A dynamical proof for the convergence of Gibbs measures at temperature zero.

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Abstract

We give a dynamical proof of a result due to J. Brémont in [4]. It concerns the problem of maximizing measures for some given observable ϕ: for a subshift of finite type, and when ϕ only depends on a finite number of coordinates, it was proved in [4] that the unique Equilibrium State associated to $\beta\phi$ converges to some measure when β goes to $+\infty$. This measure has maximal entropy among the maximizing measures for ϕ. We give here a dynamical proof of this result and we improve it. We prove that for any Hölder continuous function (not necessarily locally constant), f, the unique Equilibrium State associated to $f + \beta\phi$ converges to some measure with maximal f-pressure among the maximizing measures. Moreover we also identify the limit measure.

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1 Introduction and statement of results.

1.1 Presentation.

In this article we deal with the so-called *ground state* in physic. In mathematic it concerns the problem of maximizing measures for some given observable. Let (Ω, φ) be some dynamical system, ϕ be some function from Ω to $\mathbb{R}$ and μ be some Equilibrium State associated to ϕ. With a physical point of view, the probability measure μ means that some equilibrium has been reached for the system. When the temperature decreases to zero, the equilibrium state changes: at the limit, some transitions disappear and some independent clusters appear.

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Recently, J. Brémont published a result in [4] on this topic; he proved that when (Ω, φ) is a semi-infinite mixing subshift of finite type, and when ϕ is locally constant, and if $\widehat{\mu}_{\beta\phi}$ denotes the unique Equilibrium State associated to $\beta\phi$ (and $\beta \in \mathbb{R}$), then the family $\widehat{\mu}_{\beta\phi}$ converges as β goes to $+\infty$ to some limit measure $\widehat{\mu}$. Here the parameter β denotes the inverse of the temperature (this parameter was denoted by t in [4]). It is known that this limit measure must be a maximizing measure for ϕ, *i.e.* $\int \phi\, d\widehat{\mu}$ *is maximal among the* φ*-invariant probabilities*. Moreover $\widehat{\mu}$ has maximal entropy among the maximizing measures.
Brémont's proof is based on a general statement of Analytic Geometry, hence the theorem can be seen as a general result for matrices and not only a result in ergodic theory. However, for a dynamical point of view, the proof does not explain the phenomenon. The first goal of our article is to give a dynamical proof of Brémont's result.
If the function ϕ is constant, then every invariant measure is a maximizing measure. However, the question becomes more difficult as soon as ϕ is not constant. For the Uniformly Hyperbolic case, and when ϕ is Hölder continuous, it is known that there exists some invariant compact set $\mathbb{K}_\phi$ such that a probability measure μ is a maximizing measure for ϕ if and only if its support is included in $\mathbb{K}_\phi$ (see [15] and [7]). Hence, knowing this compact set $\mathbb{K}_\phi$ gives information about the maximizing measures for ϕ. However, the general structure for $\mathbb{K}_\phi$ is not actually known; this implies that even the general Hölder case is difficult. It has been proved (see [10] and [2]) that for the symbolic case, and when ϕ is locally constant, $\mathbb{K}_\phi$ is a subshift of finite type. When $\mathbb{K}_\phi$ is irreducible, there exists a unique measure with maximal entropy. Hence, in that case, the limit measure is well identified. However, it can happen that $\mathbb{K}_\phi$ is not irreducible, and it has thus more than one measure with maximal entropy. This corresponds to the independent clusters we mentioned above. In that case Brémont's result doesn't give any information to identify the limit measure.
In our article we noticeably extend Brémont's result: for any Hölder continuous function f, we prove that $\widehat{\mu}_{f+\beta\phi}$ converges to some limit measure in $\mathbb{K}_\phi$. This limit measure is an Equilibrium State associated to the potential f (which was $\equiv 0$ in [4]) for the subshift $\mathbb{K}_\phi$; moreover the limit measure is well identified as some special barycenter between all the ergodic Equilibrium States associated to f in $\mathbb{K}_\phi$.

1.2 Statement of results.

Throughout, (Σ, σ) will denote an aperiodic subshift of finite type; ϕ will be some locally constant function from Σ to $\mathbb{R}$. We also pick some Hölder continuous function f (which has not to be locally constant), and we denote by $\widehat{\mu}_\beta$ the only Equilibrium State for Σ associated to $f + \beta\phi$. As we said above, $\mathbb{K}_\phi$ is the σ-invariant compact set such that a probability μ is maximizing for ϕ if and only if its support is included in $\mathbb{K}_\phi$. Then our result is the following:

Theorem. *The family of measures* $\widehat{\mu}_\beta$ *converges to some measure* $\widehat{\mu}_\infty$ *in* $\mathbb{K}_\phi$ *when* β

goes to $+\infty$*. Moreover* $\widehat{\mu}_\infty$ *is an Equilibrium State in* $\mathbb{K}_\phi$ *associated to the potential* f*. If* $\widehat{\mu}^1, \dots \widehat{\mu}^q$ *are the* q *ergodic Equilibrium States associated to* f *in* $\mathbb{K}_\phi$*, there exist* q *real numbers* $c_1, \dots, c_q$ *in* $[0,1]$ *which can be explicitly computed such that*

$$\lim_{\beta\to+\infty} \widehat{\mu}_\beta = \sum c_i \widehat{\mu}^i.$$

A more precise statement for the c_i will be given later (see section 4, Theorem' page **??**). Indeed, it would have been very long to exactly explain which irreducible components in $\mathbb{K}_\phi$ contribute the the limit measure, and how do they do.
Nevertheless, we have to put the notion of "explicit" in perspective: in our case $\mathbb{K}_\phi$ is quite simple to determine; thus the irreducible components (see section 2) are not so difficult to identify, and then, we can also identify the irreducible components which will have positive limit measure (see section 4 and theorem' page **??**).But the exact values for the c_i's seem difficult to compute, even if we can theoretically prove their existence.
Our proof is based on the classical tools in Dynamical Systems and Ergodic Theory. It may appear very technical, but it explains why the $\widehat{\mu}_\beta$'s converge. The key point is the induction. For every β and for every set A with positive $\widehat{\mu}_\beta$-measure we have

$$\widehat{\mu}_\beta(A) = \frac{1}{\int r_A \, d\nu_\beta},$$

where ν_β is the conditional measure $\dfrac{\widehat{\mu}_\beta(.\cap A)}{\widehat{\mu}_\beta(A)}$ and r_A is the first return time in A. The idea is just to identify which A can have positive measure at the limit, and to prove that for these A there is convergence.
Actually, we do not know if induction is really needed for a dynamical proof of the Theorem. However we think it makes things easier.
On one hand, induction allows us to control the Gibbs constant. The unique Equilibrium State is also defined as the unique invariant probability measure such that

$$e^{-C} \le \frac{\widehat{\mu}_\beta(C_n(x))}{e^{S_n(f)(x)-\beta S_n(\phi)(x)-n\mathcal{P}_\beta}} \le e^C$$

for some constants C and $\mathcal{P}_\beta$ and for every n and every x (where $C_n(x)$ is a n-cylinder, see section 2). But the constant C deeply depends on β, and goes $+\infty$ as β goes to $+\infty$. Induction and the method which was introduced in [13] allow us to get some control on the constant C when we only consider the induced Dynamical System.
On the other hand, periodicity plays an important role in the problem of maximizing measures. Even if the Birkhoff average of the function ϕ can fluctuate along a periodic orbit, it is fixed at each complete-period. Then induction allows us to control what happens at each period without controlling what happens "during" a

period.
We also want to mention that our method has some chances to be extend to the general case of ϕ Hölder-continuous and not locally constant. For this general case, the first problem would be to know the structure of $\mathbb{K}_\phi$. However if one can prove that $\mathbb{K}_\phi$ has some nice structure, we believe that our method could be used, as soon as we can identify which A can have positive measure at the limit.

In [4] it is also proved that the number of maximizing measures is finite when ϕ describes the set of functions which only depend on a fixed number of coordinates. This fact can also be checked in our proof:

Corollary. *There are only finitely many maximizing measures with maximal f-pressure when ϕ describes the functions constant on every 1-cylinders*[1]*.*

Outline of the article. In section 2 we precise some vocabulary and hypotheses we can make for the functions ϕ and f. In section 3 we recall some ideas from [13] which where improved in [11] about induction and reduced dynamical systems. In section 4 we explain how $\mathbb{K}_\phi$ looks, and explain which irreducible components have necessarily null limit measure. In section 5 we define two parameters, the transition cost and the isolation rate, which determine which components have positive limit measure. In section 6 we finish to prove the Theorem and we prove the corollary on the finiteness.

2 Notations, hypotheses and spectral decomposition for subshifts.

2.1 Notations.

The set of vertices of the defining graph of (Σ, σ) is $\{1, \dots, N\}$ with $N \geq 2$. We denote by $\mathcal{A} = (a_{ij})$ the $N \times N$-transition (aperiodic) matrix associated to Σ; namely points in Σ are sequences $x = \{x_n\}_{n \in \mathbb{Z}}$ such that for every n, x_n belongs to $\{1, \dots, N\}$ and

$$a_{x_n x_{n+1}} = 1.$$

• In Σ, the **cylinder** $[i_k, \dots, i_{k+n}]$ will denote the set of points $x \in \Sigma$ such that $x_j = i_j$ (for every $k \leq j \leq k+n$). Such a cylinder will also be called a word (of length $n+1$) or equivalently a $(k, k+n)$-cylinder. If x is in Σ, the set $C_n(x)$ will denote the cylinder $[i_0, \dots, i_{n-1}]$ such that $x_j = i_j$ (for every $0 \leq j \leq n-1$. It will also be called the n- cylinder containing x.
• In Σ we **define the map** $[\ ;\ ]$ in the usual way: let $x = (x_n)$ and $y = (y_n)$ be two points in Σ such that $x_0 = y_0$. Then, $[x; y]$ denotes the point (z_n) such that $z_n = x_n$ for every $n \geq 0$ and $z_n = y_n$ for every $n \leq 0$.
• Let $\underline{x} = [x_0, \dots, x_p]$ and $\underline{y} = [y_0, \dots, y_q]$ be two finite words such that $x_p = y_0$.

[1] see section 2 for the definition of the 1-cylinder

Then $[\underline{y}, \underline{x}]$ denotes the finite word $[x_0, \ldots, x_p, y_1, \ldots, y_q]$. This operation is called the **concatenation** of the two words $\underline{x}$ and $\underline{y}$. Notice that our concatenation is not the usual one. As we are in a subshift (and not in the full-shift) some transitions are forbidden. That is why we ask for the last letter of the word $\underline{x}$ to be the first one of the word $\underline{y}$; when we concatenate these two words, this "double" letter appears only on time.
• We introduce a new operation on words, called a **looping**. Let $\underline{x} = [x_0, \ldots, x_p]$ and $\underline{y} = [y_0, \ldots, y_q]$ be two finite words such that $x_i = y_0 = y_q$. Then we introduce the word $\underline{y}$ in $\underline{x}$ in the following way :

$$[x_0, \ldots, x_{i-1}, y_0, \ldots, y_q, x_{i+1}, \ldots, x_p].$$

For any 1-cylinder $[i]$, and for any x in $[i]$ we set $W^u(x) \stackrel{def}{=} [[i]; x]$ and $W^s(x) \stackrel{def}{=} [x; [i]]$. These sets are respectively called the unstable local leaf and the stable local leaf of x. Namely points in the unstable local leaf of x have the same symbols backward, and points in the stable local leaf have the same symbols forward.

For any Hölder continuous function θ on Σ, $\mathcal{P}_\theta$ will denote the topological pressure of the Dynamical System (Σ, σ) associated to the potential θ. It will also be called the θ-pressure of Σ. In the same way if μ is some σ-invariant probability the term $h_\mu(\sigma) + \int \theta \, d\mu$ will be called the θ-pressure of the measure μ. Thus, the unique Equilibrium State in Σ associated to the potential θ is also the unique σ-invariant probability with maximal θ-pressure. For simplicity, $\mathcal{P}_\beta$ and $\beta \in \mathbb{R}$ will denote $\mathcal{P}_{f+\beta\phi}$.

2.2 Hypotheses for ϕ

The function ϕ is locally constant in the compact set Σ, which means that it only depends on a finite number of coordinates. Hence, there exists some integers p and p_- such that ϕ is constant on every $(p_-, p_- + p)$-cylinders. However, up to the fact that we use a higher-block representation of Σ (see [14]), every function which only depends on a finite number of coordinates in the Symbolic Space can be viewed as a function which only depends on one coordinate. Therefore, we will assume that ϕ is constant on every 1-cylinders. Notice that we can make this assumption without lost of generality.
Moreover if C is some real number, a σ-invariant measure μ is a maximizing measure for ϕ if and only if it is a maximizing measure for the function $\phi + C$. Therefore, we also can assume that ϕ is a positive function on Σ.
From now on till the end of the paper, ϕ is a positive function which is constant on every 1-cylinder.

2.3 Hypothesis for f

We assume that f does not depend on the past. Namely we assume that if $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$ and $y = (y_i)_{i\in\mathbb{Z}} \in S$ satisfy $x_i = y_i$ for all $i \geq 0$, then $f(x) = f(y)$.

This assumption for f is free. Indeed for any Hölder continuous function, there exists some cohomologous function with this property. Equilibrium state for one function is also an equilibrium state for the other. In the following (section 3) some function ω will be introduced (for the general case). We let the reader check (see [3] for instance) that the function ω is strongly related to the cohomologous function associated to f.However in the 2 last sections, there are long strong and complicated estimations and computations. We thus prefer to make things as simple as possible. This is the reason of this hypothesis.

2.4 Spectral decomposition.

We recall some properties of the spectral decomposition for symbolic dynamic. These properties where proved in [1], and are given as exercises in [12] on page 55. This spectral decomposition will be used in section 4.
Let B be a $n \times n$ transition matrix, *i.e.* a matrix whose entries are only 0 or 1. We assume that B has at least one 1 in each row and in each column. We denote by Σ_B the set of bi-infinite sequences $\underline{y} = (y_i)$ such that for every i $b_{y_i,y_{i+1}} = 1$, and by σ the shift on Σ_B. A symbol i is said to be *essential* if there exists some periodic point in $[i]$; two essential symbols i and j are said to be *equivalent* if there exists in Σ_B some word which has two times the symbol i and one time between the two i the symbol j (these symbols are not necessary consecutive). Any point in the ω-limit set of (Σ_B, σ) contains only essential symbols, and the set of essential symbols splits into disjoint subsets of mutually equivalent symbols. Therefore, the ω-limit set splits in disjoint σ-invariant subsets, Σ^ω_k ($k = 1, \ldots, K$ for some integer K), and in each subset there exists some dense positive semiorbit. Moreover, each Σ^ω_k can be decomposed in closed disjoint $M(k)$ subsets (for some integer $M(k)$)

$$\Sigma^\omega_k = \Sigma^\omega_{k,1} \sqcup \ldots \sqcup \Sigma^\omega_{k,M(k)},$$

such that $\sigma(\Sigma^\omega_{k,i}) = \Sigma^\omega_{k,i+1}$ (with the convention that $M(k)+1 = 1$) and the restriction of $\sigma^{M(k)}$ to each $\Sigma^\omega_{k,i}$ is topologically mixing.
The sets Σ^ω_k are called the irreducible components of Σ_B. The topological pressure of Σ_B associated to some potential θ is thus the maximal of all the θ-pressures of the irreducible components of Σ_B. Any invariant measure such that its θ-pressure is equal to the maximal of the θ-pressures of the components is an Equilibrium States for Σ_B. Moreover there exists only one ergodic (=extremal) Equilibrium State in each irreducible component with maximal θ-pressure.

3 Inductions and local equilibrium states.

In this section, we briefly recall some relevant results from [13] and [11] that we are going to use. Then we start to define and to study some good reduced systems. The main idea is the following : if A is a subset in Σ, using the first return time map,

A can be seen as a section in a flow. Then, scaling the return time by some real parameter S we define a family of flows. For each such a flow we construct some Equilibrium State; this gives a one-parameter family of measures. Among these measures, one (and only one) gives the global Equilibrium State. Notice that in the following the result and the tools are important for the proof of the Theorem.

3.1 Reduced Dynamical Systems.

We pick some 1-cylinders $[i_1], \dots [i_l]$ and in each cylinder we pick some unstable local leaf, $F_j := [[i_j]; x_j]$ for some point x_j in $[i_j]$. We set $F := \sqcup_j F_j$ and $A \stackrel{def}{=} \sqcup [i_j]$. The map $[.; x_j]$ is a projection from $[i_j]$ onto F_j that we denote by π_F. We denote by g the first return map in A by iterations of σ, and g_F is the map $\pi_F \circ g$. This defines a new dynamical System (F, g_F). Let θ be some Hölder continuous function on Σ. For x in F_j and for any x' in $[i_j]$ such that $\pi_F(x') = x$, we set

$$\omega(x, x') = \sum_{k=0}^{+\infty} \theta \circ \sigma^k(x) - \theta \circ \sigma^k(x') \ .$$

The map θ is Hölder continuous, and so, by contraction on the stable local leaves, the previous series converges. For x in F we set $\omega(x) = \omega(g_F(x), g(x))$, and we denote by $r_A(x)$ its first return time. Then we set

$$\Theta(x) = \sum_{k=0}^{r_A(x)-1} \theta \circ \sigma^k(x) + \omega(x) \ .$$

This function is defined on a set of full measure with respect to any invariant measure. The inverses branches of g_F define the family of so-called n-sets: for x in F we set

$$K_n(x) \stackrel{def}{=} \sigma^{-r_A^n(x)}(W^u(g^n(x)),$$

where $r_A^n(x)$ denotes the n^{th}-return time into A. For a given point in F, the n-sets are well-defined except for the points in F which do not return infinitely many times in A. But these points will have null-measure for all the measures we are going to consider. Hence, every n-set is a compact set and the collection of the n-sets defines a partition of F (up to the points which come back less than n times), which refines the partition in $(n-1)$-sets. We define the set of preimages of some point x in F by g_F^n, denoted by $Pre_n(x)$.

Remark 1. An important property is that all the points in the same F_j have the same "number" of preimages. Namely, if x and y are in the same F_j every n-set which contains some x', such that $g_F^n(x') \stackrel{def}{=} \pi_F \circ \sigma^{r_A^n(x')}(x') = x$, must also contain some y' such that $g_F^n(y') = \pi_F \circ \sigma^{r_A^n(x')}(y') = y$ (and $r_A^n(x') = r_A^n(y')$).

We define the Ruelle-Perron-Frobenius operator for any x

$$\mathcal{L}_S(T)(x) = \sum_{y \in Pre_1(x)} e^{\Theta(y) - r_A(y)S} T(y) \ ,$$

where S is a real parameter and T is a continuous function from F to $\mathbb{R}$. When A is a single 1-cylinder it is proved in [13] that there exists some critical value for S, such that

$$\mathcal{L}_S(\mathbb{1}_F)(x) < +\infty \quad \text{for every } S > S_c \text{ and for every } x \in F\,. \tag{1}$$

S_c is of course defined as the smallest real number with this property. Always in that case, the Markov property and the hyperbolic structure give the next important lemma:

Lemma 3.1. *There exists a positive constant C_θ which does not depend on S such that for all $x, y \in F$, $S > S_c$ and integer n*

$$\frac{1}{C_\theta}\mathcal{L}_S^n(\mathbb{1}_F)(x) \le \mathcal{L}_S^n(\mathbb{1}_F)(y) \le C_\theta \mathcal{L}_S^n(\mathbb{1}_F)(x).$$

However, the Hölder properties of θ prove that in the general case (when A is a union of several cylinders) a similar result than lemma 3.1 holds but for each F_j.

Lemma 3.2. *There exists a positive constant C_θ which does not depend on S such that for every F_j for every $x, y \in F_j$, and for every $S > S_c$ and integer n*

$$\frac{1}{C_\theta}\mathcal{L}_S^n(\mathbb{1}_F)(x) \le \mathcal{L}_S^n(\mathbb{1}_F)(y) \le C_\theta \mathcal{L}_S^n(\mathbb{1}_F)(x).$$

The independence on S of the constants C_θ in lemma (3.1) and (3.2) results from remark 1: for x and y in the same F_i (or simply in F when A is a single 1-cylinder) we associate to each preimage of x a unique preimage of y in the same 1-set. Two such preimages have the same return time r_A,which "remove" the dependence on S. Then, we simply use the Hölder continuity of θ to get the constant C_θ.

There also exists some critical values $S_c(j)$ such that for every j, $\mathcal{L}_S(\mathbb{1}_F)$ converges for every points in F_j if and only if it converges for one point in F_j. Moreover, in [11] and when A is a single cylinder, an explicit value is given for S_c . For the general case it is also proved that all the $S_c(j)$ are equal, which allows us to talk about S_c even in the general case:
Let $\mathcal{B}$ be the matrix $\mathcal{A}$ where the rows and the lines associated to the cylinders which compose A have been removed. Then S_c is the topological pressure associated to the potential θ for the subshift of finite type (Σ_B, σ). An important point is that this topological pressure is strictly smaller than the topological pressure of (Σ, σ) (see [6] or [9]).

Then we can produce local Equilibrium States:

Proposition 3.3. *There exist some measure m_S on F, some positive real number λ_S and some positive continuous function H_S on F such that*

1. $\mathcal{L}_S^*(m_S) = \lambda_S m_S$;
2. $\lambda_S \stackrel{def}{=} \int \mathcal{L}_S(\mathbb{1}_F)\, dm_S$;
3. $\mathcal{L}_S(H_S) = \lambda_S.H_S$.

See [13] for the proof. We recall that H_S is defined by the formula:

$$H_S \stackrel{def}{=} \lim_{n\to+\infty} \frac{1}{n} \sum_{0\le k<n} \frac{1}{\lambda_S^n} \mathcal{L}_S^n(\mathbb{1}_F).$$

Remark 2. When A is a single 1-cylinder, lemma 3.1 and the constructions of m_S and H_S prove that for every x in the chosen unstable leaf F

$$e^{-C_\theta} \le H_S(x) \le e^{C_\theta}, \tag{2}$$

for some positive real constant C_θ which does only depend on θ. Moreover there exists some C'_θ which does not depend on S such that for every S, $||H_S||_{H\ddot{o}l} \le C'_\theta$, where $||H_S||_{H\ddot{o}l}$ denotes the Hölder norm for the Hölder coefficient of θ.

Let us set $d\nu_S \stackrel{def}{=} H_S dm_S$. The measure ν_S is g_F-invariant. Moreover we have

Lemma 3.4. *The measure ν_S is ergodic. Moreover, λ_S is a simple eigenvalue for $\mathcal{L}_S$.*

Proof. Let us pick some $F_i \subset F$. The mixing properties of σ proves that F_i must have positive ν_S-measure: there exists at least one F_j with positive ν_S-measure and there exists at least one set $K(i,j) \subset F_i$ whose image by some iterate of g_F is exactly F_j. Now the measure ν_S is conformal in the sense that

$$\nu_S(K(i,j)) = \int_{F_j} \frac{H_S(g_F^{-n}(x))}{H_S(x)}\, e^{S_n(\Theta)(g_F^{-n}(x)) - r_A^n(g_F^{-n}(x)) - n\log\lambda_S} dm_S, \tag{3}$$

where n is the integer such that $g_F^n(K(i,j)) = F_j$ (bijectively).
The measure ν_S is g_F-invariant, thus almost every point in F_i returns infinitely many times into F_i by iteration of g_F. Let us denote by G_i this first return map from F_i into F_i. The conditional measure $\nu_S^i \stackrel{def}{=} \frac{\nu_S(.\cap F_i)}{\nu_S(F_i)}$ is G_i-invariant and conformal in the same sense than (3) (with $F_j = F_i$ and $g_F^n = G_i$). Therefore, the density theorem proves that ν_S^i is exact, hence mixing and ergodic.
Now, any g_F-invariant Borel set with positive ν_S-measure has positive ν_S^i-measure; its intersection with F_i is G^i-invariant, and have thus full ν_S^i-measure. Hence, any g_F-invariant Borel set with positive ν_S-measure has full ν_S-measure.

Let us assume that H is some Hölder continuous function on F such that $\mathcal{L}_S(H) = \lambda_S.H$. Let F_j be fixed, and let $\mathcal{L}_{S,j}$ be the Ruelle-Perron-Frobenius operator for F_j (namely with $F := F_j$). Then, writing $\mathcal{L}_S(H)(x) = \lambda_S H(x)$ for some x in F_j, we can use the global equality $\mathcal{L}_S(H) \equiv H$ in a recursive way to transform the sum $\mathcal{L}_S(H)(x)$ in a sum which only uses n-sets in F_j. Hence we get

$$\widetilde{\mathcal{L}}_{S,j}(H)(x) = \widetilde{\lambda}_S H(x),$$

for some new Ruelle-Perron-Frobenius operator on F_j. But such an operator has a simple dominating eigenvalue (due to the mixing property of the associated Gibbs measure). This finally proves that λ_S is a simple eigenvalue for $\mathcal{L}_S$. □

The measure ν_S is the unique Equilibrium State for (F, g_F) associated to $\Theta(\cdot) - S.r_A(\cdot)$. The natural extension of ν_S, denoted by ν'_S is the unique Equilibrium State associated to the potential

$$\sum_{k=0}^{r_A(x)-1} \theta \circ \sigma^k(x) - S.r_A(x)$$

for the Dynamical System (A, g). Moreover, there exists some positive constant C_1 (which can depend on S) such that for every x in A and for every integer n,

$$e^{-C_1} \leq \frac{\nu'_S(\pi_F^{-1}(K_n(\pi_F(x))))}{e^{S_{r^n_A(x)}(\theta)(x) - r^n_A(x)S - n\log\lambda_S}} \leq e^{C_1} \tag{4}$$

By definition of the natural extension, the numerator of the middle term in (4) is simply $\nu_S(K_n(\pi_F(x)))$.

Lemma 3.5. *The constant C_1 does not depend on S as soon as A is a single 1-cylinder (and of course for a given θ).*

Proof. Let $K_n(x)$ be any n-set (where x is a point in F). Then by definition we have $g^n_F(K_n(x)) = F$. Hence we get:

$$\begin{aligned} \nu_S(K_n(x)) &= \int 1\!\!1_{K_n(x)}\, d\nu_S \\ &= \int 1\!\!1_{K_n(x)} H_S\, dm_S. \end{aligned}$$

Then (2) gives $e^{-C_\theta} \leq \dfrac{\nu_S(K_n(x))}{\int 1\!\!1_{K_n(x)}\, dm_S} \leq e^{C_\theta}$. Now $\mathcal{L}^*_S(m_S) = \lambda_S.m_S$ gives

$$\int 1\!\!1_{K_n(x)}\, dm_S = \frac{1}{\lambda_S^n} \int \mathcal{L}^n_S(1\!\!1_{K_n(x)})\, dm_S.$$

But every y in F admits a unique preimage by g^n_F in $K_n(x)$. Notice that ω is upper bounded by some function which only depends on θ. Moreover, for every y in F,

$$\mathcal{L}^n_S(1\!\!1_{K_n(x)})(y) = e^{S_{r^n_A(x)}(\theta)(y') + \omega(y') - r^n_A(x)S},$$

where y' is the preimage of y by g^n_F. Thus C_1 depends only on θ. □

It is proved in [13] that for every $S > S_c$, $\mathbb{E}_{\widetilde{\nu_S}}[r_A] < +\infty$. This is a simple consequence of lemma 3.2 (with $n = 1$) and of the fact that $\lambda_S = \int \mathcal{L}_S(\mathbb{1}_F)\, d\nu_S$. Therefore the measure ν'_S can be opened out: there exists a σ-invariant and ergodic probability measure μ_S on Ω such that

$$\frac{\mu_S(.\cap A)}{\mu_S(A)} = \nu'_S(.).$$

Let P_S denote the θ-pressure of this measure μ_S. Then we have

$$P_S = h_{\mu_S}(f) + \int \theta\, d\mu_S = S + \mu_S(A)\log(\lambda_S). \tag{5}$$

A very important point for the following, is that for $S = \mathcal{P}_\theta$, $\lambda_S = 1$ and μ_S is the unique Equilibrium State in Σ associated to θ.

3.2 Some other properties for the operators $\mathcal{L}_S$.

As we said before, it is proved in [11] that S_c is the topological pressure of the System with the hole A, that is, the dynamical system in Σ of all points which do never belong to A. It is also proved that for $S = S_c$, $\mathcal{L}_S(\mathbb{1})$ diverges (for every x). Moreover we have

Lemma 3.6. *The map $\psi : S \mapsto \log(\lambda_S)$ is strictly convex and analytic on a complex neighborhood of $]S_c, +\infty[$). Moreover there exists some p such that for every S in the complex neighborhood where ψ is analytic, $\mathcal{L}_S$ has p simple dominating eigenvalues. We also have*

$$\lim_{S\to S_c} \frac{\psi'(S)}{\psi(S)} = -\infty.$$

Proof. Analyticity and strict convexity are proved in [11]. First, λ_S is a simple eigenvalue. Copying the proof of proposition 4.11 in [5] we prove that the operator $\mathcal{L}_S$ has $p(S)$ simple eigenvalues. These eigenvalues are equal to $\lambda_S.e^{\frac{2i\pi k}{p(S)}}$, where $k = 0, \dots, p(S) - 1$. Hence, $\mathcal{L}_S$ is a quasi compact operator with simple isolated dominating eigenvalues; analyticity (in some complex neighborhood of $]S_c, +\infty[$) is thus obtained via the perturbation Theorem from [8] (see Th III.8).
Now, the continuity of the map $S \mapsto p(S)$ on the connected set $]S_c, +\infty[$ means that it must be constant.
Strict convexity result from the uniqueness of the analytic continuation of analytic function and from the fact that λ_S diverges as S goes to S_c.
It remains to prove the last point, $\displaystyle\lim_{S\to S_c} \frac{\psi'(S)}{\psi(S)} = -\infty$. Let us assume that there exists $l > 0$ and some decreasing sequence (x_n) which converges to S_c such that $\psi'(x_n) \geq -l\psi(x_n)$. Let a be such that $0 < a - S_c < \frac{1}{2l}$. Then we must have for $S_c < x_n < a$,

$$-\frac{\psi(a) + (x_n - a)\psi'(x_n)}{\psi'(x_n)} \geq \frac{1}{l}.$$

Letting $n \to +\infty$ in this inequality, we arrive to some contradiction. $\square$

Lemma 3.7. *The map ψ satisfies $\psi'(S) = -\frac{1}{\mu_S(A)}$.*

See [11] for a proof. Another useful fact is proved in [11]: for the integer p which was defined in lemma 3.6 we have

$$\mathcal{L}_S(\mathbb{1}_F)^{np}(x) = \lambda_S^{np}\mathbb{1}_F(x) + \lambda_S^{np}\Psi^n(\mathbb{1}_F)(x), \tag{6}$$

where Ψ is some operator with spectral radius strictly smaller than 1. This fact will be used later. Finally, we give some consequence for the map $S \mapsto \lambda_S$:

Lemma 3.8. *The map $S \mapsto \lambda_S$ is a decreasing bijection from $]S_c, +\infty[$ onto $]0, +\infty[$.*

3.3 Applications and consequences for good θ.

Let P be any real number. We use the previous work with $\theta := f - P\phi$. An important point here is that ϕ is constant on every 1-cylinders, and so the constant which appeared and which where only depending on θ are now only depending on f. Moreover the map $\omega(x, x') = \sum_{k=0}^{+\infty} f \circ \sigma^k(x) - f \circ \sigma^k(x')$ is null because f does not depend on the future. We again set $\Theta(x) = \displaystyle\sum_{k=0}^{r_A(x)-1} f \circ \sigma^k(x)$. The new definition of the operator $\mathcal{L}_S$ is:

$$\mathcal{L}_{S,P}(T)(x) = \sum_{y \in Pre_1(x)} e^{\Theta(y) - r_A^1(y)S - P.S_{r_A(y)}(\phi)(y)} T(y).$$

Hence, as soon as $\mathcal{L}_{S,P}(\mathbb{1}_F)$ converges, proposition 3.3 is still valid, except that we have measures $m_{S,P}$ and $\nu_{S,P}$, an eigenvalue $\lambda_{S,P}$ and a function $H_{S,P}$. In the same way, lemma 3.4 and equality (6) also hold, except that we always have to add P in subscript to indicate the dependence in P.

Proposition 3.9. *There exist two convex and decreasing functions $S \mapsto P(S)$ and $S \mapsto P_l(S)$ such that*

- *For every S and P, $\mathcal{L}_{S,P}(\mathbb{1}_F)$ converges if and only if $P > P_l(S)$.*
- *For every S, the $\lambda_{S,P(S)} = 1$.*
- *The map $S \mapsto P(S)$ is real-analytic.*

Proof. We first see that for a given P, the operator $\mathcal{L}_{S,P}$ is well defined (*i.e.* $\mathcal{L}_{S,P}(\mathbb{1}_F)$ converges) for sufficiently large S in $\mathbb{R}$. Namely $\mathcal{L}_{S,P}$ is well defined as soon as $S > \mathcal{P}_{f-P\phi}(\Sigma_A)$, where Σ_A is the set of points in Σ which never enter in A under the action of σ, and $\mathcal{P}_{f-P\phi}(\Sigma_A)$ is the topological pressure of the Dynamical System (Σ_A, σ) associated to the potential $f - P\phi$. It is clear that $P \mapsto \mathcal{P}_{f-P\phi}(\Sigma_A)$ is decreasing, and it is also well known that such a map is convex. Now, P_l is exactly the inverse of the map $P \mapsto \mathcal{P}_{f-P\phi}(\Sigma_A) = S_C(P)$. Therefore, P_l is a decreasing convex map.

We now assume that for a given P, S is always bigger than $\mathcal{P}_{f-P\phi}(\Sigma_A)$ or equivalently, that for a given S, P is always bigger than $P_l(S)$.

Now, we let the reader check that the map $P \mapsto \mathcal{L}_{S,P}$ is analytic; we can again use the perturbation theorem (see again [8]) to conclude that the map $P \mapsto \lambda_{S,P}$ is analytic in some complex neighborhood of its real interval of definition. Moreover, for a given S, we have

$$\lim_{P\to+\infty} \lambda_{S,P} = 0 \text{ and } \lim_{P\to P_l(S)} \lambda_{S,P} = +\infty.$$

Now, the revisited (6) proves that

$$\frac{\partial \log \lambda_{S,P}}{\partial P} = -\frac{1}{\int \phi\, d\nu_{S,P}}.$$

Thus the map $P \mapsto \lambda_{S,P}$ is decreasing and continuous; there exists some unique value $P = P(S)$ such that $\lambda_{S,P(S)} = 1$. Moreover the implicit mapping theorem for holomorphic functions in several complex variables (see [17]) proves that the function $S \mapsto P(S)$ is analytic.

Let us consider some fixed S. By definition of $P(S)$, we have $P > P(S)$ if and only if $\lambda_{S,P} < 1$. Let us pick some $S' \neq S$ and $\alpha \in]0,1[$. For convenience we set $P = P(S)$ and $P' = P(S')$. We want to prove that $P(\alpha S + (1-\alpha)S') \leq \alpha P + (1-\alpha)P'$, which amounts to prove that

$$\lambda_{\alpha S+(1-\alpha)S', \alpha P+(1-\alpha)P'} \leq 1. \tag{7}$$

Let x be in F and n be some integer. Let us set $S'' = \alpha S + (1-\alpha)S'$ and $P'' = \alpha P + (1-\alpha)P'$. Then we have:

$$\begin{aligned}
\mathcal{L}^n_{S'',P''}(\mathbb{1}_F)(x) &= \sum_{y\in Pre_n(x)} e^{S_n(\Theta)(y) - S'' r^n_A(y) - P'' S_{r^n_A(y)}(\phi)(y)} \\
&= \sum_{y\in Pre_n(x)} e^{\alpha S_n(\Theta)(y) - \alpha S r^n_A(y) - \alpha P S_{r^n_A(y)}(\phi)(y)} e^{(1-\alpha)S_n(\Theta)(y) - (1-\alpha)S' r^n_A(y) - (1-\alpha)P' S_{r^n_A(y)}(\phi)(y)} \\
&\leq \left(\sum_{y\in Pre_n(x)} e^{S_n(\Theta)(y) - S r^n_A(y) - P S_{r^n_A(y)}(\phi)(y)} \right)^{\alpha} \left(\sum_{y\in Pre_n(x)} e^{S_n(\Theta)(y) - S' r^n_A(y) - P' S_{r^n_A(y)}(\phi)(y)} \right)^{1-\alpha},
\end{aligned}$$

where the last inequality follows from Hölder Inequality. The Birkhoff sum $S_n(\Theta)$ denotes the Birkhoff sum for the map g_F. Notice that $S_{r^n_A(y)}(\phi)$ is the Birkhoff sum for the map σ. Now exchanging n by np, and using equality (6) revisited, we can deduce

$$\log \lambda_{S'',P''} \leq 0,$$

which means that (7) holds.
The fact that the function $S \mapsto P(S)$ is decreasing follows from

$$\frac{dP(S)}{dS} = -\frac{\frac{\partial \log \lambda_{S,P}}{\partial S}}{\frac{\partial \log \lambda_{S,P}}{\partial P}} = \frac{-1}{\int \phi \, d\widetilde{\mu}_S} \tag{8}$$

where $\widetilde{\mu}_S$ is the opened-out σ-invariant measure associated to $\nu_{S,P(S)}$. □

We can now use all our machinery to recognize the measures $\widehat{\mu}_\beta$:

Proposition 3.10. *The only Equilibrium State associated to $f + \beta\phi$, $\widehat{\mu}_\beta$, is also the opened-out measure $\widetilde{\mu}_S$ when $\beta = -P(S)$.*

Proof. We first recall that any open set have positive Gibbs measure (for every potential). Let μ be some σ-invariant measure. We assume $\mu(A) > 0$. Then we can define the conditional measure $\nu := \dfrac{\mu(. \cap A)}{\mu(A)}$. This measure is g-invariant. For any β we have

$$h_\mu(\sigma) + \int f \, d\mu + \beta \int \phi \, d\mu = \mu(A) \left(h_\nu(g) + \int S_{r_A}(f) \, d\nu + \beta \int S_{r_A}(\phi) \, d\nu \right).$$

For $t = -P(S)$, the previous equality gives

$$h_\mu(\sigma) - S + \int f \, d\mu - P(S) \int \phi \, d\mu = \mu(A) \left(h_\nu(g) + \int S_{r_A}(f) - P(S) S_{r_A}(\phi) - S.r_A \, d\nu \right).$$

Now, $\nu_{S,P(S)}$ is the only Equilibrium State for (F, g_F) associated to $S_{r_A}(f) - P(S) S_{r_A}(\phi) - r_A.S$. This proves that the term in the right side of the previous equality is lower than $\mu(A) \log \lambda_{S,P(S)} = 0$, with equality if and only if $\pi_F(\nu) = \nu_{S,P(S)}$. □

4 Components without limit measure

In this section we first briefly recall the structure of the compact set $\mathbb{K}_\phi$. Then we precise the asymptotic behavior for $S \mapsto P(S)$ and $S \mapsto P_l(S)$. Hence we prove that every accumulation point for $\widehat{\mu}_\beta$ must be one Equilibrium State in $\mathbb{K}_\phi$. Finally, we give a necessarily condition on the irreducible components to have positive limit measure.

4.1 Structure for $\mathbb{K}_\phi$.

We first introduce the notion of minimal orbit:

Definition 4.1. *A periodic point x in Σ will is said to be minimal (or equivalently to have minimal period) if no 1-cylinder contains more than one element of the orbit of x.*

From [16], p.57, we have $\sup_{\mu}\int \phi d\mu = \sup_{x\in\Sigma}\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi\circ\sigma^j(x)$. We denote by $\rho(\phi)$ this supremum. Because the periodic points are dense in Ω, we also have

$$\rho(\phi) = \sup_{x\in\Sigma \text{ and } n\in\mathbb{N}^*,\ \sigma^n(x)=x}\frac{1}{n}\sum_{j=0}^{n-1}\phi\circ\sigma^j(x).$$

Now, each periodic orbit can be decomposed in "loops" of minimal orbits. Moreover, ϕ is constant on each 1-cylinder, thus $\rho(\phi)$ is also equal to the maximum of the Birkhoff averages on the set of minimal periodic orbits. Let us pick all the minimal periodic orbits x with period $n = n(x)$ such that

$$\frac{1}{n}(\phi(x) + \ldots + \phi\circ\sigma^{n-1}(x)) = \rho(\phi).$$

Each such a x gives a finite word $\underline{x}$ (of length n+1). This finite set of words is called a generating set for the compact set $\mathbb{K}_\phi$ that we are going to define. Now, the set $\mathbb{K}'_\phi$ is the set of bi-infinite words that we can construct in a recursive way by concatenation and looping of all the words of the generating set; the set $\mathbb{K}_\phi$ is the closure of $\mathbb{K}'_\phi$.
We can decompose $\mathbb{K}_\phi$ in irreducible components, as it is explained in section 2.

Lemma 4.2. *Let i be any symbol such that $[i]\cap\mathbb{K}_\phi \neq \emptyset$. Then i is essential. Moreover $\mathbb{K}_\phi$ equals its ω-limit set.*

Proof. There exists some point x in $\mathbb{K}_\phi\cap[i]$. Thus, by construction of $\mathbb{K}_\phi$, there exists some periodic point whose associated word contains i. Hence i is essential. Again by construction of $\mathbb{K}_\phi$, periodic points are dense in $\mathbb{K}_\phi$. The ω-limit set of $\mathbb{K}_\phi$ is a compact set which contains the periodic points. □

Remark 3. An important consequence is that two different irreducible components in $\mathbb{K}_\phi$ cannot share any symbol. For the same reason, this also holds for two different components in the same transitive component. For the rest of the proof, we will say that a symbol appears in $\mathbb{K}_\phi(l)$ if the 1-cylinder $[i]$ has an non-empty intersection with the irreducible component $\mathbb{K}_\phi(l)$ of $\mathbb{K}_\phi$.

4.2 Asymptotic behavior for $S\mapsto P(S)$ and $S\mapsto P_l(S)$.

As it is done in the previous section, we pick some $A = \sqcup[i_j]$ and define F_j and $\mathcal{L}_{S,P}$ and so on, as above. We notice that the function ϕ is upper bounded by some constant κ, which means that the derivative $S\mapsto P'(S)$ is increasing and upper bounded by $\frac{-1}{\kappa}$. Then it converges to some limit $-\tau<0$ as S goes to $+\infty$.

Lemma 4.3. *There exists some decreasing function χ such that $P(S) = -\tau S+\chi(S)$ and $\lim_{S\to+\infty}\frac{\chi(S)}{S} = 0$.*

Proof. The only difficult point is the second one. As the function χ decreases, it is upper bounded and we have

$$\limsup_{S\to+\infty} \frac{\chi(S)}{S} \leq 0.$$

Now, for any $\varepsilon > 0$, there exists some increasing function $\widehat{\theta}$ defined for sufficiently large S such that $P(S) = -(\tau+\varepsilon)S + \widehat{\theta}(S)$ which proves that for every $\varepsilon > 0$ and for every sufficiently large S we have $\chi(S) \geq -\varepsilon S + K$, for some constant K. This yields

$$\forall \varepsilon > 0, \quad \liminf_{S\to+\infty} \frac{\chi(S)}{S} \geq -\varepsilon.$$

$\square$

One of the difficulties now, is to identify τ. This is the goal of the next proposition:

Proposition 4.4. *With the previous definitions,* $\dfrac{1}{\tau} = \rho(\phi)$.

Proof. The definition of τ and (8) give $\dfrac{1}{\tau} = \lim\limits_{S\to+\infty} \displaystyle\int \phi \, d\widetilde{\mu}_S \leq \rho(\phi)$.
Let $y \in F_i$ be in $Fix(g_F^n)$. Then we have

$$\begin{aligned} m_{S,P(S)}(F_i) &= \int \mathbb{1}_{F_i} \, dm_{S,P(S)} \\ &= \int \mathcal{L}^n_{S,P(S)}(\mathbb{1}_{F_i}) \, dm_{S,P(S)} = \sum_j \int_{F_j} \mathcal{L}^n_{S,P(S)}(\mathbb{1}_{F_i}) \, dm_{S,P(S)} \\ &\geq e^{-C_\phi} e^{S_{r^n_A(y)}(f)(y) - (r^n_A(y) - \tau . S_{r^n_A(y)}(\phi)(y))S - S_{r^n_A(y)}(\phi)(y).\chi(S)} m_S(F_i). \end{aligned} \tag{9}$$

Hence (9) proves that the function $S \mapsto e^{S_{r^n_A(y)}(\phi)(y) S \left(\tau - \frac{r^n_A(y)}{S_{r^n_A(y)}(\phi)(y)} - \frac{\chi(S)}{S}\right)}$ is upper bounded, which forces τ to be lower than $\dfrac{r^n_A(y)}{S_{r^n_A(y)}(\phi)(y)}$ for every n-periodic point y (remember that ϕ is positive) in F. Now the mixing property yields

$$\inf_{y\in Per(F)} \left\{ \frac{r^n_A(y)}{S_{r^n_A(y)}(\phi)(y)} \right\} = \frac{1}{\rho(\phi)}.$$

$\square$

Lemma 4.5. *There exist a real number $\widetilde{\tau} \geq \tau$ and some decreasing function $\widetilde{\chi}$ such that $P_l(S) = -\widetilde{\tau} S + \widetilde{\chi}(S)$ and $\lim\limits_{S\to+\infty} \dfrac{\widetilde{\chi}(S)}{S} = 0$.*

Proof. The existence of $\widetilde{\tau}$ and $\widetilde{\chi}$ also follows from the convexity of the map $t \mapsto \mathcal{P}_{f+\beta\phi}(\Sigma_A)$. The inequality $\widetilde{\tau} \geq \tau$ follows from the fact that $S \mapsto P(S)$ is above $S \mapsto P_l(S)$. $\square$

The value of $\widetilde{\tau}$ deeply depends on A. We will see later, that for "good" A, it will be equal to τ.

4.3 Support of the accumulation points for $\widehat{\mu}_\beta$.

The goal of this subsection is to prove that any accumulation point for the family $\widehat{\mu}_\beta$ must some Equilibrium State associated to f in $\mathbb{K}_\phi$. Let $A := [i]$ be some 1-cylinder such that the symbol i does not appear in $\mathbb{K}_\phi$. Then, relation (4) (p. 10) holds for some constant C_1 which does not depend on S (see lemma 3.5).

Lemma 4.6. *With the previous notations,* $\lim\limits_{t\to+\infty} \widehat{\mu}_\beta([i]) = 0$.

Proof. Let us assume that there exists some accumulation point for the family $(\widehat{\mu}_\beta)$ such that the result does not hold. Let us denote by $\widehat{\mu}$ this accumulation point. Then $\widehat{\mu}$-almost every point in $[i]$ returns infinitely many times in $[i]$ by the iteration of σ. Let x be such a point; we also assume that x is a density point in $[i]$. Let $F := [[i], x]$. Then (4) (with $\theta = f - P(S)\phi$) holds for every S and proposition (3.10) proves that for some subfamily of S (going to $+\infty$) the numerator of the term in the middle converges to $\dfrac{\mu(\pi_F^{-1}(K_1(x)))}{\mu(A)}$ (every 1-set is a cylinder, hence its indicator map is continuous). However, no periodic orbit which belongs to $[i]$ can be in $\mathbb{K}_\phi$. Therefore $-(r_A(x) - \tau S_{r_{A}(x)}(\phi))S - S_{r_A}(x)(\phi)\chi(S)$ goes to $-\infty$ as S goes to $+\infty$, which yield some contradiction. $\square$

Let us now assume that $A := [i]$ is such that i appears in $\mathbb{K}_\phi$ but in some irreducible component with small f-pressure (*i.e.* a component such that the topological pressure associated to f is strictly smaller than the topological pressure associated to f in $\mathbb{K}_\phi$). Let $\mathbb{K}_\phi(1)$ be some irreducible component in $\mathbb{K}_\phi$ with maximal f-pressure and let $\widehat{\mu}^1$ be the unique Equilibrium State in $\mathbb{K}_\phi(1)$ associated to f. We denote by $\mathcal{P}$ this topological pressure. Then $\mathbb{K}_\phi(1)$ belongs to the hole Σ_A, and for every β, the topological pressure of Σ_A associated to $f + \beta\phi$ is larger than $h_{\widehat{\mu}^1} + \displaystyle\int f\,d\widehat{\mu}^1 + \frac{\beta}{\tau} = \mathcal{P} + \frac{\beta}{\tau}$. Then, using notations from the proof of proposition 3.9, we must have

$$S_c(P) \geq \frac{-P}{\tau} + \mathcal{P}.$$

But $S \mapsto P_l(S)$ is the inverse map of $P \mapsto S_c(P)$, which proves that $\widetilde{\tau} \leq \tau$.

Remark 4. Notice that $\widetilde{\chi}$ and χ converge when S goes to $+\infty$ (as decreasing and bounded from below functions) and their limits are bigger than $\tau \times \mathcal{P}$.

Lemma 4.7. *Let* $[i]$ *be some 1-cylinder such that* $[i] \cap \mathbb{K}_\phi \neq \emptyset$ *and the irreducible component of* $\mathbb{K}_\phi$ *which intersects* $[i]$ *has a topological pressure associated to* f *strictly lower than* $\mathcal{P}$. *Then,* $\lim_{t \to +\infty} \widehat{\mu}_\beta([i]) = 0$.

Proof. Let $\mathbb{K}_\phi([i])$ be the component of $\mathbb{K}_\phi$ which intersects with $[i]$. Then, we have just seen $\lim_S \chi(S) \geq \tau.\mathcal{P}$. Let us pick some accumulation point μ for μ_β such that $\mu([i]) > 0$. The measure μ can be seen as the opened-out measure of some accumulation point for ν'_S. Therefore, letting S go to $+\infty$ in the revisited equation (4) we get for every x in $\mathbb{K}_\phi \cap [i]$,

$$e^{-C_1}.e^{S_{r_A^n(x)}(f)(x) - r_A^n(x).\mathcal{P}'}.\mu([i]) \leq \mu(C_{r_A^n(x)}(x)) \leq e^{C_1}.e^{S_{r_A^n(x)}(f)(x) - r_A^n(x).\mathcal{P}'}.\mu([i]), \quad (10)$$

where $\mathcal{P}' = \frac{1}{\tau} \times \lim_S \chi(S)$. Notice that this implies that the projection onto F of conditional measure $\mu_{[i]} \stackrel{def}{=} \frac{\mu(.\cap[i])}{\mu([i])}$ is exact for g_F, thus mixing and thus ergodic. Remember that μ is the opened-out measure of the natural extension of $\mu_[i]$. Therefore μ is also ergodic.

For every x in $\mathbb{K}_\phi \cap [i]$ we have

$$\left| \frac{1}{r_A^n(x)} S_{r_A^n(x)}(f)(x) - \frac{1}{r_A^n(x)} \log(\mu(C_{r_A^n(x)}(x))) - \mathcal{P}' \right| \leq \frac{C_1 - \log(\mu[i])}{r_A^n(x)}.$$

But for μ-a.e x, $\frac{1}{n} S_n(f)(x)$ converges to $\int f \, d\mu$, and $-\frac{1}{n} \log \mu(C_n(x))$ converges to h_μ. Now $\mu([i])$ is positive, hence the previous convergencies hold for μ-a.e. x in $[i]$. Therefore μ admits $\mathcal{P}'$ for f-pressure. Now, $\mathcal{P}'$ is larger than the topological pressure of $\mathbb{K}_\phi$ thus strictly larger than the topological pressure in $\mathbb{K}_\phi([i])$. This yields to a contradiction. □

Remark 5. The same calculation proves that every accumulation point for μ_β is some barycenter of the ergodic Equilibriums States of $\mathbb{K}_\phi$ associated to f.

Indeed, only the irreducible components with maximal f-pressure can have positive limit measure. But restricted to each such an irreducible component, the conditional limit measure must have a f-pressure larger than $\mathcal{P}$. Hence it must be the unique Equilibrium State associated to f. We summarize it in the following way:

Proposition 4.8. *Any accumulation point for* μ_β *is the barycenter of the ergodic Equilibrium States associated to* f. *Moreover, for every* $[i]$ *such that* $\mathbb{K}_\phi \cap [i] \neq \emptyset$ *and the irreducible component in* $\mathbb{K}_\phi$ *which intersects* $[i]$ *has maximal* f*-pressure, we have*

$$\lim_S \chi(S) = \tau\mathcal{P},$$

where χ *is defined as above.*

5 Components with possible limit measure

We have just proved that only the irreducible components in $\mathbb{K}_\phi$ with maximal f-pressure can have positive limit measure (for any accumulation point of $\widehat{\mu}_\beta$). We now want to precise how much they will count in the limit measure. The goal of this section is to introduce the important parameter, called the *isolation rate* of the components which will determine which component have positive limit measure. If there is only one irreducible component in $\mathbb{K}_\phi$ with maximal f-pressure, then $\widehat{\mu}_\beta$ converges to the unique Equilibrium State of $\mathbb{K}_\phi$ when β goes to $+\infty$. In that case, the Theorem is proved (and is obvious). Therefore, and for the rest of the proof, we denote by $\mathbb{K}_\phi(1), \ldots, \mathbb{K}_\phi(q)$ the $q > 1$ irreducible components in $\mathbb{K}_\phi$ with maximal f-pressure.
We also recall that $\mathcal{P}$ denotes the f-pressure of $\mathbb{K}_\phi$.

5.1 The function χ and the transition costs

We have proved that for every 1-cylinder $[i]$ which intersects some irreducible component with maximal f-pressure, the limit of the function χ (which *a priori* depends on $[i]$) is independent of the choice of the cylinder. In fact we have some better result:

Lemma 5.1. *The function χ is independent of the choice of $\mathbb{K}_\phi(l)$ and of the choice of the 1-cylinder which intersects $\mathbb{K}_\phi(l)$.*

Proof. Pick two different 1-cylinders which intersect two different irreducible components of $\mathbb{K}_\phi$ with maximal f-pressure. Pick two unstable leaves F_i and F_j, set $A := [i] \sqcup [j]$ and $F := F_i \sqcup F_j$. Pick some real number β; then we can identify $\widehat{\mu}_\beta$ to some $\widetilde{\mu}_{S,P(S)}$ for some S (and for the map induced in F). We denote by ν_S the associated equilibrium State for (F, g_F). The main idea in the proof is to see the first return map (and projection onto F_i) as the first return map in $[i]$ by iterations of σ or by iterations of g_F. In the first case the measure $\widehat{\mu}_\beta$ gives a measure associated to the operator $\mathcal{L}_{S_i,[i]}$ for some real S_i. In the second case, F_i is a subset of F with positive ν_S-measure. Copying the method that we recalled in section 3, but with g_F instead of σ, the conditional measure $\nu_S^i \stackrel{def}{=} \dfrac{\nu_S(.\cap F_i)}{\nu_S(F_i)}$ can be found as some measure associated to the Transfer Operator $\widehat{\mathcal{L}}_{U_i,[i]}$ but for another function θ (the parameter U_i (which depends on S) plays the same role than S when we defined the local Ruelle-Perron-Frobenius operator in section 3). Namely this operator is

$$\widehat{\mathcal{L}}_{U_i,[i]}(\mathcal{T})(y) = \sum_{x \in \widehat{pre}(y)} e^{S_{r_i(x)}(f)(x) - r_i(x).S - S_{r_i(x)}(\phi)(x)P(S) - r_{i,F}(x)U_i(S)} \mathcal{T}(x),$$

where U_i is chosen such that this operator has 1 for spectral radius, and $\widehat{pre}(y)$ denote the set of preimages of y by the first return in F_i by iterations of g_F. As any iteration of g_F equals π_F composed with some iteration of σ, the set $\widehat{pre}(y)$ is exactly the set of preimages of y by the map g_{F_i} (first return in $[i]$ by iterations of σ and projection onto F_i). In the previous formula, $r_i(.)$ denotes the first return time in $[i]$ by iteration of σ, and $r_{i,F}(.)$ denotes the first return time map in F_i by iterations of g_F.

Each adjoint operator admits a unique eigenvector; these two eigenvectors are measures on F_i which are both equivalent to the g_{F_i}-invariant measure ν_S^i. Notice that the associated Radon-Nikodym derivative is the eigenvector of each operator. Therefore we obtain for almost every point x,

$$e^{-r_i(x).S-S_{r_i(x)}(\phi)(x)P(S)-r_{i,F}(x)U_i(S)} \asymp e^{-r_i(x).S_i-S_{r_i(x)}(\phi)(x)P_i(S_i)}, \tag{11}$$

where $\asymp$ means that the equality holds up to some multiplicative constant. Notice that due to the form of the functions we consider, the multiplicative constant does no depend on S or U_i (see lemma 3.5). More precisely (11) holds for every n-set for the map g_{F_i} and for every n.

Now, by proposition 3.10 we must have

$$-P_i(S_i) = \tau.S - \chi(S) = \beta = \tau.S_i - \chi_i(S_i) = -P(S). \tag{12}$$

As $[i]$ and $[j]$ are in different irreducible components, any minimal-periodic orbit which joins $[i]$ and $[j]$ has a transition cost greater than α. We pick any such periodic orbit. We can make a looping with this loop to glue on $[j]$ any minimal periodic orbit in the irreducible component $\mathbb{K}_\phi([j])$ which contains $[j]$. We can repeat this looping as many times as wanted, it will not add any transition cost, by definition (and construction) of $\mathbb{K}_\phi$. For such an orbit, there is an affine relation between $r_{i,F}$ and r_i:

when $r_{i,F} = n+2$, with n any integer, $r_i = n.L + R$, when R is the length of the minimal loop between $[i]$ and $[j]$ we are considering and L is the length of the loop in $\mathbb{K}_\phi([j])$. Then (11) and (12) yield to

$$\forall\ n \in \mathbb{N}, \quad (n+2)U_i = \frac{nL+R}{\tau}(\chi(S) - \chi_i(S_i)). \tag{13}$$

Now, remember that U_i is chosen such that the associated transfer Operator has 1 for spectral radius. Then, copying proposition 3.9, the theorem of the implicit function gives for the derivative of the map $S \mapsto U_i$:

$$\frac{dU_i}{dS} = -\frac{-\int_{F_i} r_i \, d\nu_S^i - \int_{F_i} S_{r_i}(\phi) d\nu_S^i . \frac{dP(S)}{dS}}{-\int_{F_i} r_{i,F} d\nu_S} \equiv 0.$$

This means that the function U_i is constant. Letting n goes to $+\infty$ in (13) and then letting S go to $+\infty$, we also have $S_i \to +\infty$ and thus proposition 4.8 yields to $U_i \equiv 0$. If $[i]$ and $[j]$ meet the same irreducible component we choose any $[i']$ which intersects some other irreducible component. This finishes the proof. $\square$

It makes now sense to talk about $\chi(S)$ without specifying which A is used.

Definition 5.2. *Let us pick some n-periodic orbit $\{x, \dots \sigma^{n-1}(x)\}$, in Σ. The periodic orbit will also be called the loop x. We denote by ρ_x the term $n - \tau.S_n(\phi)(x)$. We call it the transition cost of the loop.*
We denote by L the set of minimal loops which are not contained in any $\mathbb{K}_\phi(l)$.

Remark 6. If $\{x, \dots \sigma^{n-1}(x)\}$ is loop in L, its Birkhoff average is strictly smaller than $\rho(\phi)$. Then ρ_x is positive. We denote by α the minimal transition cost among all the loops in L.

Now remember that every periodic orbit can be decomposed in minimal loops and that ϕ is constant on every 1-cylinder. Thus, the transition cost for any periodic orbit can simply be defined as the sum of the transition cost for the minimal loops which compose the considered periodic orbit. With this definition, any periodic orbit in $\mathbb{K}_\phi$ has a transition cost equal to 0.

If y is a periodic point which does not belong to any $\mathbb{K}_\phi(l)$, it defines some loop in L. For the rest of the proof we will blur the point y and the associated loop in L.

Definition 5.3. *Let $\{y, \dots, \sigma^{n-1}(y)\}$ be a loop in L. We denote by $p_l(y)$ the number of $1 \leq k \leq n-1$ such that $C_1[\sigma^k(y)]$ intersects $\mathbb{K}_\phi(m)$, with $m \neq l$, and $C_2[\sigma^{k-1}(y)]$ does not intersect with $\mathbb{K}_\phi(m)$. We denote by $a_l(y)$ the term*

$$\frac{e^{-\rho_y.S}}{(\chi(S) - \tau.\mathcal{P})^{p_l(y)}},$$

where ρ_y is the transition cost of the loop

5.2 The isolation rate

Let us pick some 1-cylinder $[i]$ which intersects some $\mathbb{K}_\phi(l)$. Let us set $A := [i]$. In A we pick some F of unstable leaf. We denote by $\widetilde{\mathcal{L}}_{T,[i]}$ the Ruelle-Perron-Frobenius operator on F defined as in section 3 but where we only consider preimages y whose associated $(r_A(y)+1)$-cylinder is a word which appears in $\mathbb{K}_\phi(l)$; T is the parameter which was designed by S in section 3. By definition of the Ruelle-Perron-Frobenius operator we have

$$\begin{aligned}\mathcal{L}_S(\mathcal{T})(x) &= \sum_{y \in Pre_1(x) \cap \pi_F(\mathbb{K}_\phi(l))} e^{\Theta(y) - r_A(y).S - P(S).S_{r_A(y)}(\phi)(y)} \mathcal{T}(y) + \\ &\qquad \sum_{y \in Pre_1(x) \setminus \pi_F(\mathbb{K}_\phi(l))} e^{\Theta(y) - r_A(y).S - P(S).S_{r_A(y)}(\phi)(y)} \mathcal{T}(y).\end{aligned}$$

Remember that for any y in $Pre_1(x) \cap \pi_F(\mathbb{K}_\phi(l))$, the word of length $r_A(y)$ associated to y is a word in $\mathbb{K}_\phi(l)$. Then for such a y we have $S_{r_A(y)}(\phi)(y) = r_A(y)\rho(\phi)$.

Exchanging $-P(S)$ by its value, the first term in the right side of the previous equality is exactly

$$\widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[i]}(\mathcal{T})(x) = \sum_{y \in Pre_1(x) \cap \pi_F(\mathbb{K}_\phi(l))} e^{\Theta(y) - r_A(y).\frac{\chi(S)}{\tau}} \mathcal{T}(y).$$

We denote by $\Psi_{S,[i]}(\mathcal{T})(x)$ the second term. Then we have

$$\mathcal{L}_S(\mathcal{T})(x) = \widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[i]}(\mathcal{T})(x) + \Psi_{S,[i]}(\mathcal{T})(x). \tag{14}$$

We can now state one important result:

Proposition 5.4. *There exists some power series $\mathcal{G}_l(S)$ with non-negative coefficients in all the $e^{C'_f p_l(y)} a_l(y)$, where C'_f is a constant which depends only on f, such that*

$$||\Psi_{S,[i]}(\mathcal{T})|| \leq ||\mathcal{T}|| \mathcal{G}_l(S), \tag{15}$$

and each term in $\mathcal{G}_l(S)$ contains at least one $a_l(y)$, where y is a minimal loops which intersects $[i]$.

Corollary 5.5. *If for all y $a_l(y) \to 0$ as S goes to $+\infty$, then $\mathcal{L}_S$ converges to $\widetilde{\mathcal{L}}_{\mathcal{P},[i]}$ and the eigenvector H_S converges to the unique eigenvector (up to some multiplicative constant) associated to 1 for $\widetilde{\mathcal{L}}_{\mathcal{P},[i]}$.*

Notice that the operators $\widetilde{\mathcal{L}}_{T,[i]}$ act on the continuous functions defined on the whole unstable leaf F because it uses the preimages. Thus it makes sense to talk about the limit of the functions H_S as an eigenvector for the operator $\widetilde{\mathcal{L}}_{\mathcal{P},[i]}$.

Proof. (of proposition 5.4) We obviously have $|\mathcal{T}(x)| \leq ||\mathcal{T}|| = ||CT||.\mathbb{1}_F(x)$. Therefore we only want to give some upper bound for $\Psi_{S,[i]}(\mathbb{1}_F)(x)$. Moreover, the Hölder regularity of f and the fact that ϕ is constant on every 1-cylinders yields to

$$e^{-C_f} ||\Psi_{S,[i]}(\mathbb{1}_F)|| \leq \Psi_{S,[i]}(\mathbb{1}_F)(x) \leq e^{C_f} ||\Psi_{S,[i]}(\mathbb{1}_F)||,$$

for some constant C_f which does only depend on f. Therefore we only have to compute $\Psi_{S,[i]}(\mathbb{1}_F)(x)$ for a given x in F.

Computing $\Psi_{S,[i]}(\mathbb{1}_F)(x)$, we have to estimate and to sum expressions

$$e^{\Theta(y) - r_A(y).S - P(S).S_{r_A(y)}(\phi)(y)}$$

where y is a typical point in $Pre_1(x) \setminus \pi_F(\mathbb{K}_\phi(l))$. Let us consider some typical y as above and such that $r_i(y) = n$. The n-cylinder associated to y gives a word w which is the same than some n-periodic point in $[i]$. Moreover this word can be decomposed in minimal loops. Among these loops, some are in L and some are associated to points in $\mathbb{K}_\phi$. We associated to y its basic loop, which is the loop obtained when we remove all the subloops in w which are "in" irreducible components. This basic loop is composed by several minimal loops in L. Among them, only one contains the letter i because we are studying the first return in $[i]$. The strategy to get (15) is to count all the contributions in function on their basic loop.

Let z be some preimage of x, and let y be its basic loop. The word associated to z is the word associated to y where we eventually glue via a looping some turns in some irreducible components $\mathbb{K}_\phi(m)$ (with $m \neq l$). Due to the Hölder regularity of f and to the fat that ϕ is constant on every 1-cylinder, all the Birkhoff sums along the piece of orbits where $\sigma^s(z)$ and $\sigma^s(y)$ are in the same 1 cylinder are equal, up to the constant C_f. Let $y_1, \dots, y_t$ be the minimal loops which compose y; then when we decompose the Birkhoff sums for z in parts close to y, the constant C_f appears at most $p_l(y_1) + \ldots + p_l(y_t)$ times.

• Let us first assume, for simplicity, that y is a minimal loop and $p_l(y) = 1$. For simplicity we also assume that the loop y does not meet $\mathbb{K}_\phi(l)$ in another 1-cylinder than $[i]$ (see Figure 1) and intersects $\mathbb{K}_\phi(m)$.

Figure 1: counting orbits

The contribution of the basic loop is

$$e^{S_{n_a+n_r}(f)(y)-\rho_y.S-\chi(S).S_{n_a+n_r}(\phi)(y)}.$$

The loop starts from F, needs a time n_a to reach $\mathbb{K}_\phi(m)$ and then goes back to F, needing a time n_r. Notice that $\chi(S)$ goes to $\tau\mathcal{P}$ as S goes to $+\infty$; then, the speed of convergence for the contribution of the loop is essentially given by $e^{-\rho_y.S}$. Now, we can glue on $\{y, \dots, \sigma^{n_a}(y), \sigma^{n_a+1}(y), \dots, \sigma^{n_a+n_r-1}(y)\}$ some loop in $\mathbb{K}_\phi(m)$ via the looping principle. This loop can be as long as possible. Therefore, we will have loops with contributions

$$e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma^{n_a}(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-\chi(S)S_k(\phi)(\sigma^{n_a}(z))-\chi(S).S_{n_a+n_r}(\phi)(y)}.$$

To get this contribution, remember that ϕ is constant on every 1-cylinder. The term $e^{S_{n_a}(f)(z)}$ equals $e^{S_{n_a}(f)(y)}$ up to $e^{\pm C_f}$; in the same way $e^{S_{n_r}(f)\circ\sigma^{n_a+k}(z)}$ equals $e^{S_{n_r}(f)\circ\sigma^{n_a}(y)}$ up to $e^{\pm C_f}$. Remember that the Birkhoff sum $S_k(\sigma^{n_a}(z))$ equals $\dfrac{k}{\tau}$ because it concerns the piece of orbit in $\mathbb{K}_\phi(m)$ (we only consider complete loops).

Then, summing over all the k, the global contribution of all the loops whose basic loop is y, and which meet only on time $\mathbb{K}_\phi(m)$ (but turn a long time into it) is

$$K(y).e^{-\rho_y}.\sum_{k=0}^{+\infty} Z_k(m)e^{-k\frac{\chi(S)}{\tau}},$$

where $K(y) \leq e^{2C_f}.e^{S_{n_a+n_r}(f)(y)-\tau\mathcal{P}.S_{n_a+n_r}(\phi)(y)}$ and $Z_n(m)$ denotes the sum $\displaystyle\sum_{\sigma^n(y)=y,\ y\in\mathbb{K}_\phi(m)} e^{S_n(f)(y)}$.

Now, $Z_n(m)$ equals $e^{n\mathcal{P}}$, here again up to some multiplicative constant (this is a standard computation for uniformly hyperbolic dynamical systems). Up to the fact that we change C_f, we can assume that this constant is $e^{\pm C_f}$. For S large, $\chi(S)$ is close to $\tau.\mathcal{P}$ and thus

$$\sum_{n=0}^{+\infty} Z_n(m)e^{-n\frac{\chi(S)}{\tau}} \sim \frac{1}{\chi(S)-\tau\mathcal{P}}.$$

Therefore, the contribution of the orbits we are considering is upper bounded by

$$K.e^{3C_f}e^{S_{n_a+n_r}(f)(y)-\tau\mathcal{P}.S_{n_a+n_r}(\phi)(y)}.e^{-\rho_y}\frac{1}{\chi(S)-\tau\mathcal{P}},$$

for some universal constant K. Notice that $e^{-\rho_y}\dfrac{1}{\chi(S)-\tau\mathcal{P}}$ exactly is $a_l(y)$.

Notice also that the contribution of all the orbits whose basic loop is y is also lower bounded by $K'.e^{-3C_f}e^{S_{n_a+n_r}(f)(y)-\tau\mathcal{P}.S_{n_a+n_r}(\phi)(y)}.e^{-\rho_y}\dfrac{1}{\chi(S)-\tau\mathcal{P}}$.

• Let us now still assume that y is a minimal loop but $p_l(y)\geq 2$.
The loop can meet several others irreducible components or one other component but several times. We can again copy the previous case, except that we can now turn as long as wanted in the several others $\mathbb{K}_\phi(m)$ (or several series in the same $\mathbb{K}_\phi(m)$). Now, the basic loop is minimal, then it has a maximal length. Therefore, we can copy the previous case, and adapt it. This times, exactly $p_l(y)$ power series $\sum_k Z_k(m).e^{-k.\chi(S)/\tau}$ will appear. This give a global contribution for this loop upper bounded by

$$K.e^{3p_l(y)C_f}e^{S_{r_i}(f)(y)-\tau\mathcal{P}.S_{r_i}(\phi)(y)}a_l(y),$$

for some universal constant K.

Notice that the definition of $p_l(y)$ implies that even if the loop y meets consecutively two times the same $\mathbb{K}_\phi(m)$, no loop in this $\mathbb{K}_\phi(m)$ contains this word (of length 2). Hence we can differentiate loops starting from the first cylinder and loops starting from the second cylinder.

Notice again that the contribution is lower bounded by some $K'.e^{-3p_l(y)C_f}e^{S_{r_i}(f)(y)-\tau\mathcal{P}.S_{r_i}(\phi)(y)}a_l(y)$

• Let us now still assume that y is a minimal loop, but it meets $\mathbb{K}_\phi(l)$ at least two times. For such a basic loop, we could glue via the looping principle subloops which turns a very long time in $\mathbb{K}_\phi(l)$ without meeting $[i]$. The computation of the global contribution of such loops is on the same kind than the second case; however the difference is that one term $Z'_k(l)$ will only concern k-periodic orbits in $\mathbb{K}_\phi(l)\setminus[i]$. This quantity equals $e^{k\mathcal{P}'}$, up to some constant. But $\mathcal{P}'<\mathcal{P}$ and then

$$\frac{1}{1-e^{-\chi(S)/\tau+\mathcal{P}'}}$$

admits a finite limit as S goes to $+\infty$. Therefore the added turns in $\mathbb{K}_\phi(l)$ (but without meeting $[i]$) only introduce some new multiplicative constant but does not add new $a_l(y')$.

• If y is a minimal loop with $p_l(y) = 0$ and it does not meet $\mathbb{K}_\phi(l)$ several times, there is no way to add extra-terms to this loops without changing the global transition cost.

• recursivity:

If y is not a minimal loop, we can do such a work on all the minimal loops which compose y. All the contributions will multiply themselves together, which produces the power series in all the $e^{C_f p_l(y)} a_l(y)$ where we set $C'_f := 3C_f$. This is a power series with non-negative terms. Moreover for each minimal loop which intersects $[i]$, y, $a_l(y)$ appears in the sum with valuation 1. $\square$

Remark 7. It is not clear that the power series $\mathcal{G}_l(S)$ converges. Nevertheless, we always get lower bounds, exchanging C_f with $-C_f$ and taking lower bound for the comparison between $\dfrac{1}{1-e^{-\chi(S)/\tau+\mathcal{P}}}$ and $\dfrac{1}{\chi(S)-\tau\mathcal{P}}$ instead of upper bound. As $\Psi_{S,[i]}\mathbb{1}$ is upper bounded, it proves that this new power series converges. Thus it has a positive radius of convergence. Therefore the power series $\mathcal{G}(S)$ also have a positive radius of convergence. Now an important point will be to prove that all the $a_l(y)$ go to 0 as S goes to $+\infty$, which will prove the convergence for the power series $\mathcal{G}_l(S)$ for large enough S.

5.2.1 The main tool.

We can now introduce the main tool to determine which components have positive limit measure. The first step is to get some lower and upper bound for $\Psi_{S,[i]}(r_i)$ in the same way than (15).

Proposition 5.6. *Let us set* $A \stackrel{def}{=} \left| \int_F r_{[i]}\, d\nu_S - \int_F \widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[i]}(r_{[i]}H_S)\, d\nu_S \right|$. *Then, there exist two power series such that*

$$\sum_{\vec{n}\in\mathbb{N}^{\#L}} a_{\vec{n}} \frac{1}{(\chi(S)-\tau.\mathcal{P})^{q_l(\vec{n})}} \prod_{y\in L} (a_l(y))^{\vec{n}_y} \leq A \leq \sum_{\vec{n}\in\mathbb{N}^{\#L}} b_{\vec{n}} \frac{1}{(\chi(S)-\tau.\mathcal{P})^{q_l(\vec{n})}} \prod_{y\in L} (a_l(y))^{\vec{n}_y}, \tag{16}$$

where $\vec{n}_y$ is the coordinate of $\vec{n}$ associated to the loop y in L, $q_l(\vec{n}) = 1$ if there exists some y such that $\vec{n}_y \geq 1$ and $p_l(y) \geq 1$, and $q_l(\vec{n}) = 0$ otherwise. Moreover, the $a_{\vec{n}}$ and the $b_{\vec{n}}$ are non-negative, and $a_{\vec{n}} = 0$ if and only if $b_{\vec{n}} = 0$.

Proof. The first return time is constant on every 1 sets, and m_S is $\mathcal{L}^*_S$-invariant. Therefore we get

$$\int_F r_i\, d\nu_S = \int_F r_i.H_S\, dm_S = \int_F \mathcal{L}_{S,P(S)}(r_i H_S)\, dm_S,$$

where $\mathcal{L}_S(r_i H_S)$ has the same expression than for any continuous function.

Then we use (14) to get some estimate like (15): the term A in (16) equals $\int \Psi_{S,[i]}(r_i H_S)\, dm_S$ which is positive as a positive sum. We thus use (2) (p.9) to give some upper and lower bound for H_S. Therefore it amounts on giving estimates for $\displaystyle\int \Psi_{S,[i]}(r_i)\,dm_S$. Copying the proof of (15), we decompose the sum on the sum of the contributions of the basic loops. To get an upper bound for A, it is thus sufficient to sum upper bounds for the terms in $\displaystyle\int \Psi_{S,[i]}(r_i)\,dm_S$ or equivalently for just one $\Psi_{S,[i]}(r_i)(x)$ (up to some e^{C_f}). To get lower bound for A it is sufficient to give lower bounds for the terms in $\displaystyle\int \Psi_{S,[i]}(r_i)\,dm_S$. But notice that is a sum of positive terms, and thus 0 is an obvious lower bound for each term.

• Let us assume that the basic loop y is minimal and satisfies $p_l(y) = 1$. We have to compute the sum over k of

$$\int (n_a+k+n_r)e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma^{n_a}(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-\chi(S)S_k(\phi)(\sigma^{n_a}(z))-\chi(S).S_{n_a+n_r}(\phi)(y)}\,d\nu_S(x),$$

where z is the preimage of the considered point x whose piece of orbits admits y as basic loop and turns for a time k in $\mathbb{K}_\phi(m)$. The arguments are the same as before, except that we have to care about the coefficient n_a+k+n_r in front of the exponential.

However y is a minimal loop, and so n_a+n_r is upper bounded by N. Therefore, when we sum over k, the global contribution $C_1(y)$ of

$$\int (n_a+n_r)e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma^{n_a}(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-\chi(S)S_k(\phi)(\sigma^{n_a}(z))-\chi(S).S_{n_a+n_r}(\phi)(y)}\,d\nu_S(x),$$

is upper bounded by $K.e^{3C_f}.a_l(y)$, as it was proved above. It is also lower bounded by $K'e^{-3C_f}.a_l(y)$.

Now, summing over k the global contribution $C_2(y)$ of

$$\int ke^{S_{n_a}(f)(z)+S_k(f)\circ\sigma^{n_a}(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-\chi(S)S_k(\phi)(\sigma^{n_a}(z))-\chi(S).S_{n_a+n_r}(\phi)(y)}\,d\nu_S(x),$$

has order $\displaystyle e^{-\rho_y.S}\sum_k kZ_m(k).e^{-\chi(S)/\tau}$. Here "order" means equal up to some multiplicative constant $e^{\pm 2C_f}$. Hence, the global contribution has order $\displaystyle\frac{a_l(y)}{(\chi(S)-\tau.\mathcal{P})}$. As $\chi(S)-\tau.\mathcal{P}$ goes to zero as S goes to $+\infty$, $C_2(y)/C_1(y)$ goes to $+\infty$ as S goes to $+\infty$. This proves that the global contribution of all the preimages which have y as basic loop has order $\displaystyle\frac{a_l(y)}{(\chi(S)-\tau\mathcal{P})}$.

• Let us assume that y is a minimal loop but $p_l(y) \geq 2$. Let $\mathbb{K}_\phi(m_t)$ $1 \leq t \leq r$ be the r visited irreducible components (ordered by the order of visits). Then the global contribution is proportional to

$$e^{-\rho_y.S}.\sum_{k=0}^{+\infty}\sum_{k_1+\ldots+k_r=k}(k+l(y))\prod_t Z_{m_t}(k_t)e^{-k\chi(S)/\tau},$$

where $Z_{m_t}(k_t)$ is the number of k_t-periodic points in $\mathbb{K}_\phi(m_t)$ and $l(y)$ is the length of the basic loop y. Now, the sum

$$\sum_{k=0}^{+\infty} \sum_{k_1+\ldots+k_r=k} k \prod_t Z_{m_t}(k_t) e^{-k\chi(S)/\tau}$$

is proportional to $\dfrac{1}{(\chi(S)-\tau.\mathcal{P})^{p_l(y)+1}}$, and the sum

$$\sum_{k=0}^{+\infty} \sum_{k_1+\ldots+k_r=k} l(y) \prod_t Z_{m_t}(k_t) e^{-k\chi(S)/\tau}$$

is proportional to $\dfrac{1}{(\chi(S)-\tau.\mathcal{P})^{p_l(y)}}$. This proves that the global contribution of all these loops is "proportional" to $\dfrac{a_l(y)}{(\chi(S)-\tau.\mathcal{P})}$.

• Let us assume that y is a minimal loop but it meets $\mathbb{K}_\phi(l)$ several times. Then the situation is the same than the previous case, except that when $m_t = l$, we have

$$Z_l(k_t) \asymp e^{k.\mathcal{P}'}.$$

As $\mathcal{P}' < \mathcal{P}$, if $p_l(y) = 0$ (the loop does not meet any other $\mathbb{K}_\phi(m)$), then the global contribution is proportional to $a_l(y)$. If $p_l(y) > 0$, the sum without the excursions in $\mathbb{K}_\phi(l) \setminus [i]$ is smaller than the sum with the excursion. However, the convergence of the series $\sum e^{s(\mathcal{P}'-\mathcal{P})}$ proves that the sum with the excursions is upper bounded by $\dfrac{1}{(1-e^{\mathcal{P}'-\mathcal{P}})^d}$ multiplied by the sum without the excursions (in $\mathbb{K}_\phi(l) \setminus [i]$), where d is the number of possible excursions. Therefore in the global sum, we can remove the terms with the excursions, in the lower bound and in the upper bound for A.

• If y is a minimal loop with $p_l(y) = 0$ and which does not meet $\mathbb{K}_\phi(l)$ several times, then we upper bound its length by N and lower bound it by 1.

• Recursivity. Let y be any basic loop which is not a minimal loop. Then it can be decomposed in finitely many minimal loops. As in the proof of (15) the individual contributions multiply themselves to produce the power series. However, it is important to notice that the extra term $\dfrac{1}{\chi(S)-\tau.\mathcal{P}}$ appears if and only if we are computing a term

$$\sum_{k=0}^{+\infty} \sum_{k_1+\ldots+k_r=k} k. \prod_t Z_{m_t}(k_t) e^{-k\chi(S)/\tau},$$

with $m_t \neq l$, where the $\mathbb{K}_\phi(m_t)$ are the visited irreducible components by the basic loop. This proves that the extra term $\dfrac{1}{\chi(S)-\tau.\mathcal{P}}$ appears if and only if $\vec{n}_y \geq 1$ for some $y \in L$ satisfying $p_l(y) \geq= 1$.

The last point to check is that $a_{\vec{n}} = 0$ if and only if $b_{\vec{n}} = 0$. But notice that $b_{\vec{n}}$ has been chosen by taking some upper bound for the contributions of the minimal loops which compose the "loop" $\vec{n}$. The term $a_{\vec{n}}$ is obtained by taking lower bounds for the minimal loops which compose $\vec{n}$. However, for minimal loops, the different cases has been studied, and the coefficient for the lower bound in $a_l(y)$ equals 0 if and only if the coefficient for the upper bounded equals 0. This finishes the proof of the proposition. $\square$

Remark 8. We have $a_{\vec{n}} = 0$ as soon as $n_y \geq 2$ for any basic (and minimal) loop y which intersects $[i]$. Indeed, we are considering the first return in $[i]$!

Lemma 5.7. *There exists some constant $C = C_f$ which does not depend on S such that if $\sum_{\vec{n}} a_{\vec{n}} \prod |X_y|^{\vec{n}_y}$ converges, then $\sum_{\vec{n}} b_{\vec{n}} \prod \left(e^{-C}|X_y|\right)^{\vec{n}_y}$ also converges.*

Proof. This simply follows from the definition of $a_{\vec{n}}$ and $b_{\vec{n}}$: there exists some constant which only depend on f such that

$$e^{-|\vec{n}|C} \leq \frac{a_{\vec{n}}}{b_{\vec{n}}} \leq 1,$$

where $|\vec{n}| = \sum_y \vec{n}_y$. $\square$

The terms of the two power series in (16) which contains some $\dfrac{1}{\chi(S) - \tau.\mathcal{P}}$ are on the form

$$\frac{e^{-\beta.S}}{(\chi(S) - \tau.\mathcal{P})^{\gamma}} = \left(\frac{e^{-\beta/\gamma.S}}{\chi(S) - \tau.\mathcal{P}}\right)^{\gamma},$$

where γ is an integer and β is the transition cost of the basic loop. Namely γ is the number of visited irreducible components different from $\mathbb{K}_\phi(l)$, plus eventually 1 if this number is positive. Notice that the other terms are on the form $e^{-\delta.S}$ for some positive δ and comes from loops which does not meet any $\mathbb{K}_\phi(m)$ with $m \neq l$. As S goes to $+\infty$ the contribution of these terms goes to 0.

Definition 5.8. *We keep the previous notations. The number $\mathcal{I}(l) \stackrel{def}{=} \inf \dfrac{\beta}{\gamma}$ is well defined. It is called the isolation rate of the irreducible component $\mathbb{K}_\phi(l)$.*

We can now explain which irreducible components with maximal f-pressure can have positive limit measure:

Proposition 5.9. *Only irreducible components with maximal isolation rate can have positive limit measure.*

Proof. Let us assume that $\mathbb{K}_\phi(1)$ has not a maximal isolation rate; let us assume that for some subsequence of S (or equivalently of β) $\mathbb{K}_\phi(1)$ has positive limit measure. For simplicity we will write $S \to +\infty$ instead of taking the limit along the good subsequence. We use the notations from above.

• We claim that the term A in (16) is upper bounded (as $S \to +\infty$), thus the right side term is upper bounded.

Indeed, the first term in A is $1/\mu_\beta([i])$ which converges to some positive limit. The function H_S is bounded above and below from 0 by some constant which does not depend on S (see 2). Moreover the critical "T_c" for the operator $\widetilde{\mathcal{L}}_{T,[i]}$ is $\mathcal{P}'$, the f-pressure of $\mathbb{K}_\phi(1)\setminus[i]$. We still have $\mathcal{P}' < \mathcal{P}$, and $\chi(S)$ decrease to $\tau.\mathcal{P}$. Therefore, for every x in F, $\widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[i]}(r[i])$ is upper bounded by

$$e^{C_f}\widetilde{\mathcal{L}}_{\mathcal{P},[i]}(r_i)(x) \leq e^{2C_f}\frac{1}{\mu^1([i])}.$$

As m_S is a probability measure, we get the result.

• The fact that the term A in (16) is upper bounded implies that the lower bound in (16) is also upper bounded. But this lower bound is a countable sum of term on the form

$$\left(\frac{e^{-\mathcal{I}(1).S-\varepsilon.S}}{\chi(S)-\tau.\mathcal{P}}\right)^n,$$

where ε is non-negative. By definition of $\mathcal{I}(l)$, for every positive ε, a term $\dfrac{e^{-(\mathcal{I}(1)+\varepsilon').S}}{\chi(S)-\tau.\mathcal{P}}$ appears (to some power) at least one time in the sum, with some $0<\varepsilon'<\varepsilon$. This term is thus upper bound. Therefore, every term on the form $\left(\dfrac{e^{-\mathcal{I}(1).S-\varepsilon.S}}{\chi(S)-\tau.\mathcal{P}}\right)^n$ with $\varepsilon>0$ goes to 0 as S goes to $+\infty$.

• Let us now assume that $\mathbb{K}_\phi(2)$ has a maximal isolation rate. Let us pick some 1-cylinder $[j]$ which intersects $\mathbb{K}_\phi(2)$; let $\mathcal{F}$ be some unstable leaf in $[j]$. Then we can write a revisited (16) with a middle term

$$A' = \left|\int_{\mathcal{F}} r_{[j]}\,d\nu'_S - \int_{\mathcal{F}} \widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[j]}(r_{[j]}H_{S,j})\,d\nu'_S\right|.$$

We also have terms $a'_{\vec{n}}$ and $b'_{\vec{n}}$ instead of $a_{\vec{n}}$ and $b_{\vec{n}}$. But $\chi(S)$ does not depend on the 1-cylinder and then $\dfrac{e^{-\mathcal{I}(2).S}}{\chi(S)-\tau.\mathcal{P}}$ goes to 0 as $S\to+\infty$. Therefore, all the terms $\dfrac{(a_2(y))^{\vec{n}_y}}{(\chi(S)-\tau.\mathcal{P})^{q_2(y)}}$ uniformly go to zero as S goes to $+\infty$. Hence lemma 5.7 implies that the power series with the $b'_{\vec{n}}$ goes to 0. Thus, the term A' goes to 0. Now corollary 5.5 proves that the term $\displaystyle\int_{\mathcal{F}} \widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[j]}(r_{[j]}H_{S,j})\,d\nu'_S$ converges to $\dfrac{1}{\mu^2([j])}$. Doing the same for every $[j]$ which intersects $\mathbb{K}_\phi(2)$, we get that $\mathbb{K}_\phi(1)$ has positive limit measure and that the limit measure for $\mathbb{K}_\phi(2)$ equals 1. Therefore, the total limit measure is strictly larger than 1. This yields to a contradiction. $\square$

6 End of the proofs

6.1 Existence of the limit.

We can now finish the proof of the Theorem. We recall that proposition 5.9 proves that if there is only one irreducible component in $\mathbb{K}_\phi$ with maximal f-pressure and maxima isolation rate, then $\widehat{\mu}_\beta$ converges to the unique Equilibrium State of this irreducible component $\mathbb{K}_\phi$ when β goes to $+\infty$. In that case, the Theorem is proved (and is obvious). Therefore, and for the rest of the proof, we denote by $\mathbb{K}_\phi(1),\ldots,\mathbb{K}_\phi(q)$ the $q>1$ irreducible components in $\mathbb{K}_\phi$ with maximal f-pressure and with maximal isolation rate.

We already know any accumulation point μ of μ_β satisfies

$$\mu=\sum_j c^j\mu^j,$$

where μ^j is the unique equilibrium state associated to f in $\mathbb{K}_\phi(j)$. As any 1-cylinder meets at most only one irreducible component, the previous equality gives

$$\mu([i])=c^j\mu^j([i]),$$

for any 1-cylinder $[i]$ which intersects $\mathbb{K}_\phi(j)$. But $\mu^j([i])$ is well-determined. Thus, it is sufficient to prove that for any 1-cylinder $\mu_\beta([i])$ converges to prove the Theorem.

Let $\mathcal{I}$ be the maximal isolation rate. Our goal is to give an expression of any accumulation point of $\mu_\beta([i])$ in function of the associated accumulation point for $\dfrac{e^{-\mathcal{I}.S}}{\chi(S-\tau.\mathcal{P})}$

Proposition 6.1. *Let $\mathcal{I}$ be the maximal isolation rate. Let $[i]$ and l be such that the irreducible component $\mathbb{K}_\phi(l)$ intersects $[i]$ and has maximal isolation rate. Then there exists some $L_l\in\mathbb{R}^+\cup\{+\infty\}$ and some non-negative increasing function $F_l:[0,+\infty[\to\mathbb{R}^+\cup\{+\infty\}$ satisfying $F_l(x)=+\infty$ as soon as $x>L_l$ (when it makes sense) and $F_l(0)=0$, such that for every accumulation point for $\dfrac{e^{-\mathcal{I}.S}}{\chi(S)-\tau\mathcal{P}}$, L, we have*

$$\lim_\beta\mu_\beta([i])=\frac{\widehat{\mu}^l([i])}{1+\widehat{\mu}^l([i])F_l(L)},$$

where $F_l(L)$ belongs to $R^+\cup\{+\infty\}$ and where $\lim_\beta$ means following the family of S which gives the accumulation point L.

Before proving this proposition, we explain how we can deduce the Theorem from this result. We must have for every β

$$\sum_i\mu_\beta([i])=1,$$

because μ_β is a probability measure. Keeping only the irreducible components which have maximal f-pressure and maximal isolation rate we must thus have

$$1 = \sum_{l,i} \frac{\widehat{\mu}^l([i])}{1 + \widehat{\mu}^l([i])F_l(L)}. \tag{17}$$

The fact that each function is increasing and non-negative implies that there are only finitely many possible values for L such that (17) effectively holds. But L is an accumulation point for the continuous function $S \mapsto \dfrac{e^{-\mathcal{I}.S}}{\chi(S) - \tau\mathcal{P}}$; therefore the set of accumulation points for this function is a non-empty interval which contains only finitely many points. It is thus reduced to a single point; this proves that $\dfrac{e^{-\mathcal{I}.S}}{\chi(S) - \tau\mathcal{P}}$ converges to the unique possible L, and then that $\mu_\beta([i])$ has a unique possible accumulation point. Hence, it converges.

The rest of this subsection is devoted to the proof of proposition 6.1. The goal is to obtain some relation like (16) but with an exact equality. Let us assume that $\mathbb{K}_\phi(1)$ has a maximal isolation rate; let us pick some $[i]$ which intersects $\mathbb{K}_\phi(1)$. We want to compute $\int_F \mathcal{L}_{S,P(S)}(\mathbb{1}_C r_i.H_S)\, dm_S$ for each 1-set C in F, where F is some fixed unstable leaf in $[i]$. Notice that this first return time is constant on this 1-set. Hence we associate to each 1-set a unique piece of orbit (*ie* its word). As above, some of these words are "in" $\mathbb{K}_\phi(1)$, some others are pieces of orbits which leave $[i]$ reach some other irreducible component (and may be several) and then return in $[i]$. Orbits in $\mathbb{K}_\phi(1)$ produce the second term in the left side term in (16). Hence we only consider pieces of orbits on the second form.

6.1.1 Counting loops

The main point is to compute which orbits produce the $\mathcal{I}$. If a, b, c, d, e are positive real numbers, the function $x \mapsto \dfrac{d + ax}{e + xc + 1}$ increases (in x) if $\dfrac{a}{c} > \dfrac{d}{e+1}$, decreases (in x) if $\dfrac{a}{c} < \dfrac{d}{e+1}$ and is constant if $\dfrac{a}{c} = \dfrac{d}{e+1}$. This also holds if $e = 0$. Therefore three possibilities can occur:

1. $\mathcal{I} = \dfrac{d}{e+1}$,with $e > 0$, where there exists some basic and minimal loop y starting from $[i]$ which satisfies $a_l(y) = \dfrac{e^{-d.S}}{(\chi(S) - \tau.\mathcal{P})^e}$.

2. $\mathcal{I} = \dfrac{a}{c}$, with $c > 0$, where there exists some minimal loop y with empty intersection with $[i]$ which satisfies $a_l(y) = \dfrac{e^{-a.S}}{(\chi(S) - \tau.\mathcal{P})^c}$.

3. $\mathcal{I} = \dfrac{d}{e+1} = \dfrac{a}{c}$, where there exist two minimal loops y and z which satisfy $a_l(y) = \dfrac{e^{-d.S}}{(\chi(S)-\tau.\mathcal{P})^e}$ and $a_l(z) = \dfrac{e^{-a.S}}{(\chi(S)-\tau.\mathcal{P})^c}$, the loop y intersects $[i]$ but the loop z have an empty intersection with $[i]$ and both loops intersects the same 1-cylinder. In that case e can equal 0.

Notice that the three cases can occur in the same time. Moreover the last possibility can occur in a more complicated way, where the loop y meets several loops "z" or even the same loop several times. However the principle is the following :

1. the term $e^{-\mathcal{I}.S}$ due to the considered loop appears exactly one time in the power series and all the other terms are on the form $e^{-(\mathcal{I}+\varepsilon).S}$.

2. The term $e^{-\mathcal{I}.S}$ due to the considered loop never appears, but the more a basic loop uses z, the more the associated contribution is closed to $e^{-\mathcal{I}.S}$.

3. The term $e^{-\mathcal{I}.S}$ due to the considered loop appear infinitely many often in the power series.

6.1.2 Proof in the first case

Namely we are considering that $\mathcal{I}$ is only produced by basic and minimal loops y such that $\mathcal{I} = \dfrac{d}{e+1}$,with $e > 0$ and $a_1(y) = \dfrac{e^{-d.S}}{(\chi(S)-\tau.\mathcal{P})^e}$. The fact that e is positive means that the loop y meets some $\mathbb{K}_\phi(l)$ with $l \neq 1$.

•Let us first consider a loop y which meets only one other irreducible component $\mathbb{K}_\phi(2)$, and only one time. The contribution of al the loops with basic loop y is the sum over k of

$$\int (n_a+k+n_r)e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma_a(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-k\frac{\chi(S)}{\tau}-\chi(S)S_{n_a+n_r}(\phi)(y)}H_S(\xi)\,d\nu_S(\xi),$$

where n_a is the time that the point y needs to reach $\mathbb{K}_\phi(2)$ and n_r is the time that $\sigma^{n_a}(y)$ needs to reach $[i]$ and z is the associated preimage of ξ. Here the loop is exactly $y,\dots,\sigma^{n_a+n_r-1}(y)$. Let us denote by $[j]$ the 1-cylinder which contains $\sigma^{n_a}(y)$. Let F_j be any unstable leaf in $[j]$. The map $\sigma_a := \pi_{F_j}\circ\sigma^{n_a}$ is a bijection from the projection on F of the $(0,n_a)$-cylinder which contains y onto F_j. We denote by $F(2)$ the projection on F of the $(0,n_a)$-cylinder which contains y. Notice that this cylinder is a disjoint union of 1-sets. In the same way, there is a bijection σ_r from the projection on F_j of the $(0,n_r)$-cylinder which contains $\sigma^{n_a}(y)$ onto F.
Therefore any loop z whose basic loop is y and which have an excursion of length k in $\mathbb{K}_\phi(2)$ is a point which belongs to $F(2)$ and such that $\sigma^{n_r+n_a+k}(z) = \sigma_r(z')$ where $z' = \pi_{F_j}(\sigma^{n_a+k}(z))$.

Figure 2: computing orbits

However, the estimations done in the proof of proposition 5.6 proves that the global contribution of the terms in

$$\int (n_a + n_r) e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma_a(z)+S_{n_r}\circ\sigma^{n_a+k}(z)-\rho_y.S-k\frac{\chi(S)}{\tau}-\chi(S)S_{n_a+n_r}(\phi)(y)} H_S(\xi)\, d\nu_S(\xi),$$

is negligible in front of the contribution due to terms in

$$\int k e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma_a(z)+S_{n_r}\circ\sigma^{n_a+k}(z)-\rho_y.S-k\frac{\chi(S)}{\tau}-\chi(S)S_{n_a+n_r}(\phi)(y)} H_S(\xi)\, d\nu_S(\xi).$$

We now use the fact that $\widetilde{\mathcal{L}}_{\mathcal{P},[j]}$ has a spectral gap. Considering the semi-infinite subshift of finite type $\mathbb{K}_\phi(2)^+$ instead of the bi-infinite subshift $\mathbb{K}_\phi(2)$ means that we choose some unstable leaf in each 1-cylinder which intersects $\mathbb{K}_\phi(2)$ and we consider the projection onto these unstable leaves. We can do it such that we choose F_j in $[j]$. Then, we denote by $\widehat{\mathcal{L}}_{\mathcal{P},2}$ the Ruelle-Perron-Frobenius associated to f in the semi-infinite subshift of finite type $\mathbb{K}_\phi(2)^+$. This operator has a unique dominating eigenvalue, which is $e^{\mathcal{P}}$. It has thus a spectral gap, which ensures the exponential convergence of the iterations of this operator. Hence, lemma 1.12 in [3] applied to $\mathbb{1}_{[j]}$ yields that there exists some constant $0 < \lambda(2) < \mathcal{P}$ such that for every ξ in F_j and for every continuous function $\mathcal{T}$ on F_j

$$\left| \widehat{\mathcal{L}}^k_{\mathcal{P},2}(\mathcal{T})(\xi) - \int \mathcal{T}\, d\widetilde{\nu}^2 e^{k\mathcal{P}} H_2(\xi) \right| \leq ||\mathcal{T}||_\infty e^{k\lambda(2)}, \tag{18}$$

where $\widetilde{\nu}^2$ is equivalent to the projection onto $\mathbb{K}_\phi(2)^+$ of the measure μ^2 (namely $\widetilde{\nu}^2$ is the eigenvector for $\mathcal{L}^*_{\mathcal{P},2}$ associated to $\mathcal{P}$) and H_2 is the normalized eigenvector for $\mathcal{L}_{\mathcal{P},2}$. Notice that the Markov property means that (18) effectively holds for every ξ in F_j, even if the point does not belong to $\mathbb{K}_\phi(2)^+$; as it was said after corollary 5.5, it make sense to talk about H_2 in the whole F_j and not only in $F_j \cap \mathbb{K}_\phi(2)^+$.

Applying (18) with $\mathcal{T} := e^{S_{n_a}(f)\circ\sigma_a^{-1}}$ and multiplying by $e^{S_{n_r}(f)(.)-\rho_y.S} H_S \circ \sigma_r(.)$, we get

$$\left| \sum_z e^{S_{n_a+k+n_r}(f)(z)-\rho_y.S} H_S(\xi) - \left(\int e^{S_{n_a}(f)\circ\sigma_a^{-1}}\, d\widetilde{\nu}^2 \right) e^{k\mathcal{P}-\rho_y.S} e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)} H_2 \circ \sigma_r^{-1}(\xi) H_S(\xi) \right| \leq$$
$$||e^{S_{n_a}(f)\circ\sigma_a^{-1}}||_\infty e^{k\lambda(2)-\rho_y.S} e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)} H_S(\xi), \tag{19}$$

where ξ is any point in F, and the sum is over the points z such that their basic loop is y and $r_{[i]}(z) = n_a + k + n_r$ and $g_F(z) = \xi$. Therefore (19) yields to

$$\Big| k\sum_z e^{S_{n_a+k+n_r}(f)(z)-\rho_y.S-k\frac{\chi(S)}{\tau}-S_{n_a+n_r}(\phi)(y)\chi(S)}H_S(\xi)-$$
$$\left(\int e^{S_{n_a}(f)\circ\sigma_a^{-1}}\,d\widetilde{\nu}^2\right)ke^{-k(\frac{\chi(S)}{\tau}-\mathcal{P})-\rho_y.S-S_{n_a+n_r}(\phi)(y)\chi(S)}e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)}H_2\circ\sigma_r^{-1}(\xi)H_S(\xi)\Big| \leq$$
$$||e^{S_{n_a}(f)\circ\sigma_a^{-1}-S_{n_a+n_r}(\phi)(y)\chi(S)}||_\infty ke^{-k(\frac{\chi(S)}{\tau}-\lambda(2))-\rho_y.S}H_S(\xi)e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)}. \quad (20)$$

Remember that $\chi(S)\to\tau\mathcal{P}$ when S goes to $+\infty$, and that $\lambda(2)<\mathcal{P}$. Thus, integrating (20) and summing over k we get

$$\sum(k+n_a+n_r)\int e^{S_{n_a}(f)(z)+S_k(f)\circ\sigma_a(z)+S_{n_r}(f)\circ\sigma^{n_a+k}(z)-\rho_y.S-k\frac{\chi(S)}{\tau}-\chi(S)S_{n_a+n_r}(\phi)(y)}H_S(\xi)\,d\nu_S(\xi)=$$
$$\left(\int e^{S_{n_a}(f)\circ\sigma_a^{-1}}\,d\widetilde{\nu}^2\right).e^{-S_{n_a+n_r}(\phi)(y)\chi(S)}\left(\int e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)}H_2\circ\sigma_r^{-1}(\xi)H_S(\xi)\,d\nu_S\right)\frac{e^{-(\frac{\chi(S)}{\tau}-\mathcal{P})-\rho_y.S}}{(1-e^{-(\frac{\chi(S)}{\tau}-\mathcal{P})})^2}$$
$$+o(S), \quad (21)$$

with $\lim_{+\infty}o(S)=0$. Notice that

$$e^{-C(n_a+n_r)}.e^{-\rho_y.S}e^{-S_{n_a+n_r}(\phi)(y)\chi(S)}\leq o(S)\leq e^{C(n_a+n_r)}.e^{-\rho_y.S}e^{-S_{n_a+n_r}(\phi)(y)\chi(S)},$$

where C does only depend on f. This can be easily checked by using upper or lower bounds for f.

Notice that $\int e^{S_{n_a}(f)\circ\sigma_a^{-1}}\,d\widetilde{\nu}^2$ does not depend on S, that $e^{-S_{n_a+n_r}(\phi)(y)\chi(S)}$ converges when S goes to $+\infty$. Remember that ν_S is a probability measure on F and that H_S is upper bounded by some constant which does not depend on S. Now we have

$$\frac{e^{-(\frac{\chi(S)}{\tau}-\mathcal{P})-\rho_y.S}}{(1-e^{-(\frac{\chi(S)}{\tau}-\mathcal{P})})^2}=\frac{\tau^2e^{-\rho_y.S}}{(\chi(S)-\tau.\mathcal{P})^2}+o(S),$$

when S goes to $+\infty$. Therefore the contribution of all the loops whose basic loop is y exactly equals $A_y(S).\dfrac{a_l(y)}{\chi(S)-\tau.\mathcal{P}}+o_0(S)$, with

$$A_y(S)=\tau^2.\left(\int e^{S_{n_a}(f)\circ\sigma_a^{-1}}\,d\widetilde{\nu}^2\right).e^{-S_{n_a+n_r}(\phi)(y)\chi(S)}.\left(\int e^{S_{n_r}(f)\circ\sigma_r^{-1}(\xi)}H_2\circ\sigma_r^{-1}(\xi)H_S(\xi)\,d\nu_S\right),$$

and $o_0(S)\approx a_1(y)$. Also remember that $\dfrac{a_l(y)}{\chi(S-\tau.\mathcal{P})}=\left(\dfrac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}\right)^2$.

• If the loop y meets several other irreducible component, the same kind of computation can be made. We let the reader check that if $\mathbb{K}_\phi(l_1),\dots,\mathbb{K}_\phi(l_t)$ are the ordered visited irreducible components, then the global contribution of all the loops whose basic loop is y equals $A_y(S).\dfrac{a_l(y)}{\chi(S)-\tau.\mathcal{P}}+o_1(S)$, with

$$A_y(S)=\tau^{p_l(y)+1}.\left(\int e^{S_{n_{0,1}}(f)\circ\sigma_{0,1}^{-1}}\,d\widetilde{\nu}^{l_1}\right).e^{-S_n(\phi)(y)\chi(S)}.\prod_{k=1}^{t-1}\left(\int e^{S_{n_{k,k+1}}(f)\circ\sigma_{k,k+1}^{-1}(\xi)}H_k\circ\sigma_{k,k+1}^{-1}\,d\widetilde{\nu}^{l_{k+1}}\right)$$
$$\left(\int e^{S_{n_{t,t+1}}(f)\circ\sigma_{t,t+1}^{-1}(\xi)}H_k\circ\sigma_{t,t+1}^{-1}H_S(\xi)\,d\nu_S\right)+o(S), \qquad (22)$$

where n is the length of the loop y, $n_{k,k+1}$ is the time the loop y needs to go from $\mathbb{K}_\phi(l_k)$ to $\mathbb{K}_\phi(l_{k+1})$, with the convention $l_0=1=l_{t+1}$, H_k are functions which do not depend on S and $\circ\sigma_{k,k+1}$ is the bijection from the subset in unstable leaf in the 1-cylinder which intersects $\mathbb{K}_\phi(l_k)$ to unstable leaf in the 1-cylinder which intersects $\mathbb{K}_\phi(l_{k+1})$ as before σ_a and σ_r: namely, if $t=1$ $\sigma_{0,1}=\sigma_a$ and $s_{1,2}=\sigma^r$.
This claim is proved by induction. It was proved for $t=1$ above; the case $t=2$ is obtained in the same way, but we have to multiply the formula (18) by $e^{S_{n_{1,2}}(f)(.)-\rho_y.S}H_2\circ\sigma_{1,2}(.)$, to apply the operator and then to multiply by $e^{S_{n_{2,3}}(f)(.)-\rho_y.S}H_S\circ\sigma_{2,3}(.)$, and so on for the other t. The term $o_1(S)$ has order $e^{C(n_a+n_r)}.a_l(y)$. Here again C does only depend on f. The term $o_1(S)$ is thus much smaller than the first one for large enough S. More precisely we have

$$e^{-C(n_a+n_r)}.a_l(y)e^{-S_{n_a+n_r}(\phi)(y).\chi(S)}\leq o_1(S)\leq e^{C(n_a+n_r)}.a_l(y)e^{-S_{n_a+n_r}(\phi)(y).\chi(S)}.$$

Remark 9. Notice that when compute the sum over k, we have to compute the sum over all the $k_1+\ldots+k_{p_1(y)}=k$. This produces some $\frac{n^{p_1(y)-1}}{(p_1(y)-1)!}$. Then computing $\sum n^{p_1(y)}e^{n(\mathcal{P}-\chi(S)/\tau)}$ we produce some $p_1(y)!$; both terms are balanced.

In the previous calculation we have computed the dominating terms. Relation (16) proves that all the contribution of all the other terms goes to zero when S goes to $+\infty$. Moreover, each dominating term is associated to some "small" perturbation: there are the correcting terms "$o_0(S)$" and "$o_1(S)$". However each dominating term due to a loop y only introduces finitely many perturbations, which have order $(a_1(y))^{p_1(y)}$ (compared to $\dfrac{a_1(y)}{\chi(S)-\tau.\mathcal{P}}$ which is the order of the dominating term).

Therefore, if only basic and minimal loops produce the term $\mathcal{I}$, then the term $\mathcal{I}$ only appears finitely many times in the global sum. There are only finitely many dominating terms in the global contribution, thus finitely many perturbations, which are effectively much smaller than the dominating terms (remember that $\chi(S)\to\tau.\mathcal{P}$). In that case we have

$$\left|\int_F r_{[i]}\,d\nu_S-\int_F\widetilde{\mathcal{L}}_{\frac{\chi(S)}{\tau},[i]}(r_{[i]}H_S)\,d\nu_S\right|=F_{1,S}\left(\frac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}\right)(1+o(S)), \qquad (23)$$

where $F_{1,S}(.)$ is a polynomial with positive coefficient, and $o(S)$ goes to 0 when S goes to $+\infty$. Namely, the coefficients of $F_{1,S}(.)$ are the terms $A_y(S)$, where y describes the (finite) set of minimal basic loops which produce the $\mathcal{I}$. Remember that H_S is uniformly bounded (see (2)). This proves that all the coefficient are positive (the null coefficient have no contribution !). Now, we re considering families of S such that $\lim_\beta \mu_\beta([i])$ exists. As all the H_S are equicontinuous (see remark 2)), we always can assume that the H_S converge for some subfamily as Hölder continuous functions. This proves that the coefficients of $F_{1,S}$ converge to positive coefficients. Hence we set $L_1 := +\infty$

Let $L < +\infty$ be any accumulation point for $\dfrac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}$. For every basic loop, y such that $q_1(y) = 1$ we have

$$0 \leq a_1(y) \leq (\chi(S)-\tau.\mathcal{P})^{q_1(y)} \left(\frac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}\right)^{p_1(y)+1},$$

which yields to $\lim_S a_1(y) = 0$. This also holds for every basic and minimal loop y such that $q_1(y) = 0$ because in that case $a_1(y) = e^{-\rho_y.S}$. Therefore corollary 5.5 holds, which yield to the fact that H_S converge to "H_1". This proves that the limit F_1 does not depend on L. It is a polynomial with positive coefficient. Moreover the limit does not depend on $[i]$ but only on $\mathbb{K}_\phi(1)$, because for any $[i']$ which intersects $\mathbb{K}_\phi(1)$, there exists loop in $\mathbb{K}_\phi(1)$ which joins $[i']$ and $[i]$. Therefore, for any $[i']$ and for any loop y for $[i]$, we can construct one loop y' for $[i']$ by doing the looping of the loop y with the loop which joins $[i']$ and $[i]$.

Now, again corollary 5.5 yields to the fact that the second term in the left side of (23) converges to $\dfrac{1}{\mu^1([i])}$. Hence (23) is equivalent to

$$\lim_\beta \frac{1}{\mu_\beta([i])} = \frac{1}{\mu^1([i])} + F_1(L),$$

where L is any (finite) accumulation point for $\dfrac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}$.

Notice that the polynomial F_1 has a positive valuation, because for every basic loop y introduces a contribution proportional to $L^{p_1(y)+1}$. This proves proposition 6.1 in that case.

6.1.3 Proof in the second or the third case

We are now considering a basic loop y which can be decomposed in two parts. Some loop which intersects $[i]$, namely y' and some minimal loop z which intersects the irreducible component which produces the $\mathcal{I}$ (see fig. 3). The loop y is obtained by concatenating several times the loop z, and doing a looping with this result and the loop y'. We denote by y_N the loop obtained when we concatenate N times the loop z.

Figure 3: Second case

• Let us first assuming that the minimal loop y' does not meet any $\mathbb{K}_\phi(l)$. Let us denote by $[i']$ the 1-cylinder where we do the looping with z and y'. We fix some integer N and compute all the contributions due to the loop y_N.

We can copy (and adapt) what has been done before. Let us fix $k_1,\dots,k_N$ integers such that $k_1+\dots+k_N=k$. We claim that the sum over all the possibilities of $\displaystyle\int e^{S_n(f)(\xi)+\beta.S_n(\phi)(\xi)-nS}H_S(\zeta)d\nu_S(\zeta)$, where ξ is a preimage of ζ, is

$$e^{-S_{p+Nt}(\phi)(y_N)\chi(S)}\sum\int P_1(k_1)(\zeta)\dots P_N(k_N)(\zeta)H_S(\zeta)d\nu_S(\zeta)e^{-(\rho_{y'}.N\rho_z)S}, \tag{24}$$

where $P_i(k_i)(\zeta)$ is a polynomial in k_i of degree $p_1(z)-1$, $\sum k_i=k$, p is the length of y', t is the length of z, and ξ is the associated preimage of the point ζ. The claim is proved by using (18) by induction ; in the loop z (to obtain a relation closed to (22) it produces $P_i(k_i)(\zeta)$. More precisely the polynomial $P_i(k_i)$ equals

$$\alpha(p_1(z),k_i)e^{k_i(\mathcal{P}-\chi(S)/\tau)}A_i+T_i(\zeta)e^{k_i(\mathcal{P}-\chi(S)/\tau)}+Q_i(\zeta),$$

where $\alpha(p_1(z),k_i)$ is the number of possibilities such that $t_1+\dots t_{p_1(z)}=k_i$ (where the t_l are integers). The term A_i is the product of integrals like in (22), but for the considered loop. Notice that except for P_1 and P_N, all the A_i are equal. For A_1 there is the first part of the orbit, namely the loop y' till the 1-cylinder $[i']$ and for P_N there is the last part of the loop y' from $[i']$ back to $[i]$. The terms $T_i(\zeta)$ and $Q_i(\zeta)$ are the correcting terms, which can be upper and lower bounded by using (18). This bounds do not depend on ζ. A bound for the term $|T_i|$ is obtained by using (18), and when we take at least one time in all the possibilities some dominating term in $e^{t_l\mathcal{P}}$ (still with $t_1+\dots t_{p_1(z)}=k_i$). It is thus a polynomial of degree $p_1(z)-2$ in k_i. Except for 1 and N, this polynomial does not depend on i. The term $Q_i(\zeta)$ is obtained when we never take any dominating term is the previous sum. Thus, $|Q_i(\zeta)|$ is lower than $\alpha(p_1(z),k_i)e^{k_i(\lambda-\chi(S)/\tau)}C_i$ for some $\lambda<\mathcal{P}$ and some constant C_i. Here again, except for 1 and N, al the C_i are equal.

Upper bound for the contribution Taking such upper bounds, we get for upper bound the term

$$e^{-S_{p+Nt}(\phi)(y_N)\chi(S)}e^{-(\rho_{y'}.N\rho_z)S}$$
$$\prod_i(\alpha(p_1(z),k_i)e^{k_i(\mathcal{P}-\chi(S)/\tau)}A_i+B_i(k_i)e^{k_i(\mathcal{P}-\chi(S)/\tau)}+\alpha(p_1(z),k_i)e^{k_i(\lambda-\chi(S)/\tau)}C_i), \tag{25}$$

where $B_i(k_i).e^{k_i(\mathcal{P}-\chi(S)}$ is the upper bound for $T_i(\zeta)$. Notice that the first part of this term does not depend on k. Multiplying by k, and summing over k, we exactly get for the "product"

$$\frac{\partial}{\partial u}\prod_i\left(\sum_n(\alpha(p_1(z),n)A_i+B_i(n)+\alpha(p_1(z),n)C_ie^{-n(\mathcal{P}-\lambda)})e^{nu}\right)_{|u=\mathcal{P}-\chi(S)/\tau}. \quad (26)$$

Now, remember that B_i is a polynomial in n with degree $p_1(z)-2$ and that the dominating term for $u(p_1(z),n)$ is $\frac{n^{p_1(z)-1}}{(p_1(z)-1)!}$. Then we have

$$\sum_n(\alpha(p_1(z),n)A_i+B_i(n)+\alpha(p_1(z),n)C_ie^{-n(\mathcal{P}-\lambda)})e^{n.(\mathcal{P}-\chi(S)/\tau)}=\frac{A_i+o_i(S)}{(1-e^{\mathcal{P}-\chi(S)/\tau})^{p_1(z)}},$$

where $o_i(S)\to 0$ as S goes to $+\infty$. In the same way we have

$$\frac{\partial}{\partial u}\sum_n(\alpha(p_1(z),n)A_i+B_i(n)+\alpha(p_1(z),n)C_ie^{-n(\mathcal{P}-\lambda)})e^{nu}_{|u=\mathcal{P}-\chi(S)/\tau}=p_1(z)\frac{A_i+o_i'(S)}{(1-e^{\mathcal{P}-\chi(S)/\tau})^{p_1(z)+1}},$$

where $o_i'(S)$ also goes to 0 as $S\to+\infty$. Hence the global contribution of the loops y_N is upper bounded by

$$\frac{e^{-S_p(\phi)(y')-\rho_{y'}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\sum_N\left(Np_1(z)e^{-NS_t(\phi)(z)}\left(\frac{e^{-\frac{\rho_z}{p_1(z)}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^{Np_1(z)}\prod_{i=1}^N(A_i+o"_i(S))\right)+$$
$$\frac{e^{-S_p(\phi)(y')-\rho_{y'}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\sum_N\left((Nt+p)e^{-NS_t(\phi)(z)}\left(\frac{e^{-\frac{\rho_z}{p_1(z)}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^{Np_1(z)-1}\prod_{i=1}^N(A_i+o"_i(S))\right), \quad (27)$$

where $o"_i(S)\to_{S\to+\infty}0$. The first term is due to the sum over the length of the excursions (which was denoted by k above) and the second term is due to the rest of the length of the loops y_N, namely the length of y', p and N times the length of z, t. However this second term is infinitely many smaller than the first one, when S goes to $+\infty$. Now remember that except for A_1 and A_N, all the A_i are equal. Moreover $\rho_z/p_1(z)=\mathcal{I}$. Therefore the radius of convergence of this power series in $\left(\frac{e^{-CI.S}}{1-e^{CP-\chi(S)/\tau}}\right)$ is $L_1(z)\stackrel{def}{=}e^{S_t(\phi)(z)}/A$.

Lower bound To get a lower bound, we would like to exchange $B_i(k_i)$ and C_i by $-B_i(k_i)$ and $-C_i$. However this does not work so directly. Indeed, $B_i(k_i)$ and C_i are dominated terms for large enough k_i. In may happen that for small k_i there are no possible excursion with length k_i in any irreducible component ! However the bounds works for large enough k_i. Therefore we get a lower bound like (26), exchanging $B_i(n)$ and C_i with $-B_i(n)$ and $-C_i$ but for large enough n. Namely we have

$$\frac{\partial}{\partial u}\prod_i\left(\sum_{n\geq n_i}(\alpha(p_1(z),n)A_i-B_i(n)-\alpha(p_1(z),n)C_ie^{-n(\mathcal{P}-\lambda)})e^{nu}\right)_{|u=\mathcal{P}-\chi(S)/\tau}.$$

Here again n_i does not depend on i for every $2 \leq i \leq N-1$. We can thus do the same work as before ; the global contribution of the loops y_N is lower bounded by

$$\frac{e^{-S_p(\phi)(y')-\rho_{y'}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\sum_N\left(Np_1(z)e^{-NS_t(\phi)(z)}\left(\frac{e^{-\frac{\rho_z}{p_1(z)}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^{Np_1(z)}\prod_{i=1}^N(A_i+o_i'''(S))\right), \tag{28}$$

where $o_i'''(S) \to_{S\to+\infty} 0$.This power series has the same radius of convergence than the upper bound.

We can now finish the proof in that case. We thus assume that all the orbits which produce $\mathcal{I}$ are either on the first form, or on the second form, but with the assumption that every such loop is on the form on y_N. We set $L_1 \stackrel{def}{=} \min \dfrac{(L_1(z))^{\frac{1}{p_1(z)}}}{\tau}$, where the minimum is taken over all the z which produce $\mathcal{I}$.

We claim that $\limsup_S \dfrac{e^{-\mathcal{I}.S}}{\chi(S-\tau.\mathcal{P})}$ is lower than L_1. Indeed, if this does not holds, there are infinitely many S such that one of the power series diverges (and equals $+\infty$ as the sum of positive numbers). This would implies that $\mu_\beta([i]) = 0$ for infinitely many β. However we know that for every Gibbs measure ν, every cylinder has positive measure. This produces a contradiction.

As we still have

$$a_1(y) \leq (\chi(S)-\tau.\mathcal{P})\left(\frac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}\right)^{p_1(y)+1},$$

for any minimal loop y such that $q_1(y) = 1$, and $a_1(y) = e^{-\rho_y.S}$ if $q_1(y) = 0$, then $a_1(y)$ converges to 0 for any y, as S goes to $+\infty$. Here again we can use corollary 5.5 to get that H_S converges to some fixed function H_1 which does not depend on S. We now can copy the end of the proof in the first case. Notice that we now have power series, but all of them have positive valuation and positive coefficients. However we can precise how do these power series influence the limit:

1. If $\rho_{y'}/(p_1(y')+1)$ is strictly larger than $\mathcal{I}$ (the case where $\mathcal{I}$ never appears in the power series), then the global contribution due to this loop z is null because $\dfrac{e^{-\rho_{y'}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}$ goes to 0 when S goes to $+\infty$.

2. If $\rho_{y'}/(p_1(y')+1) = \mathcal{I}$, this introduces some $(\tau L)^{p_1(y')}$ in front of the previous power series.

Notice that the case $\rho_{y'}/(p_1(y')+1) < \mathcal{I}$ can not occur, otherwise the loop y would not produce any $\mathcal{I}$.

Remark 10. We can also get an explicit and simpler expression for the contribution of the loops y_N. Indeed the power series is a geometric power series (up to some

terms), which converges to $\dfrac{p_1(z)(\tau.X)^{p_1(z)}}{1-Ae^{-S_t(\phi)(z)\chi(S)}(\tau.X)^{p_1(z)}}\dfrac{A_a.A_r}{A^2}\dfrac{\tau.e^{-S_p(\phi)(y')\chi(S)-\rho_{y'}.S}}{\chi(S)-\tau.\mathcal{P}}$,
where $X=\dfrac{e^{-\mathcal{I}.S}}{\chi(S)-\tau.\mathcal{P}}$; A is the product of integrals which appear along the loop z:

$$A=\prod_{k=1}^{g}\left(\int e^{S_{n_{k,k+1}}(f)\circ\sigma_{k,k+1}^{-1}(\xi)}H_k\circ\sigma_{k,k+1}^{-1}\,d\widetilde{\nu}^{l_{k+1}}\right),$$

as in (22). A_a and A_r are the same term than A, except that the first integrals for A_a "starts" from $[i]$ and the last integral for A_r returns to $[i]$. Remember that in that integral, H_S has disappear because it uniformly converges to some function H_1.

• Other cases for the loop y can happen. The loop y' can meet other irreducible components. In that case we have to consider the possible excursions in irreducible components during the loop y'. Summing over all the possibilities, the same argument just introduces the polynomial $A_{y'}L^{p_1(y')}$ (before the derivative).

The loop y' can also meet several loops "z" which produce the $\mathcal{I}$. Here two possibilities appear. The order for these loops z is fixed. Then the global power series is the product of the powers series due to each "z_g". Notice that the length of y' is upper bounded because it is a minimal loop, and two extra-loops z_a and z_b cannot share any symbol, otherwise we could combine them in any order we want. The second possibility occurs when we can combine the loops in ay order we want (up to some starting and finishing constraints). It happens when the loops z_g share some symbol. However we recall that the loops z_g are minimal loops and thus there are only finitely many such loops. If $z_1,\dots,z_g$ are g such loops that we can combine in any order, we let the reader check that the power series which appears in that case is the derivative of the power series

$$\sum_n\left(\left(A_1\frac{e^{-\mathcal{I}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^{p_1(z_1)}+\ldots+\left(A_g\frac{e^{-\mathcal{I}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^{p_1(z_g)}\right)^n,$$

where the dominating power series for z_j is $\sum\left(A_j\frac{e^{-\mathcal{I}.S}}{1-e^{\mathcal{P}-\chi(S)/\tau}}\right)^n$. As there are only finitely possible A_j, this proves that the radius of convergence of such a power series is positive;

6.2 About finiteness.

As we said in the introduction, this method can also prove the finiteness of the number of the possible limit measures. Remember that every locally constant function ϕ can be seen as constant on every 1-cylinder, up to some higher block representation. Then the corollary concerns all the locally-constant functions.

Proof. For such a ϕ, $\mathbb{K}_\phi$ is constructed from the minimal periodic orbits. There are finitely many minimal periodic orbits, thus there are only finitely many ways to

associate them. Therefore there are only finitely possible “$\mathbb{K}_\phi$’s”. But each possible $\mathbb{K}_\phi$ admits finitely many ergodic Equilibrium States. This proves that there are only finitely many possible ergodic. The last point to check is that there are only finitely many possible barycenters of these ergodic Equilibrium States. This results from our construction:
Using the formula given along the way (for instance in remark 10 or (22)), we let the reader check that as soon as the minimal loops which produce the $\mathcal{I}$ has been chosen, the relation (17) can be written in the following way:

$$1 = \sum_i \frac{a_i(\tau.L.e^{-\mathcal{I}.\mathcal{P}})^{r_i}}{1 - c_i(\tau.L.e^{-\mathcal{I}.\mathcal{P}})^{p_i}}, \tag{29}$$

where the sum is done over the loops which produce the $\mathcal{I}$ and a_i, c_i, r_i and p_i do only depend on the loop. Indeed the main point to understand is that for such a loop z, and with the notations from above, $\mathcal{I} = (t - S_t(\phi)(z)\tau)/p_1(z)$, which yields to the fact that $S_t(\phi)(z)\chi(S)$ converges to $\mathcal{P}.(t - p_1(z).\mathcal{I})$ as S goes to $+\infty$. Notice that $p_1(z)$ does not depend on ”ϕ” as soon as the irreducible components and the loops have been chosen. In the same way, if $e^{-\rho_{y'}.S}/\chi(S) - \tau.\mathcal{P}$ does not converge to 0, it means that it converges to L, and so on.

Therefore we claim that, as soon as the irreducible components which have positive limit measure have been chosen (finitely many choices) and as soon as the loops which produce the $\mathcal{I}$ have been chosen (finitely many choices), to solve (17) is equivalent to solve

$$1 = \sum_i \frac{a_i X^{r_i}}{1 - c_i X^{p_i}}. \tag{30}$$

Such an equation admits only one solution X_0. Now each $\mu([k])$ is $\sum_i \frac{a_i X_0^{r_i}}{1-c_i X_0^{p_i}}$ where we only consider the terms due to this 1-cylinder.

□

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