $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$, by induction:

$$p_0 = 0,\ p_1 = 1,\ q_0 = 1,\ q_1 = a_1$$

$$\forall k \in \mathbb{N},\ p_{k+2} = a_{k+2}p_{k+1} + p_k,\ q_{k+2} = a_{k+2}q_{k+1} + q_k.$$

It's easy to see, by induction, that for every $k > 0$, $q_k \geq 1$. Using $a_{2k+1} \geq 1$, we easily get $q_{2k+1} \geq k$, and then $q_{2k} \geq A.k$, where A is a positive lower bound for all the a_{2j}'s. Therefore, q_k goes to $+\infty$ as k increases to $+\infty$.

If we set $u_k = p_{k+1}q_k - p_kq_{k+1}$, then $u_{k+1} = -u_k$ for every k. We claim that $r_k = \dfrac{p_k}{q_k}$. Then, the two subsequences (r_{2k}) and (r_{2k+1}) are mutually adjacent and converge to the same limit. We leave the reader check that the even sequence (r_{2k}) increases and the odd sequence (r_{2k+1}) decreases. $\square$

Let $\gamma > 0$ be a real number. We define $g : (0, \infty) \to \mathbb{R}$, given by

$$g_\gamma(z) = g(z) = \frac{1}{(1 + \frac{1}{z})^\gamma - 1}.$$

For a fixed γ, and when the meaning is clear, we omit the subscribe γ in g_γ, in order to make the formulas simpler.

We have for every $z \in (0, +\infty)$, $g'(z) = \dfrac{\gamma}{z^2} \dfrac{1}{((1 + \frac{1}{z})^\gamma - 1)^2}(1 + \dfrac{1}{z})^{\gamma - 1}$, hence g is increasing. Moreover, $\lim_{z\to 0} g(z) = 0$ and $\lim_{z\to+\infty} g(z) = +\infty$. Therefore, for any given $y \in (0, \infty)$, there exists a $n_y \in \mathbb{N}$, such that

$$g(n_y) \le y < g(n_y + 1). \tag{14}$$

Moreover, $g(z) = z^\gamma + o(z^\gamma)$ when z is close to 0, and, $g(z) = \dfrac{z}{\gamma} - \dfrac{\gamma - 1}{2} + O(\dfrac{1}{z})$, when z is close to $+\infty$.

Lemma 4.2. *The map g_γ is convex for $\gamma > 1$, and is concave for $\gamma < 1$.*

Proof. To prove this lemma, first note that $g'(z) = \dfrac{\gamma}{z^2 + z}\left(g(z) + g^2(z)\right)$. This yields

$$g''(z) = -\gamma \frac{2z + 1}{(z^2 + z)^2}\left(g(z) + g^2(z)\right) + \frac{\gamma}{z^2 + z}\left(g'(z) + 2g'(z)g(z)\right).$$

If we replace in this last expression the value of $g'(z)$ in function of z and $g(z)$, we get

$$g''(z) = \gamma^2 \frac{g(z) + g^2(z)}{(z^2 + z)^2}\left(g(z) - \left(\frac{z}{\gamma} - \frac{\gamma - 1}{2}\right)\right).$$

Note that $\frac{z}{\gamma} - \frac{\gamma - 1}{2}$ is the asymptote of g close to $+\infty$. Then, the convexity of the map depends on the position of the graph with respect to the asymptote. It's convex when the graph is above the asymptote, and it's concave when the graph is below the asymptote. Now, recall that a convex map has a non-decreasing derivative, and a concave map has a non-increasing derivative. Therefore, easy considerations on the relative position of the graph with respect to the asymptote prove that the graph cannot cross the asymptote. Hence the map is convex for $\gamma > 1$, and concave for $\gamma < 1$. □

Note that $g_\alpha(1) = \frac{1}{2^\alpha - 1}$. Therefore, $g_\alpha(1) < 1$, for $\alpha > 1$, and $g_\beta(1) > 1$, for $\beta < 1$.

Given $\bar{x} = (\underbrace{0, \ldots, 0}_{n_0}, \underbrace{1, \ldots, 1}_{n_1}, \underbrace{0, \ldots, 0}_{n_2}, 1, \ldots) \in \Sigma = \{0, 1\}^{\mathbb{N}}$, we claim (and let the reader check) that the sequence defined by $a_{2k} = g_\alpha(n_{2k-1} + a)$ and $a_{2k+1} = g_\beta(n_{2k})$ satisfies the properties of Lemma 4.1. Therefore the real number $[0, a_1, a_2, \ldots]$ is well-defined.

This allows to define

$$\theta_{\alpha,\beta,a}(\bar{x}) = \cfrac{1}{\cfrac{(n_0+1)^\beta}{(n_0+2)^\beta-(n_0+1)^\beta} + \cfrac{1}{\cfrac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha-(n_1+a)^\alpha} + \cfrac{1}{\cfrac{n_2^\beta}{(n_2+1)^\beta-n_2^\beta} + \cfrac{1}{\cfrac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha-(n_3+a)^\alpha}+\dots}}}}$$

We now claim that $\theta_{\alpha,\beta,a}(\bar{x})$ belongs to $[0, 2^\beta - 1]$. Indeed, the odd subsequence (r_{2k+1}) decreases and the even subsequence (r_{2k}) increases. To minimize the value of $\theta_{\alpha,\beta,a}(\bar{x})$, it is necessary and sufficient to maximize n_0. On the other hand, to maximize the value of $\theta_{\alpha,\beta,a}(\bar{x})$, it is necessary and sufficient to minimize n_0 and to maximize n_1. Therefore, for every $\bar{x}$,

$$0 = \theta_{\alpha,\beta,a}(\bar{0}^\infty) \leq \theta_{\alpha,\beta,a}(\bar{x}) \leq \theta_{\alpha,\beta,a}(\bar{1}^\infty) = 2^\beta - 1.$$

Remark 3. The number a does not need to be in $\mathbb{N}$ to define $\theta_{\alpha,\beta,a}$, but in $\mathbb{R}^+$. This restriction is due to the fact we want to see a as a parameter of the renormalization.

4.1.2. *Lexicographic order and values for θ_β.* We now present a technical Lemma which gives inequalities with respect to the lexicographic order in Σ. For this, we introduce a new notation: from now until the end, the term $0^n 1^m$ shall denote the cylinder

$$(\underbrace{0,\dots,0}_{n\ terms},\underbrace{1,\dots,1}_{m\ terms}).$$

We shall also extend it in the natural way to describe more complicated cylinders and in particular use it with n or m equal to $+\infty$, and also

Lemma 4.3. *We have the following inequalities for $n_0 \geq 0$:*

$$\begin{aligned}\theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}},1,0^\infty) &\leq \theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}}1^{n_{2p+1}},0,\dots) \leq \theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}},1^\infty)\\ \theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}}1^{n_{2p+1}}0^\infty) &\leq \theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}}1^{n_{2p+1}},0,\dots) \leq \theta(0^{n_0}1^{n_1}\dots 0^{n_{2p}}1^{n_{2p+1}},0,1^\infty).\end{aligned}$$

Let us set $a_n = a_n(\beta) := -2\log\dfrac{g_\beta(n+1)}{g_\beta(n)}$ *and* $b_n = b_n(\beta) := -2\log\frac{g_\beta(n+1)+\left(1+\frac{1}{1+a}\right)^\alpha-1}{g_\beta(n)+\left(1+\frac{1}{1+a}\right)^\alpha-1}$. *Then for* $(\underbrace{00...00}_{n}\ 1\,0^\infty) \leq w < (\underbrace{00...00}_{n}\ 1^\infty)$ *and $n > 0$ we have*

$$a_n \leq \phi_\beta(w) \leq b_n.$$

Let us set $u_m = u_m(\beta) := -2\log\dfrac{g_\alpha(m+a)+2^\beta-1}{g_\alpha(m-1+a)+2^\beta-1}$ *and* $v_m = v_m(\beta) := -2\log\frac{g_\alpha(m+a)+2(2^\beta-1)}{g_\alpha(m-1+a)+2(2^\beta-1)}$. *Then for* $(\underbrace{11...11}_{m}\ 0^\infty) \leq w < (\underbrace{11...11}_{n}\ 0,1^\infty)$ *and $m > 1$ we have*

$$u_m \leq \phi_\beta(w) \leq v_m.$$

If $m = 1$ we have

$$u_1 := -2\log\left(1+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right) \leq \phi_\beta(w) \leq -2\log\left(\left(2+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)\left(\frac{2^\beta-\left(\frac{3}{2}\right)^\beta}{2^\beta-1}\right)\right) =: v_1.$$

Proof. For w satisfying $(\underbrace{00...00}_{n}\,1\,0^\infty) \leq w < (\underbrace{00...00}_{n}\,1^\infty)$ and $n > 0$, we have

$$\phi_\beta(w) = -2\log\frac{\theta_\beta\circ\sigma(w)}{\theta_\beta(w)}.$$

We set $\theta(w) = \dfrac{1}{g_\beta(n+1)+r}$ and we have $\theta\circ\sigma(w) = \dfrac{1}{g_\beta(n)+r}$. Here we use $n > 0$. We thus have to give bounds for

$$\frac{\frac{1}{g_\beta(n+1)+r}}{g_\beta(n)+r} = 1 + \frac{g_\beta(n+1)-g_\beta(n)}{g_\beta(n)+r}.$$

A bound from above is obtained when $r = 0$ and a bound from below is obtained for $r = \left(1+\frac{1}{1+a}\right)^\alpha - 1$.

For w satisfying $(\underbrace{11...11}_{m}\,0^\infty) \leq w < (\underbrace{11...11}_{m}\,0,1^\infty)$ and $m > 1$ we first recall that we have $\theta(w) = \dfrac{1}{\frac{1}{2^\beta-1}+\frac{1}{g_\alpha(m+a)+r}}$. Then a simple computation gives

$$2^\beta - 1 - \theta(w) = \frac{1}{\frac{1}{2^\beta-1}} - \theta(w) = \frac{(2^\beta-1)^2}{g_\alpha(m+a)+r+2^\beta-1}.$$

□

4.2. **Parameters.**

4.2.1. *Choices for parameters α, β and a.* We first check that conditions (4) are compatible with our assumptions $\alpha \geq 1$ and $\beta \leq 1$. Remember that (4) means:

$$\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta-1} = (1+\frac{1}{a+1})^\alpha - 1. \quad \text{(4a)}$$

$$\frac{1}{\alpha} = 2^\beta - 1. \quad \text{(4b)}$$

Note that $\beta \leq 1$ yields $2^\beta - 1 \leq 1$, and, then, we indeed have $\alpha \geq 1$.

We now want to solve a (from the two equations) as a function of β. For this we have to consider the map

$$\beta \mapsto a(\beta) + 1 := \frac{1}{\left(\frac{1}{(3/2)^\beta-1} - \frac{1}{2^\beta-1} + 1\right)^{2^\beta-1} - 1}.$$

Lemma 4.4. *The map* $A : x \to \dfrac{1}{\left(\frac{1}{(3/2)^x-1} - \frac{1}{2^x-1} + 1\right)^{2^x-1} - 1} - 1$ *is a decreasing bijection from* $]0,1[$ *onto* $]1,+\infty]$.

Proof. We first prove that the function A is one-to-one.

Let us pick some $a > 0$, and set $C := 1 + \frac{1}{1+a}$. Note that C belongs to the interval $]1,2[$.

We set $\varphi(x) = C^{\frac{1}{2^x-1}} - 1 - \frac{1}{\left(\frac{3}{2}\right)^x - 1} - \frac{1}{2^x-1}$. Hence we have

$$A(x) = a \iff \varphi(x) = 0.$$

We thus want to prove that there exists a unique x in $]0,1[$ such that $\varphi(x) = 0$. Note that $\varphi(1) = C - 2 < 0$. Moreover $\frac{1}{2^x-1} = \frac{\log 2}{x} + o(\frac{1}{x})$ close to 0.

Therefore for x close to 0 we have $\varphi(x) = e^{\log C(\frac{\log 2}{x} + o(\frac{1}{x}))} - 1 - \dfrac{\log\left(\frac{3}{2}\right)}{x} + \dfrac{\log 2}{x} + o(\dfrac{1}{x})$. This yields that

$$\lim_{x\to 0^+} \varphi(x) = +\infty.$$

As the function is continuous on the interval $]0,1]$, there exists at least one x such that $\varphi(x) = 0$. We thus want to prove the uniqueness of this solution.

Claim 1. *The function φ is either decreasing on $]0,1]$ or the exists $c \in]0,1[$ such that φ is decreasing on $]0,c[$ and increasing on $]c,1[$.*

We first explain why Claim 1 gives our result: indeed, the variations of φ and the fact that $\varphi(1) < 0$ imply that there can be at most one solution for the equation

$$\varphi(x) = 0.$$

We now prove Claim 1. Note that φ is C^∞ and we have

$$\varphi'(x) = \frac{\log\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^x}{\left(\frac{1}{(3/2)^x-1}\right)^2} - \frac{\log 2\, 2^x}{\left(\frac{1}{2^x-1}\right)^2}\left(1 + \log C e^{\frac{1}{2^x-1}\log C}\right).$$

We thus want to know where we have

$$\frac{\log\left(\frac{3}{2}\right)(2^x-1)^2\left(\frac{3}{2}\right)^x}{\log 2\left(\left(\frac{3}{2}\right)^\beta - 1\right)^2 2^x} \leq 1 + \log C e^{\frac{1}{2^x-1}\log C}. \tag{15}$$

Claim 2. *The function $x \mapsto 1 + \log C e^{\frac{1}{2^x-1}\log C}$ is decreasing.*

Indeed, $x \mapsto \dfrac{1}{2^x-1}$ is decreasing and $x \mapsto e^x$ is increasing and C is larger than 1.

Claim 3. *The function $x \mapsto \dfrac{(2^x-1)^2\left(\frac{3}{2}\right)^x}{\left(\left(\frac{3}{2}\right)^\beta - 1\right)^2 2^x}$ is increasing.*

We first explain how these two claims proves that Claim 1 is correct. Note that for $x = 1$

$$\frac{(2^x-1)^2\left(\frac{3}{2}\right)^x}{\left(\left(\frac{3}{2}\right)^\beta - 1\right)^2 2^x} = 3.$$

On the other hand note, note that $1 + \log C e^{\frac{1}{2^x-1}\log C} \sim \log C e^{\frac{\log C \log 2}{x}}$ close to 0 and then

$$\lim_{x\to 0^+} 1 + \log C e^{\frac{1}{2^x-1}\log C} = +\infty.$$

We remind that $\sim$ means that the quotient goes to 1. Hence, Claims 2 and 3 yield that there exists at most one real number $c\in]0,1]$ such that for $x=c$ we have

$$\frac{\log\left(\frac{3}{2}\right)(2^x-1)^2\left(\frac{3}{2}\right)^x}{\log 2\left(\left(\frac{3}{2}\right)^\beta-1\right)^2 2^x}=1+\log Ce^{\frac{1}{2^x-1}\log C}.$$

Moreover, for $x<c$, (15) holds and it does not hold for any $x>c$. If the real c does not exists, (15) holds for every $x\in]0,1]$. This prove that Claim 1 is correct.

We now prove Claim 3. It is sufficient to prove that $\psi:=x\mapsto\log\dfrac{(2^x-1)^2\left(\frac{3}{2}\right)^x}{\left(\left(\frac{3}{2}\right)^\beta-1\right)^2 2^x}$ increases. Equivalently, we want to prove that ψ' is positive on $]0,1]$. We have

$$\psi'(x)=\frac{2\log 2\,2^x}{2^x-1}-\frac{2\log\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^x}{\left(\frac{3}{2}\right)^\beta-1}-\log 2+\log\left(\frac{3}{2}\right).$$

Hence $\psi'(x)>0$ is equivalent to

$$\begin{aligned}
\log 2\left(\frac{2\,2^x}{2^x-1}-1\right) &> \log\left(\frac{3}{2}\right)\left(\frac{2\left(\frac{3}{2}\right)^x}{\left(\frac{3}{2}\right)^\beta-1}-1\right)\\
\log 2\frac{2^x+1}{2^x-1} &> \log\left(\frac{3}{2}\right)\frac{\left(\frac{3}{2}\right)^x+1}{\left(\frac{3}{2}\right)^\beta-1}\\
\frac{x\log 2}{2}\left(\frac{e^{x\log 2}+1}{e^{x\log 2}-1}\right) &> \frac{x\log\left(\frac{3}{2}\right)}{2}\left(\frac{e^{x\log\left(\frac{3}{2}\right)}+1}{e^{x\log\left(\frac{3}{2}\right)}-1}\right)\\
\frac{x\log 2}{2}\left(\frac{e^{\frac{x\log 2}{2}}+e^{-\frac{x\log 2}{2}}}{e^{\frac{x\log 2}{2}}-e^{-\frac{x\log 2}{2}}}\right) &> \frac{x\log\left(\frac{3}{2}\right)}{2}\left(\frac{e^{\frac{x\log\left(\frac{3}{2}\right)}{2}}+e^{-\frac{x\log\left(\frac{3}{2}\right)}{2}}}{e^{\frac{x\log\left(\frac{3}{2}\right)}{2}}-e^{-\frac{x\log\left(\frac{3}{2}\right)}{2}}}\right)
\end{aligned}$$

$$\frac{x\log 2}{2}\coth\frac{x\log 2}{2} > \frac{x\log\left(\frac{3}{2}\right)}{2}\coth\frac{x\log\left(\frac{3}{2}\right)}{2}. \tag{16}$$

We now let the reader check that the function $x\mapsto x\coth x$ is increasing on $\mathbb{R}_+$. Therefore (16) holds and Claim 3 is correct. This finish to prove that the function A is one-to one.

We let the reader check that, close to 1 we have $A(x)=(x-1)(2\log 2-4\log^2(2)-6\log\dfrac{3}{2})+O((x-1)^2)$.

In the other hand, close to 0 we have $A(x)=\dfrac{\frac{1}{\log\frac{3}{2}}-\frac{1}{\log 2}}{x\log x\log 2}[1+\kappa.\dfrac{1}{\log x}]+O(1)$.

The function A is one-to-one and the limits on the boundaries yield it is a decreasing bijection from $]0,1[$ on its image $]0,+\infty[$.

□

From the lemma above we get the property that each positive integer value of a can be reached. In this way, several renormalization operators, with different values $a \in \mathbb{N}$, can be considered in our future reasoning. For each such value a, we have the corresponding values α_a and β_a. We point out, however, that it also has meaning to consider real values of a (any positive real is possible) in several of our results (which are not related to the renormalization operator for the shift).

5. Proof of Theorem C

The proof has 3 main steps. In the first subsection we give an important result on the control of the variation of the potential on cylinders. In the second subsection we recall the construction of Gibbs states obtained by induction on a cylinder. We recall and use the method that was introduced in [19]. In particular, we introduce a one-family parameter of Transfer Operator, introduced the critical parameter S_c and we show that existence of Gibbs state is dependent of the fact that the operator has spectral radius larger than 1 or not close to the critical value S_c. In the last section we study the realization of this condition for our family of potentials in function of the parameters.

5.1. **Distortion on cylinders.**

We recall that the potential ϕ_β is defined as follows:

$$\begin{aligned}\phi_\beta(x) &= -2\log\left(\frac{\theta_\beta\circ\sigma(x)}{\theta_\beta(x)}\right) \text{ if } x\in[0],\\ \phi(x)_\beta &= -2\log\left(\frac{2^\beta-1-\theta_\beta\circ\sigma(x)}{2^\beta-1-\theta_\beta(x)}\right) \text{ if } x\in[1].\end{aligned}$$

The theory of equilibrium state has been developed for various type of dynamics and various potentials. It is however noteworthy that in every case, one of the main point is to control the distortion of Birkhoff sum of the potential on cylinders.

Proposition 5.1. *There exists a positive real number $\mathfrak{A}$ such that for every k in $\mathbb{N}^*$, for every w and w' in $01^{m_1}0^{n_1}1^{m_2}0^{n_2}\dots1^{m_k}0^{n_k}1$ (with $0<m_i,n_i<+\infty$) we have*

$$\left|S_{|\vec{m}|+|\vec{n}|}(\phi_\beta)(w)-S_{|\vec{m}+\vec{n}|}(\phi_\beta)(w')\right|\le\mathfrak{A},$$

where $|\vec{m}|+|\vec{n}|:=\sum_i m_i+n_i$.

Proof. The proof has three steps. In the first step we recall some simple facts on analysis. In the second step we do explicit computations for $k=2$. In particular we emphasize a general form of the difference of the two Birkhoff sums that can be used to estimate it by induction. In the last step we give the value for $\mathfrak{A}$.

Some usual analysis arguments. Given $A_0>0$, R_1 and R_2 non-negatives, then

$$\left|\frac{1}{A_0+R_1}-\frac{1}{A_0+R_2}\right|\le\frac{|R_2-R_1|}{A_0^2}$$

The repeated use of this fact yields

$$\left|\frac{1}{A_0+\frac{1}{A_1+\cdots+\frac{1}{A_n+R_1}}}-\frac{1}{A_0+\frac{1}{A_1+\cdots+\frac{1}{A_n+R_2}}}\right|\le\frac{1}{A_0^2A_1^2\cdots A_n^2}|R_2-R_1| \tag{17}$$

We shall use several times estimates of the form

$$\left|\log\frac{A}{B}\right|=|\log A-\log B|\le\frac{1}{\min(A,B)}\,|A-B|. \tag{18}$$

In particular, to get abound from above for (18) we need to get a bound from below for A and B.

The case $k=2$. We set

$$w=(0,\underbrace{1,\dots,1}_{m_1},\underbrace{0,\dots,0}_{n_1},\underbrace{1,\dots,1}_{m_2},\underbrace{0,\dots,0}_{n_2},1,W,\dots).$$

and

$$w'=(0,\underbrace{1,\dots,1}_{m_1},\underbrace{0,\dots,0}_{n_1},\underbrace{1,\dots,1}_{m_2},\underbrace{0,\dots,0}_{n_2},1,W',\dots).$$

We want to estimate $|S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w)-S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w')|$. For simplicity we drop the indexes β. Note that we have

$$\begin{aligned}S_{m_1+n_1+m_2+n_2}(\phi)(w) &= \phi(w)+S_{m_1}(\phi)(\sigma(w))+S_{n_1}(\phi)(\sigma^{m_1+1}(w))\\ &\quad +S_{m_2}(\phi)(\sigma^{m_1+n_1+1}(w))+S_{n_2-1}(\phi)(\sigma^{m_1+n_1+m_2+1}(w)).\end{aligned}$$

Owing to the definition of ϕ we thus get

$$\begin{aligned}S_{m_1+n_1+m_2+n_2}(\phi)(w) &= 2\log\frac{\theta\circ\sigma(w)}{\theta(w)}+2\log\frac{2^\beta-1-\theta\circ\sigma^{m_1+1}(w)}{2^\beta-1-\theta\circ\sigma(w)}\\ &\quad +2\log\frac{\theta\circ\sigma^{m_1+n_1+1}(w)}{\theta\circ\sigma^{m_1+1}(w)}+2\log\frac{2^\beta-1-\theta\circ\sigma^{m_1+n_1+m_2+1}(w)}{2^\beta-1-\theta\circ\sigma^{m_1+n_1+1}(w)}\\ &\quad +2\log\frac{\theta\circ\sigma^{m_1+n_1+m_2+n_2}(w)}{\theta\circ\sigma^{m_1+n_1+n_2+1}(w)}.\end{aligned}$$

We shall thus get bounds for the 10 terms of the form $\log\theta\circ\sigma\dots(w)-\log\theta\circ\sigma\dots(w)$ or $\log(2^\beta-1-\theta\dots(w))-\log(2^\beta-1-\theta\dots(w'))$.

- Estimation for $\log\theta(w)-\log\theta(w')$. Using (18) we need to get a lower bound for $\theta(w)$ and $\theta(w')$. Lemma 4.3 gives

$$\theta(w),\theta(w')\ge\theta(010^\infty)=\frac{1}{g_\beta(2)+\frac{1}{g_\alpha(1+a)}}=\frac{1}{\frac{1}{(3/2)^\beta-1}+\frac{1}{(3/2)^\beta-1}-\frac{1}{2^\beta-1}}.$$

Now, w and w' coincides for the first $1+m_1+n_1+m_2+n_2$ symbols. Then (17) gives

$$|\theta(w)-\theta(w')| = \left|\frac{1}{g_\beta(2)+\frac{1}{g_\alpha(m_1+a)+\cdots}} - \frac{1}{g_\beta(2)+\frac{1}{g_\alpha(m_1+a)+\cdots}}\right| \leq \frac{1}{g_\beta(2)^2}\frac{1}{g_\alpha(m_1+a)^2}\frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}|R-R'|.$$

Here $|R-R'|$ is of the form $\left|\frac{1}{g_\alpha()+\dots} - \frac{1}{g_\alpha()+\dots}\right|$, hence is lower than $\left(1+\frac{1}{1+a}\right)^\alpha - 1 \leq \frac{1}{\left(\frac{3}{2}\right)^\beta-1} - \frac{1}{2^\beta-1}$. Therefore we get

$$|\log\theta(w)-\log\theta(w')| \leq \left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1} - \frac{1}{2^\beta-1}\right)\left(\frac{2}{\left(\frac{3}{2}\right)^\beta-1} - \frac{1}{2^\beta-1}\right)\left(\left(\frac{4}{3}\right)^\beta-1\right)^2 \frac{1}{g_\alpha(m_1+a)^2}\frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}. \tag{19}$$

• Estimation for $\log\theta\circ\sigma(w)-\log\theta\circ\sigma(w')$. Note that we have

$$\theta(\sigma(w)),\theta(\sigma(w')) = \frac{1}{g_\beta(1)+\frac{1}{g_\alpha(m_1+a)+\dots}}.$$

Copying what we have done just above we get

$$|\log\theta\circ\sigma(w)-\log\theta\circ\sigma(w')| \leq \left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1} - \frac{1}{2^\beta-1}\right)\left(\left(\frac{3}{2}\right)^\beta-1\right)^2 \frac{1}{g_\alpha(m_1+a)^2}\frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}. \tag{20}$$

• Estimation for $\log(2^\beta-1-\theta\circ\sigma(w))-\log(2^\beta-1-\theta\circ\sigma(w'))$. Again we have

$$\theta(\sigma(w)),\theta(\sigma(w')) = \frac{1}{g_\beta(1)+\frac{1}{g_\alpha(m_1+a)+\dots}}. = \frac{1}{\frac{1}{2^\beta-1}+\frac{1}{R}}.$$

Note that we get

$$\begin{aligned} 2^\beta-1-\frac{1}{\frac{1}{2^\beta-1}+\frac{1}{R}} &= \frac{1}{\frac{1}{2^\beta-1}} - \frac{1}{\frac{1}{2^\beta-1}+\frac{1}{R}} \\ &= \frac{\frac{1}{R}}{\left(\frac{1}{2^\beta-1}\right)\left(\frac{1}{2^\beta-1}+\frac{1}{R}\right)} = \frac{\left(2^\beta-1\right)^2}{R+2^\beta-1}. \end{aligned}$$

Therefore we get $\frac{1}{2^\beta-1-\theta\circ\sigma(w)} \leq \frac{R+2^\beta-1}{\left(2^\beta-1\right)^2}$, and we get a bound from above for this last expression if we get a bound from above for R.

Now, $R = g_\alpha(m_1+a) + \cfrac{1}{g_\beta(n_2) + \cfrac{1}{\ddots}}$, therefore we get

$$R \leq g_\alpha(m_1+a) + 2^\beta - 1.$$

This yields

$$\frac{1}{2^\beta - 1 - \theta\circ\sigma(w)} \leq \frac{1}{(2^\beta-1)^2}\left[2(2^\beta-1) + g_\alpha(m_1+a)\right]. \tag{21}$$

This also holds for w'. Now, $\left|\log(2^\beta - 1 - \theta\circ\sigma(w)) - \log(2^\beta - 1 - \theta\circ\sigma(w'))\right| \leq \dfrac{|\theta\circ\sigma(w) - \theta\circ\sigma(w')|}{2^\beta - 1 - \min(\theta\circ\sigma(w), \theta\circ\sigma(w'))}$, and we finally get

$$\begin{aligned}\left|\log(2^\beta - 1 - \theta\circ\sigma(w)) - \log(2^\beta - 1 - \theta\circ\sigma(w'))\right| \leq\ & \frac{1}{(2^\beta-1)^2}\left[2(2^\beta-1) + g_\alpha(m_1+a)\right] \\ & \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\left(\frac{3}{2}\right)^\beta - 1\right)^2 \\ & \frac{1}{g_\alpha(m_1+a)^2}\frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}.\end{aligned} \tag{22}$$

• Estimation for $\log(2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w)) - \log(2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w'))$. We copy what we have just done to get (22). Note however that $\sigma^{m_1+1}(w)$ starts with 0 and it is thus lower than 01^∞. Note also it belongs to $0^{n_1}1^{m_2}0^{n_2}$. We thus get

$$\begin{aligned}\left|\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w)}{2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w')}\right| \leq\ & \frac{1}{2^\beta - \left(\frac{3}{2}\right)^\beta}\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right) \\ & \frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}.\end{aligned} \tag{23}$$

• Estimation for $\log(\theta\circ\sigma^{m_1+1}(w)) - \log(\theta\circ\sigma^{m_1+1}(w'))$. Following (18) we have to find a bound from below for $\theta\circ\sigma^{m_1+1}(w)$ and $\theta\circ\sigma^{m_1+1}(w')$. Note that we have

$$\theta\circ\sigma^{m_1+1}(w), \theta\circ\sigma^{m_1+1}(w') = \cfrac{1}{g_\beta(n_1) + \cfrac{1}{g_\alpha(m_2+a) + \cfrac{1}{\ddots}}}.$$

We thus have to get a bound from below for $g_\alpha(m_2+a)+\ldots$. Hence, we claim that we have

$$\frac{1}{\theta\circ\sigma^{m_1+1}(w)} \leq g_\beta(n_1) + \left(1 + \frac{1}{1+a}\right)^\alpha - 1 = g_\beta(n_1) + \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right).$$

Applying (17) we get

$$\left|\log \frac{\theta\circ\sigma^{m_1+1}(w)}{\theta\circ\sigma^{m_1+1}(w')}\right| \leq \left(g_\beta(n_1)+\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)$$
$$\frac{1}{g_\beta(n_1)^2}\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}. \tag{24}$$

From now on, the other bounds are similar to those already computed. We let the reader check that we have

(25)

$$\left|\log \frac{\theta\circ\sigma^{m_1+n_1+1}(w)}{\theta\circ\sigma^{m_1+n_1+1}(w')}\right| \leq \left(\left(\frac{3}{2}\right)^\beta-1\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2},$$

obtained similarly than (20). A similar computation than (22)

$$\left|\log \frac{2^\beta-1-\theta\circ\sigma^{m_1+n_1+1}(w)}{2^\beta-1-\theta\circ\sigma^{m_1+n_1+1}(w')}\right| \leq \frac{2(2^\beta-1)+g_\alpha(m_2+a)}{(2^\beta-1)^2}\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\left(\left(\frac{3}{2}\right)^\beta-1\right)^2$$
$$\frac{1}{g_\alpha(m_2+a)^2}\frac{1}{g_\beta(n_2)^2}. \tag{26}$$

Copying (23) we get

(27)

$$\left|\log \frac{2^\beta-1-\theta\circ\sigma^{m_1+n_1+m_2+1}(w)}{2^\beta-1-\theta\circ\sigma^{m_1+n_1+m_2+1}(w')}\right| \leq \frac{1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\frac{1}{g_\beta(n_2)^2}.$$

Copying (24) we get

(28)

$$\left|\log \frac{\theta\circ\sigma^{m_1+n_1+m_2+1}(w)}{\theta\circ\sigma^{m_1+n_1+m_2+1}(w')}\right| \leq \left(g_\beta(n_2)+\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\frac{1}{g_\beta(n_2)^2}.$$

And finally we get

(29)

$$\left|\log \frac{\theta\circ\sigma^{m_1+n_1+m_2+n_2}(w)}{\theta\circ\sigma^{m_1+n_1+m_2+n_2}(w')}\right| \leq \left(\frac{1}{\left(\frac{4}{3}\right)^\beta-1}+\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta-1}-\frac{1}{2^\beta-1}\right)\left(\left(\frac{4}{3}\right)^\beta-1\right)^2.$$

Collecting all inequalities (19), (20), (22), (23), (24), (25), (26), (27), (28) and (29), we get

$$
\begin{aligned}
&|S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w) - S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w')| \leq \\
&\left(\frac{1}{\left(\frac{4}{3}\right)^\beta - 1} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2 + \\
&\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\frac{1}{g_\beta(n_2)}\left[1 + \frac{1}{g_\beta(n_2)}\left[\frac{1}{2^\beta - \left(\frac{3}{2}\right)^\beta} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1} + \right.\right. \\
&\frac{1}{g_\alpha(m_2+a)}\left[\left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2 + \frac{1}{g_\alpha(m_2+a)}\left[\left(\frac{3}{2}\right)^\beta - 1 + 2\left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2(2^\beta - 1) + \right.\right. \\
&\frac{1}{g_\beta(n_1)}\left[1 + \frac{1}{g_\beta(n_1)}\left[\frac{1}{2^\beta - \left(\frac{3}{2}\right)^\beta} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1} + \right.\right. \\
&+\frac{1}{g_\alpha(m_1+a)}\left[\left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2 + \frac{1}{g_\alpha(m_1+a)}\left[\left(\frac{3}{2}\right)^\beta - 1 + 2\left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2(2^\beta - 1) + \right.\right. \\
&\left.\left.\left.\left.\left.\left.\left.\left.\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2\left(\frac{2}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\right]\right]\right]\right]\right]\right]\right]\right].
\end{aligned}
\tag{30}
$$

Let us now set :

$$
\begin{aligned}
F_{1,\beta}(x) &:= \left(\frac{3}{2}\right)^\beta - 1 + 2\left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2(2^\beta - 1) + x, \\
F_{2,\beta}(x) &:= \left(\frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2 + x, \\
F_{3,\beta}(x) &:= \frac{1}{2^\beta - \left(\frac{3}{2}\right)^\beta} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1} + x, \\
F_{4,\beta}(x) &:= 1 + x, \\
G_{\beta,n}(x) &:= \frac{x}{g_\beta(n)}, \\
G_{\alpha,m}(x) &:= \frac{x}{g_\alpha(m+a)}.
\end{aligned}
$$

Then (30) yields

$$|S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w) - S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w')| \leq \tag{31}$$
$$\left(\frac{1}{\left(\frac{4}{3}\right)^\beta - 1} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2 +$$
$$\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right) G_{\beta,n_2} \circ F_{4,\beta} \circ G_{\beta,n_2} \circ F_{3,\beta} \circ G_{\alpha,m_2} \circ F_{2,\beta} \circ G_{\alpha,m_2} \circ F_{1,\beta} \circ$$
$$G_{\beta,n_1} \circ F_{4,\beta} \circ G_{\beta,n_1} \circ F_{3,\beta} \circ G_{\alpha,m_1} \circ F_{2,\beta} \circ G_{\alpha,m_1} \circ F_{1,\beta}\left(\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2\left(\frac{2}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\right)$$

Note that all the functions $F_{i,\beta}$ are of the form $x \mapsto x + A_{i,\beta}$ with $A_{i,\beta} > 0$. Now g_β and g_α are increasing functions thus

$$\frac{1}{g_\beta(n)} \leq \left(\frac{3}{2}\right)^\beta - 1, \quad \frac{1}{g_\alpha(m+a)} \leq \left(1 + \frac{1}{1+a}\right)^\alpha - 1.$$

Let us set $G_\beta(x) := (\left(\frac{3}{2}\right)^\beta - 1)x$ and $G_\alpha(x) := \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)x$. We finally set

$$F := G_\beta \circ F_{4,\beta} \circ G_\beta \circ F_{3,\beta} \circ G_\alpha \circ F_{2,\beta} \circ G_\alpha \circ F_{1,\beta}.$$

Let us set $A(\beta) := \left(\frac{1}{\left(\frac{4}{3}\right)^\beta - 1} + \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2$ and $X(\beta) := \left(\left(\left(\frac{4}{3}\right)^\beta - 1\right)^2\left(\frac{2}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)\right)$. Then (31) gives

$$|S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w) - S_{m_1+n_1+m_2+n_2}(\phi_\beta)(w')| \leq A(\beta) + \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right) F^2(X(\beta)). \tag{32}$$

We emphasize that the "2" in F^2 in (32) exactly is the "2" from the case $k = 2$.

End of the proof. Existence and value for $\mathfrak{A}$. From (32) we claim that for general k we have

$$|S_{\vec{m}+\vec{n}}(\phi_\beta)(w) - S_{\vec{m}+\vec{n}}(\phi_\beta)(w')| \leq A(\beta) + \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right) F^k(X(\beta)). \tag{33}$$

The function F is of the form $x \mapsto b(\beta)x + c(\beta)$. Computing $b(\beta)$ we get

$$b(\beta) = \left(1 - \frac{\left(\frac{3}{2}\right)^\beta - 1}{2^\beta - 1}\right)^2 < 1.$$

Therefore, for any initial point X_0, the sequence $(F^n(X_0))_n$ converges to the unique fixed point $L(\beta) := \dfrac{c(\beta)}{1 - b(\beta)}$.

For β close to 0, e recall that $x^\beta - 1 = \beta \log x + o(\beta)$. We now want to estimate $c(\beta)$. This depends continuously on β. We shall thus estimate this value for β close to 0 to know if it bounded from below or not when β describes $]0, 1]$.

The following scheme indicates the dominating of the constant iterating the maps $F_{i,\beta}$ of G_j.

$$\xrightarrow{F_1} \beta \xrightarrow{G_\alpha} 1 \xrightarrow{F_2} 1 \xrightarrow{G_\alpha} \frac{1}{\beta} \xrightarrow{F_3} \frac{1}{\beta} \xrightarrow{G_\beta} 1 \xrightarrow{F_4} 1 \xrightarrow{G_\beta} \beta.$$

This means that $L(\beta)$ is in $O(\beta)$. Note that $X(\beta)$ is also in $O(\beta)$. Therefore, the sequence $F^n(X(\beta))$ is bounded from above by some constant of the form $B\beta$, with B independent of β. Note that $A(\beta)$ of bounded when β describes $]0, 1]$, and $\left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right)$ is in $O(\frac{1}{\beta})$. We can thus find some $\mathfrak{A}$ such that for every k and for every β,

$$A(\beta) + \left(\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1}\right) F^k(X(\beta)) \leq \mathfrak{A}.$$

This achieves the proof of the proposition. □

5.2. **Construction of Gibbs states.** In this section we recall the method of construction of Gibbs state done in [19] and developed in further later works of R. Leplaideur. We denote by h_μ the Kolmogorov entropy of the invariant probability μ.

First note that the potential $\gamma\phi_\beta$ is a continuous function, hence the variational principle proves that there exists an equilibrium state associated to this potential. We recall that given a function $\varphi : \Sigma \to \mathbb{R}$, an invariant *probability* measure μ is called an equilibrium state for the *potential* φ if it satisfies

$$h_\mu + \int \varphi \, d\mu = \sup_\nu \left\{ h_\nu + \int \varphi \, d\nu \right\}.$$

We denote by $\mathcal{P}(\gamma, \beta)$ the associated pressure.

We now consider the first return map g in the cylinder 01. For y in 01, $r(y)$ denotes the first return time in 01 of y by iterations of σ. For a real number Z, for x in 01 and for ψ a continuous function from 01 to $\mathbb{R}$, we define

$$\mathcal{L}_{Z,\beta,\gamma}(\psi)(x) \sum_{y,\ g(y)=x} e^{S_{r(y)}(\gamma\phi_\beta)(y) - Zr(y)} \psi(y).$$

This is the usual transfer operator for the map g and associated to the potential $S_{r(.)}(\gamma\phi_\beta)(.) - Zr(.)$. We study this operator, for fixed γ and β and for large enough Z. Namely, we set

$$Z_c = Z_c(\gamma, \beta) := \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{x = g(y),\ r(y) = n} e^{S_n(\gamma\phi_\beta)(y)} \right).$$

Note that for every $Z > Z_c$, the series $\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{01})(x)$ converges for every x in 01. Moreover, for every $Z < Z_c$ and for every x Proposition 5.1 proves that the same series diverge.

Using Proposition 5.1 we get

$$Z_c \leq \mathcal{P}(\gamma,\beta).$$

The main problem is to check if we get the strict inequality $Z_c < \mathcal{P}(\gamma,\beta)$ or not. In the rest of the proof we may omit γ and β when they are not necessary.

We claim that for every $Z > Z_c$ the spectral radius λ_Z of the two adjoint operators $\mathcal{L}_Z$ and $\mathcal{L}_Z^*$ is a simple and dominating eigenvalue. If ν_Z is the associated eigen-measure and if h_Z is the associated eigen-function (characterized by $\int h_Z\, d\nu_Z = 1$), then the measure μ_Z defined by

$$d\mu_Z := h_Z d\nu_Z,$$

is the unique equilibrium state associated to $S_{r(.)}(\gamma\phi_\beta)(.) - Zr(.)$ for the dynamical system $(01, g)$ (see Propositions 4.5, 4.8, 5.7 and 5.9 in [19]). We indicate that the main argument to get this result is to use the Ionescu-Tulcea and Marinescu theorem for the operator $\mathcal{L}_Z$. The large Banach space is the set of continuous functions on 01 and the small Banach space is the set of functions with the same module of continuity than the potential (see Subsection 4.3 in [19]).

Moreover the pressure of the equilibrium state is $\log\lambda_Z$ (see proposition 5.9 in [19]). This is a convex and decreasing function on $]Z_c, +\infty[$ (see Theorem 2.1 and its proof in section 4 in [3]). Furthermore, for every $Z > Z_c$, there exists a unique σ-invariant probability measure $\widehat{\mu}_Z$ such that its restriction and renormalization to the cylinder 01 is the measure μ_Z (see Proposition 6.8 in [19]). For this we use the fact that the expectation of the return time is proportional to $\mathcal{L}_Z(r)(x)$ (for any x in 01). This last term is equal to $-\dfrac{\partial\mathcal{L}_Z(\mathbb{1})(x)}{\partial Z}$ (see Lemma 3.7 in [20]). Now, note that a power series and its derivative have the same radius of convergence. A simple computation gives (See Proposition 6.8 in [19])

$$h_{\widehat{\mu}_Z}(\sigma) + \int \gamma\phi_\beta\, d\widehat{\mu}_Z = Z + \widehat{\mu}_Z(01)\log\lambda_Z. \tag{34}$$

Concerning the conformal property, Proposition 5.1 yields that for every k for every γ and β, for every $Z > Z_c(\gamma,\beta)$, and for every x in $01^{m_1}0^{n_1}\dots 1^{m_k}0^{n_k}1$

$$e^{-\mathfrak{A}} \leq \frac{\nu_Z(01^{m_1}0^{n_1}\dots 1^{m_k}0^{n_k}1)}{e^{S_{|\vec{m}|+|\vec{n}|}(\gamma\phi_\beta)(x) - k\log\lambda_{Z,\gamma,\beta}}} \leq e^{\mathfrak{A}}. \tag{35}$$

The same holds for μ_Z exchanging $\mathfrak{A}$ with $2\mathfrak{A}$ (see Lemma 5.10 in [19]). Therefore, the measure ν_Z can be $\gamma\phi_\beta$-conformal if and only if $\log\lambda_Z = 0$, namely if $\lambda_Z = 1$. To prove Theorem C, hence to get conformal and quasi-conformal measures, it is sufficient to prove

$$\lim_{Z\downarrow Z_c} \lambda_Z > 1. \tag{36}$$

If inequality (36) holds, there exists a unique Z_0 such that $\lambda_{Z_0} = 1$. This furnishes unique conformal and quasi-conformal measures (note that we have $Z_0 > Z_c$), and it it natural to ask for if the quasi-conformal measure $\widehat{\mu}_{Z_0}$ is an equilibrium state for $\gamma\phi_\beta$ or not.

Let us denote by δ_0 the Dirac measure at the fixed point 0^∞; similarly the Dirac measure at 1^∞ is denoted δ_1 . We point out that every ergodic probability measure different from δ_0 and δ_1 gives positive weight to the cylinder 01. Now, we claim that (34) yields that Z_0 is the maximum of the pressures of the potential $\gamma\phi_\beta$ among the measure different from δ_0 and δ_1. To see this, pick any such measure $\widehat{\mu}$, consider its restriction-renormalization μ to 01 and check that we have

$$\widehat{\mu}(01)\left(h_\mu(g)+\int S_{r(.)}(\gamma\phi_\beta)(.)\,d\mu\right)\leq\widehat{\mu}(01)Z_0\int r(.)\,d\mu+\widehat{\mu}(01)\log\lambda_{Z_0}=Z_0,$$

with equalities if and only if $\mu=\mu_{Z_0}$. Note that $\phi_\beta(0^\infty)=\phi_\beta(1^\infty)=0$. Therefore, if we have

$$Z_c\geq 0, \tag{37}$$

then, we automatically get $Z_0>0$. This yields that $\mathcal{P}(\gamma,\beta)=Z_0$, and that the unique $\gamma\phi_\beta$-quasi-conformal invariant probability is also the unique equilibrium state associated to $\gamma\phi_\beta$.

5.3. **Conditions hold.** In this subsection we prove that the two conditions (37) and (36) hold. This shall achieve the proof of Theorem C.

Proposition 5.2. *For every γ and β we have $Z_c(\gamma,\beta)=0$.*

Proof. We recall that the transfer operator is defined by

$$\mathcal{L}_Z(\mathbb{1}_{01})(w):=\sum_{v,\ g(v)=w}e^{S_{r(v)}(\gamma\,\phi)(v)-nZ},\quad w\in 01.$$

The point v is of the form $v=01^m0^{n-1}w$. In that case we have $r(v)=1+m+n-1+1=m+n+1$. Therefore we get

$$\mathcal{L}_Z(\mathbb{1}_{01})(w)=\sum_{n\geq 1}\sum_{m\geq 1}e^{S_{1+m+n}(\gamma\,\phi)(01^m0^{n-1}w)-nZ}.$$

Now we have

$$S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w))=\phi_\beta(01^m0^{n-1}w))+S_m(\phi_\beta)(1^m0^{n-1}w))+\phi_\beta(10^nw)+S_{n-1}(\phi_\beta)(0^{n-1}w)).$$

Using Lemma 4.3 we get for $m \geq 2$

$$\begin{aligned} S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w)) &\geq \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{m} u_k \\ &\geq -2\log(g_\beta(n+1)) - 2\log\left(\frac{g_\alpha(m+a)+2^\beta-1}{g_\alpha(1+a)+2^\beta-1}\right) + \\ &\quad -2\log\left((2^\beta-1)\left(2+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)\right). \end{aligned} \tag{38}$$

$$\begin{aligned} S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w)) &\leq \sum_{k=1}^{n-1} b_k + \sum_{k=1}^{m} v_k \\ &\leq -2\log\left(g_\beta(n+1) + \left(1+\frac{1}{1+a}\right)^\alpha - 1\right) - 2\log\left(\frac{g_\alpha(m+a)+2(2^\beta-1)}{g_\alpha(1+a)+2(2^\beta-1)}\right) \\ &\quad -2\log\left(\left(\frac{3}{2}\right)^\beta - 1\right) - 2\log\left(\left(2+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)\left(\frac{2^\beta-\left(\frac{3}{2}\right)^\beta}{2^\beta-1}\right)\right). \end{aligned} \tag{39}$$

Let us set

$$A(\beta) := \frac{1}{(2^\beta-1)\left(1+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)},$$

$$B(\beta) := \frac{2^\beta-1}{(\left(\frac{3}{2}\right)^\beta-1)(2^\beta-1+2^\beta-\left(\frac{3}{2}\right)^\beta)}.$$

Then we have

$$\begin{aligned} \mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{01})(w) &\geq A^{2\gamma}(\beta)e^{-Z}\left[\sum_{n=1}^{+\infty}\left(\left(1+\frac{1}{n+1}\right)^\beta-1\right)^{2\gamma}e^{-nZ}\right] \\ &\quad \left[1+\left(1+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)^{2\gamma}\sum_{m=2}^{+\infty}\left(\frac{\left(1+\frac{1}{m+a+1}\right)^\alpha-1}{\frac{1}{2^\beta-1}+\left(1+\frac{1}{m+a+1}\right)^\alpha-1}\right)^{2\gamma}e^{-mZ}\right] \end{aligned} \tag{40}$$

$$\begin{aligned} \mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{01})(w) &\leq B^{2\gamma}(\beta)e^{-Z}\left[\sum_{n=1}^{+\infty}\left(\frac{\left(1+\frac{1}{n+1}\right)^\beta-1}{1+\frac{\left(1+\frac{1}{n+1}\right)^\beta-1}{g_\alpha(1)}}\right)^{2\gamma}e^{-nZ}\right] \\ &\quad \left[1+\left(2+\frac{\left(\frac{3}{2}\right)^\beta-1}{2^\beta-\left(\frac{3}{2}\right)^\beta}\right)^{2\gamma}\sum_{m=2}^{+\infty}\left(\frac{\left(1+\frac{1}{m+a+1}\right)^\alpha-1}{\frac{1}{2^\beta-1}+2\left(1+\frac{1}{m+a+1}\right)^\alpha-2}\right)^{2\gamma}e^{-mZ}\right] \end{aligned} \tag{41}$$

Now, the four series have a general term equivalent to $\frac{1}{n^{2\gamma}}e^{-nZ}$ or $\frac{1}{m^{2\gamma}}e^{-mZ}$ when n or m go to $+\infty$. Hence we get $Z_c = 0$ and the proposition is proved. □

Proposition 5.3. *For any $\gamma \leq \frac{1}{2}$, for any $\beta \leq 1$ and for any w in* 01 *we have*

$$\lim_{Z \downarrow 0} \lambda_Z = +\infty.$$

For any $\gamma > \frac{1}{2}$, there exists $\beta_c = \beta_c(\gamma)$ such that for any $\beta < \beta_c$ and for any w in 01 *we have*

$$\lim_{Z \downarrow 0} \lambda_Z > 1.$$

Proof. We start with the case $\gamma \leq \frac{1}{2}$. Note that for n large we have

$$\left(1 + \frac{1}{n}\right)^{\beta} - 1 = \frac{\beta}{n} + O(\frac{1}{n^2}).$$

The series is increasing as Z decreases to 0.

We now deal with the case $\gamma > \frac{1}{2}$. The function $x \mapsto (1+x)^{\beta} - 1 - \frac{\beta}{2}x$ is increasing on the interval $[0, 2^{\frac{1}{1-\beta}} - 1]$. This interval contains $[0, 1]$. Therefore, for every $\beta < 1$ and for every $n \geq 1$,

$$\left(1 + \frac{1}{n}\right)^{\beta} - 1 \geq \frac{\beta}{2n}.$$

Therefore, we get

$$A^{2\gamma}(\beta) \left[\sum_{n=1}^{+\infty} \left(\left(1 + \frac{1}{n+1}\right)^{\beta} - 1\right)^{2\gamma}\right] \geq \left(\frac{\beta}{2^{\beta} - 1}\right)^{2\gamma} \frac{1}{2^{2\gamma}\left(2 + \frac{\left(\frac{3}{2}\right)^{\beta} - 1}{2^{\beta} - \left(\frac{3}{2}\right)^{\beta}}\right)^{2\gamma}} (\zeta(2\gamma) - 1).$$

All the terms from the right hand side are bounded from below away from 0 when β describe $[0, 1]$. This also holds for $\dfrac{\left(\frac{3}{2}\right)^{\beta} - 1}{2^{\beta} - \left(\frac{3}{2}\right)^{\beta}}$.

Let us set $H(Z) := \sum_{m=1}^{+\infty} \left(\dfrac{\left(1 + \frac{1}{m+a+1}\right)^{\alpha} - 1}{\frac{1}{2^{\beta}-1} + \left(1 + \frac{1}{m+a+1}\right)^{\alpha} - 1}\right)^{2\gamma} e^{-mZ}$. Note that $H(0)$ converges. Therefore we have to show that the term $H(0)$ goes to $+\infty$ when β goes to 0. Hence, we now analyze for $\beta > 0$ the function

$$S(\beta, \gamma) = S(\beta, \alpha, \gamma, a) = \sum_{m=1}^{\infty} \left(\frac{(1 + \frac{1}{m+a+1})^{\alpha} - 1}{(\frac{1}{2^{\beta}-1} + [(1 + \frac{1}{m+a+1})^{\alpha} - 1])}\right)^{2\gamma},$$

for fixed values of β, α, γ.

We remind the reader that when $\beta \to 0$ we have that $\alpha \to \infty$ and $a \to \infty$.

We are interested now in the upper bound.

Note that

$$S(\beta, \alpha) = \sum_{m=1}^{\infty} \left(1 - \frac{1}{1 + (2^{\beta} - 1)\,[(1 + \frac{1}{m+a+1})^{\alpha} - 1]}\right)^{2\gamma}.$$

Consider

$$u(\alpha,m,a) = (1+\frac{1}{m+a+1})^\alpha - 1 = e^{\alpha\,\log(1+\frac{1}{m+a+1})} - 1.$$

As $\log(x) \geq 1-\frac{1}{x}$, we get that

$$u(\alpha,m,a) \geq e^{\alpha\,\log(1+\frac{1}{m+a+1})} - 1 \,\geq e^{\alpha(1-\frac{1+m+a}{2+m+a})} - 1 \geq e^{\alpha(\frac{1}{2+m+a})} - 1.$$

In this way

$$S(\beta,\alpha) \geq \sum_{m=1}^{\infty} \left(1 - \frac{1}{1+(2^\beta-1)\,[e^{\alpha(\frac{1}{2+m+a})}-1]}\right)^{2\,\gamma}.$$

From elementary calculus we get that last summation is, up to a multiplicative constant, of the same order as the integral

$$\int_0^\infty \left(1 - \frac{1}{1+(2^\beta-1)\,[e^{\alpha(\frac{1}{2+t+a})}-1]}\right)^{2\,\gamma} dt.$$

Consider the change of variable $s = e^{\alpha(\frac{1}{2+t+a})} - 1$. Then

$$ds = -\frac{\alpha}{(2+t+a)^2} e^{\alpha(\frac{1}{2+t+a})}\,dt = -\frac{1}{\alpha}\,(s+1)\log^2(s+1)\,dt.$$

Note that when $t\to\infty$, we have $s\to 0$, and when $t\to 0$, we get that $s\to e^{\frac{\alpha}{2+a}}-1$.

We claim that when $\beta\to 0$, we get that $e^{\frac{\alpha}{2+a}}-1 \sim \frac{C}{\beta}$, for some constant C (again, we remind that $\sim$ means that the quotient goes to 1). Indeed, $(1+\frac{1}{1+a})^\alpha - 1 = \frac{1}{(3/2)^\beta-1} - \frac{1}{2^\beta-1}$ behaves like $\frac{\log(2)-\log(3/2)}{\log(2)\,\log(3/2)}\,\frac{1}{\beta} \;=\frac{1}{\beta}1.02361...$, when β goes to 0.

As $\alpha\log(1+\frac{1}{1+a}) \sim \frac{\alpha}{1+a}$, when a and α are large, then $e^{\frac{\alpha}{1+a}}-1 \sim \frac{C}{\beta}$.

Finally, from $\frac{\alpha}{2+a} = \frac{\alpha}{1+1+a} = \frac{\alpha}{1+a}\,\frac{1}{\frac{1}{1+a}+1}$, we get the claim.

We return to our main estimation.

After that change of variables we get for some fixed constants $0 < C' < C$

$$\int_0^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)\,s}\right)^{2\,\gamma} [\frac{1}{\alpha}\,(s+1)\log^2(s+1)]^{-1}\,ds \geq$$

$$\int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)\,s}\right)^{2\,\gamma} [\frac{1}{\alpha}\,(s+1)\log^2(s+1)]^{-1}\,ds.$$

It is easy to see that for any fixed $0\leq\gamma\leq 1$, and any s such that $C'/\beta \leq s \leq C/\beta$, the expression

$$\left(1 - \frac{1}{1+(2^\beta-1)\,s}\right)^{2\,\gamma},$$

uniformly in β, is bounded above and is far away from zero.

Therefore, up to a multiplicative positive constant

$$\int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)\,s}\right)^{2\,\gamma} [\frac{1}{\alpha}\,(s+1)\log^2(s+1)]^{-1}\,ds \geq$$

$$K\int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} [\frac{1}{\alpha}\,(s+1)\log^2(s+1)]^{-1}\,ds \geq$$

$$K\alpha[\frac{-1}{\log(s+1)}]_{\frac{C'}{\beta}}^{\frac{C}{\beta}} = K\alpha[\frac{1}{\log(\frac{C'}{\beta}+1)} - \frac{1}{\log(\frac{C}{\beta}+1)}] =$$

$$K\alpha\,[\frac{\log(\,\frac{(\frac{C}{\beta}+1)}{(\frac{C'}{\beta}+1)}\,)}{\log(\frac{C'+\beta}{\beta})\,\log(\frac{C+\beta}{\beta})}] \sim$$

$$K\frac{1}{\beta}\,\frac{C_3}{(A_1-\log(\beta))\,(A_2-\log(\beta))} \sim \frac{C_4}{\beta\,\log^2(\beta)}.$$

Therefore, for fixed γ, we have that $S(\beta,\gamma)\to\infty$ when $\beta\to 0$. □

Now, Propositions 5.2 and 5.3 prove Theorem C. The quantity $\beta_c(\gamma)$ is chosen such that for every w in 01, $\mathcal{L}_{0,\gamma,\beta_c(\gamma)}(\mathbb{1}_{01})(w) > 1$.

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