On t-conformal measures and Hausdorff dimension for a family of non-uniformly hyperbolic horseshoes

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Abstract

In this paper we consider horseshoes with homoclinic tangencies inside the limit set. For a class of such maps, we prove the existence of a unique equilibrium state μ_t, associated to the (non-continuous) potential $-t \log J^u$. We also prove that the Hausdorff dimension of the limit set, in any open piece of unstable manifold, is the unique number t_0 such that the pressure of μ_{t_0} is zero. To deal with the discontinuity of the jacobian, we introduce a countable Markov partition adapted to the dynamics, and work with the first return map defined in a rectangle of it.

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1 Introduction and statement of results

In this paper we study a family of surface diffeomorphisms displaying a non-uniformly hyperbolic horseshoe Λ. The lack of uniform hyperbolicity, here, is due to the presence of an internal homoclinic tangency, that is, a tangency in the closure of the set of periodic points of Λ. We prove the existence and uniqueness of equilibrium states for the family $-t \log J^u$, where $J^u(x)$ is the Jacobian of the diffeomorphism in the unstable direction $E^u(x)$. In the context we are considering, the existence and uniqueness of equilibrium states were proved for Hölder continuous potentials in [18]. Due to the presence of the tangency, the unstable jacobian of the system fails to be continuous, and so it needs special attention.

For any $t \geq 0$, the equilibrium state associated to $-t \log J^u$ is usually referred to as the t-conformal measure. In hyperbolic dynamics, this family of measures is meaningful, and carries a lot of information. Indeed, when the SRB-measure exists,

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it is the 1-conformal measure. Still in the hyperbolic setting, in the codimension one case, if $\mathcal{P}(t)$ denotes the pressure of the t-conformal measure, the map $t \mapsto \mathcal{P}(t)$ vanishes for some special value t_0; this t_0 is the Hausdorff dimension of the hyperbolic set in the unstable direction. These results are quite general, holding for various classes of hyperbolic dynamical systems. In [20], the t-conformal measures were studied for the Axiom-A case. They were also studied in the context of complex dynamics, see *e.g.* [8]. Some literature is also available for the one-dimensional and non-uniformly hyperbolic case. We refer, for instance, to [7] and [4]. More recently, Pesin and Senti studied the one-dimensional case, and a class of multi-dimensional systems admitting "Young" towers (see *eg* [23]). Note that for these systems, the limit set has positive Lebesgue measure in the unstable direction. This does not occur in our case (see [25]).

In the specific case of horseshoes, Palis and Yoccoz study systems with heteroclinic bifurcations in [21]. In the class of maps they consider, as the (external) tangency unfolds, new orbits are being added to an underlying uniformly hyperbolic horseshoe. They show that the Hausdorff dimension increases after the bifurcation. Though in this present work we do not unfold the tangency, we believe that the kind of tangency we have corresponds to a further destruction of dynamics.

As far as we know, there is no general theory to deal with the different causes of loss of uniform hyperbolicity in abstract. Most of the existing results are based on examples or models. The horseshoes we deal with in this paper were first introduced in [25]. There, it was proved that they are at the boundary of the set of uniformly hyperbolic systems. The hyperbolic splitting was also constructed there. Although the model represents a somehow simplified situation (one single orbit of homoclinic tangency as the reason for the lack of uniform hyperbolicity), it illustrates many of the difficulties that appear in the general setting.

The results here are very similar to those in the uniformly hyperbolic case. However, the usual tools do not apply directly, and we have to adapt them to this case. We refer the reader to the works of Sinai, Ruelle and Bowen (see [28], [27] and [2]) for an overview on transfer operators and equilibrium states which are used in depth here.

One main difficulty here is to control the distortion of $\log J^u$ between pieces of two different orbits. Usually, the hyperbolicity of the map and the regularity of the potential help one to do that. Here the homoclinic tangency and the discontinuity of the potential make the estimates harder. The main strategy to bypass the difficulties is to construct a countable Markov partition whose elements are smaller near the orbit of tangency. The partition is constructed so that the first-return map for any of its elements is hyperbolic. We then use the machinery constructed in [16] to produce local equilibrium states.

The next problem is to open-out the measure constructed for the induced map. Following [9], this can be done if and only if the expectation of the first return-time is finite. To get this latter condition, one usually uses that $\mathcal{P}(t)$ is strictly inside the domain of convergence of some zeta function (see [3]). Note that, in some cases, it is equivalent to the uniqueness of the equilibrium state (see again [3], Proposition 2.3). In [23], for instance, they consider a different class of examples, and prove, for the multidimensional case, the condition on $\mathcal{P}(t)$ for values of t close to 1. In our case, we use a different method, and the existence of the equilibrium state will imply its uniqueness. This applies to an interval of values of t containing 0 and 1. As a by-product, we get that $\mathcal{P}(t)$ is strictly inside the mentioned domain of convergence, and that $t \mapsto \mathcal{P}(t)$ is analytic.

Finally, we want to point out that, due to the discontinuity of the map $\log J^u$, even the existence of the equilibrium state is not obvious. We are able to prove existence (and uniqueness) only for t such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. In, some cases, though, the constant $\frac{1}{2}\log\sigma$ is optimal, as we prove in Subsection 2.4. This kind of condition was also stated in [19]. There, Makarov and Smirnov prove that, for a rational map on $\mathbb{C}$, the function $t \mapsto \mathcal{P}(t)$ is real analytic as long as $\mathcal{P}(t) > -kt$ for some positive k. They claim that their arguments could probably be adapted to other kind of maps. In some sense, it is what we do here, though the present paper was written independently.

We would like to point out some differences between their methods and ours. First, the kind of maps we deal with is more general. Also, in our case, we do not have the "Yoccoz puzzle". In fact, part of our work consists in constructing a good Markov partition. The fact that our Markov partition is geometrically defined, and the fact that a similar approach works for a different setting (rational maps on $\mathbb{C}$ in [19]), suggest that this method is very powerful and could work for a large class of non-uniformly hyperbolic dynamics. The last difference with [19] is that we deal with measures. Indeed, they prove analyticity of the pressure using the fact that $\mathcal{P}(t)$ is strictly inside a domain of convergence of some zeta-function. In our case, analyticity comes from the existence (and the uniqueness) of the equilibrium state. Finally, Makarov and Smirnov prove that $t \mapsto \mathcal{P}(t)$ is not necessarily analytic on $[0, +\infty[$, and that it could have a phase transition. It is a very interesting question to see whether a similar result holds in our case.

1.1 Preliminaries

Here we define the maps we are considering, and summarize the main properties and results that we use throughout this paper.

1.1.1 The map

Let $\lambda < 1/3$, $\sigma > 3$. For $c > 0$ be large, we construct a one-to-one $\mathcal{C}^2$ map f from $\mathcal{Q} = [0,1]^2$ into $\mathbb{R}^2$ satisfying the following conditions (see Figure 1):

a) $f(x,y) = (\lambda x, \sigma y)$, if $0 \le y \le \sigma^{-1}$ (region R_1).

b) $f(x,y) = (\lambda x + (1-\lambda), \sigma y - (\sigma - 1))$ if $1 - \sigma^{-1}2/3 \le y \le 1$ (region R_5).

c) There exists a horizontal strip (region R_3) contained in $[0,1]\times[1/3,1]$, depending on c, which is mapped affinely into a vertical strip, parallel to the image of the region R_5. The derivative of f at points of this region is

$$Df(x,y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\sigma \end{pmatrix}.$$

d) Points of $\mathcal{Q}$ which are between R_1 and R_3 (region R_2) are mapped outside $\mathcal{Q}$.

e) There exists, between R_3 and R_5, a region R_4, bounded by two disjoint curves of the form $\{y = \psi(x) : x \in [0,1]\}$, in which the map is not affine. In this region we have:

i) The top and bottom sides of R_4 are mapped into R_2, outside the image of R_1.

ii) $f\left[\{(0,y) : y \in \mathbb{R}\} \cap R_4\right]$ is contained in the graph of the map $f_0(x) = c(x-q)^2$, with $\|\frac{\partial f}{\partial y}(0,y)\| \geq \sigma$, where $q \in (2/3,1)$

iii) For every x_0 in $[0,1]$, $f\left[\{(x_0,y) : y \in \mathbb{R}\} \cap R_4\right]$ is contained in the graph of the map $f_{x_0}(x) = c(x-q)^2 - \lambda x_0$, with

$$\left\langle \frac{\partial f}{\partial y}(x,y), \frac{\partial f}{\partial x}(x,y) \right\rangle = 0$$

and

$$\|\frac{\partial f}{\partial x}(f^{-1}(q,0))\| = \lambda.$$

Note that we want the image of $[\{(0,y) : y \in \mathbb{R}\} \cap R_4]$ not to intersect the right side of $\mathcal{Q}$.

f) Points between R_3 and R_5 which are outside R_4, are mapped inside region R_2 with second coordinate greater than σ^{-1}. We just ask the map to be smooth at this points, and globally one-to-one.

For the construction and good hyperbolic properties, we also ask in [18] that, when we move the parameters λ and σ, they always satisfy

$$b^{-1} < -\frac{\log \lambda}{\log \sigma} < b$$

for some fixed b. This condition is compatible with the hypotheses in [25].

In Figure 1, $R_i' = f(R_i)$ for $i = 1, \ldots, 5$. Note that f can be extended to $\mathbb{R}^2$ in such a way that $(0,0)$ is a hyperbolic fixed point. The left side and the bottom side of $\mathcal{Q}$ are contained, respectively, in its unstable and stable manifolds. That implies that $Q = (q,0)$ is a point of homoclinic tangency.

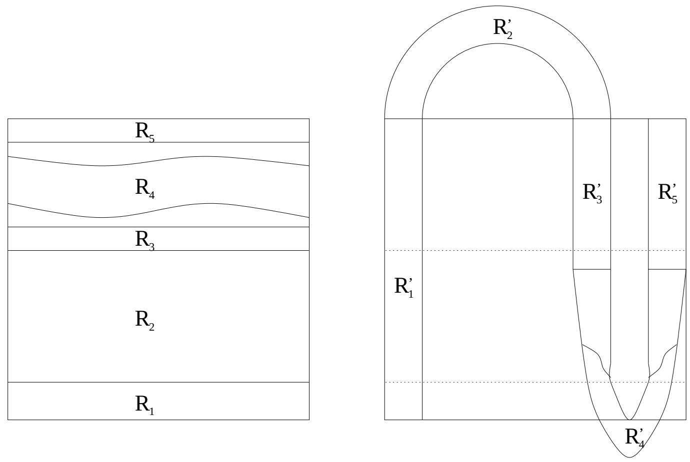

Figure 1: The map f

Notation Throughout the paper, for $n \in \mathbb{Z}$ and $M \in \Lambda$, M_n shall denote $f^n(M)$.

1.1.2 The hyperbolic splitting

Let Λ denote the maximal invariant set of f in $\mathcal{Q}$, $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{Q})$. Following [18], through each point $M \in \Lambda$ outside the critical orbit (the orbit of tangency), there are one-dimensional tangent spaces $E^{u,s}(M)$ and local $C^{1+\varepsilon}$ manifolds $W^{u,s}_{loc}(M)$ satisfying the following:

- $W^u_{loc}(M)$ is tangent to $E^u(P)$ for each $P \in W^u_{loc}(M)$;
- $W^s_{loc}(M)$ is tangent to $E^s(P)$ for each $P \in W^s_{loc}(M)$;
- $W^u_{loc}(M)$ is the graph of a $(C, 1/2)$-Hölder continuous function $x = g^u_M(y)$, $y \in [0,1]$ and $W^s_{loc}(M)$ is the graph of a $(C, 1/2)$-Hölder continuous function $y = g^s_M(x)$, $x \in [0,1]$;
- $f(W^u_{loc}(M)) \supset W^u_{loc}(f(M))$ and $f^{-1}(W^s_{loc}(M)) \supset W^s_{loc}(f^{-1}(M))$.

For $M, M' \in \Lambda$, we denote by $\lfloor M; M' \rfloor$ the unique point in $W^u_{loc}(M) \cap W^s_{loc}(M')$. We call a *rectangle* a set $R \subset \mathcal{Q}$, such that, for any M and M' in R, $\lfloor M; M' \rfloor$ also belongs to R. If R is a rectangle, we set, for $i = u, s$ and $M \in R$,

$$W^i(M, R) = W^i_{loc}(M) \cap R.$$

A rectangle is said to be *proper* if it is the closure of its interior for the topology induced in Λ. We also say that $\{R_j\}$ is a *Markov cover* if all the R_j are proper rectangles and for any M

$$M \in R_j \cap f^{-1}(R_k) \Longrightarrow \begin{cases} W^u(f(M), R_k) \subset f(W^u(M, R_j)), \\ W^s(f(M), R_k) \supset f(W^s(M, R_j)). \end{cases}$$

Finally, we say that $\{R_j\}$ is a Markov partition if it is a Markov cover and a partition.

1.1.3 The critical zone and the potential $\log J^u$

Let A be the intersection of the region R'_4 (the image of R_4) with the horizontal region R_1. We call the *escape time* of a point M in A, the biggest positive integer n such that for every $0 < k < n$, $M_k \in R_1$.

If $M = (x, y)$ is in A, we set $l(M) = |x - q|$. If M is in $\mathcal{Q} \setminus A$, we set $l(M) = \sup_{\xi \in A} l(\xi)$.

Recall that, by definition, the images by the map f of the vertical lines intersected to A are pieces of parabolas that are called *local parabolas* in A. For $M = (x, y)$ in A, the stable direction $E^s(M)$ is almost horizontal; the slope of the unstable direction (with respect to the two vectors $(1, 0)$ and $(0, 1)$) is $2cCx$, with C in $[\frac{1}{3}, 3]$ (see [18] for a proof). Note that $M \mapsto E^{u,s}(M)$ can be defined by continuity for every point M in the critical orbit: if $M = Q_n$ with $n \geq 0$, then we set $E^u(M) = E^s(M) = (1, 0)$, and if $M = Q_n$ with $n < 0$, then we set $E^u(M) = E^s(M) = (0, 1)$. We point out that with this (natural) definition, the map $M \mapsto \log J^u(M)$ is defined on Λ and continuous in $\Lambda \setminus (0, 0)$.

1.1.4 The map F

In [18], we introduced a new map F defined as follows:

- if M belongs to R_3 or R_5, then $F(M) = f(M)$,
- if M belongs to A with escape time n, and $f^n(M) \in R_3 \cup R_5$, then $F(M) = f^n(M)$,
- if M belongs to A with escape time n, and $f^n(M) \in A' = f^{-1}(A)$, then $F(M) = f^{n+1}(M)$,
- and if M belongs to $R_4 \setminus A'$, then $F(M) = f(M)$.

The map F is "uniformly hyperbolic" in the following sense: if $F(M) = f^n(M)$, then $|DF(x)_{|E^u(x)}| \geq e^{n\frac{1}{2}\log\sigma}$ and $|DF(x)_{|E^s(x)}| \leq e^{n\frac{1}{2}\log\lambda}$.

Note that the set where F is defined is contained in $A \cup R_3 \cup R_5 \cup (R_4 \setminus A')$. The map F is not defined on the segment $[0, 1] \times \{0\} = W^s_{loc}(0)$, and the map F^{-1} is not defined on the segment $\{0\} \times [0, 1] = W^u_{loc}(0)$. We denote by Λ_F the set of points in Λ for which F is defined and which do not belong to $W^u_{loc}(0) \cup W^s_{loc}(0)$. In other words, points in Λ_F are points whose f-orbit can be decomposed in a F-orbit. Note that Λ_F is not compact.

1.1.5 The dynamical partition

For $n, m \in \mathbb{N}$, we define the (degenerated) partition $\mathcal{G}^n_m$ as follows. First, note that $f^n(\mathcal{Q})$ intersects $\mathcal{Q}$ in a set of 3^n stripes crossing $\mathcal{Q}$ from the bottom to the top. From these stripes, there are n pairs connected at some forward image of the homoclinic tangency $(q, 0)$. Analogously, $f^{-m}(\mathcal{Q})$ intersects $\mathcal{Q}$ along 3^m horizontal stripes, m pairs of them connecting at preimages of the homoclinic tangency.

The rectangles of the partition $\mathcal{G}_m^n$ are determined by the intersection of one of the vertical stripes with one of the horizontal stripes. Note that there are $n+m$ pairs of rectangles that are not disjoint, but each pair intersects at exactly one (critical) point. The other rectangles are open in the topology of Λ, and all of the rectangles have for boundary pieces of the stable and unstable manifolds of the fixed saddles $(0, 0)$ and $(1, 1)$. We also have that, for each rectangle R in $\mathcal{G}_m^n$, $f^m(R)$ is a stripe crossing $\mathcal{Q}$ from bottom to top, and $f^{-n}(R)$ is a stripe crossing $\mathcal{Q}$ from left to right.

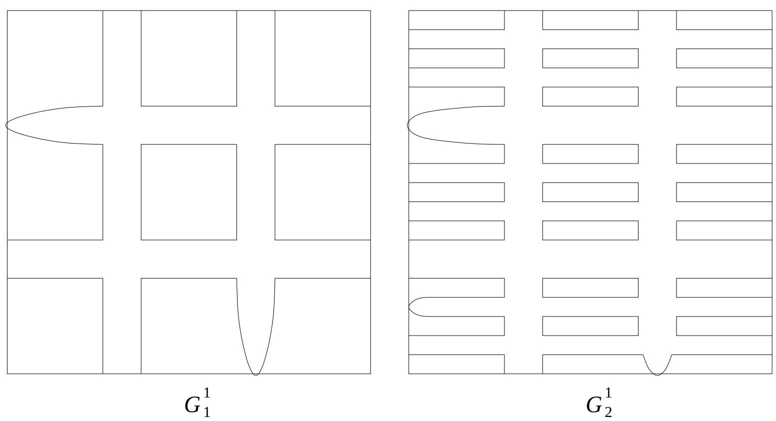

Figure 2: the partitions $\mathcal{G}_1^1$ and $\mathcal{G}_2^1$

In [18] we used the partition $\mathcal{G}_0^1$ to construct a finite-to-one semi-conjugacy $\Theta \colon \Sigma_3 \to \Lambda$ between the full 3-shift and the horseshoe Λ. More precisely, Θ is one-to-one, except for the critical orbit on Λ where it is two-to-one. We also showed that Θ is Hölder continuous, and used this fact to prove the existence and uniqueness of the equilibrium states (for Hölder continuous potentials).

1.2 Statement of the results

Let $\mathcal{M}_f$ be the set of f-invariant probability measures. We define

$$\mathcal{P}(t) = \sup_{\mu \in \mathcal{M}_f} \left\{ h_\mu(f) - t \int \log J^u(x)\, d\mu(x) \right\},$$

where $h_\mu(f)$ denotes the entropy of the measure μ. The quantity $\mathcal{P}(t)$ is called the *pressure* for the potential $-t \log J^u$. The number $\mathcal{P}(t)$ will also be called the *t-pressure*, since $-t \log J^u$ is the only potential considered here. Note that, though the map J^u is not continuous, it is bounded. Moreover, for any ergodic measure μ, the term $\int \log J^u\, d\mu$ denotes the unstable Lyapunov exponent of f for μ. It is proved in [5] that, for any ergodic μ in $\mathcal{M}_f$,

$$\frac{1}{2}\log\sigma \le \int \log J^u \, d\mu \le \log(2\sigma). \tag{1}$$

We also recall that, for any μ in $\mathcal{M}_f$, μ-almost every point in $\mathcal{Q}$ has only one preimage by the map Θ. Therefore, $0 \le h_\mu(f) \le \log 3$, and the t-pressure is well defined. We also have that $\mathcal{P}(0) = h_{top}(f) = \log 3$. Convexity of $\mathcal{P}(t)$ is a consequence of the definition.

The inequality (1) gives

$$\log 3 - t\log(2\sigma) \le \mathcal{P}(t) \le \log 3 - \frac{t}{2}\log\sigma.$$

Therefore, $t \mapsto \mathcal{P}(t)$ is a decreasing function and $\lim_{t\to+\infty}\mathcal{P}(t) = -\infty$.

We recall that an *equilibrium state* associated to the potential $-t\log J^u$ is a measure $\mu \in \mathcal{M}_f$ such that $\mathcal{P}(t) = h_\mu(f) - t\displaystyle\int \log J^u \, d\mu$. It means that the supremum is reached, and thus it is a maximum.

Theorem A. *For non negative t, and as long as $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$, the map $t \mapsto \mathcal{P}(t)$ is analytic, and there exists a unique equilibrium state associated to the potential $-t\log J^u$.*

From now on, t shall denote a non negative real parameter.

The lower bound $-t\frac{1}{2}\log\sigma$ is, in some cases, an optimal bound, in the sense that $\mathcal{P}(t) \ge -t\frac{1}{2}\log\sigma$ for every t, and there is no $k < \frac{1}{2}\log\sigma$ such that $\mathcal{P}(t) \ge -kt$ for all $t > 0$. We will prove that this happens when $\sigma.\lambda \le 1$ (see Subsection 2.4). When the bound is not optimal, the theorem yields the existence and uniqueness of the t-conformal measure for every t in some interval $[0, t_{max}[$. We point out that, even in the "optimal" case, existence and uniqueness may occur only for $t < t_{max}$: it could happen that $\mathcal{P}(t) \equiv -t\frac{1}{2}\log\sigma$ for t large enough.

For t such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$, the unique equilibrium state associated to $-t\log J^u$ will be denoted by μ_t. It is well-known that, for any $\mu \in \mathcal{M}_f$, we have $h_\mu(f) \le \int \log J^u \, d\mu$. Moreover (see [14]), $h_\mu(f) = \int \log J^u \, d\mu$ holds if and only if μ is a u-Gibbs state. This means that the disintegration of μ along the unstable leaves (see [26] or [22]) is absolutely continuous with respect to the induced Lebesgue measure Leb^u.

Note that the unstable Hausdorff dimension of the horseshoe considered here is strictly less than 1, if we assume that σ and λ^{-1} are big enough (see [25] for a proof that the limit capacity of this set is small). Therefore, the u-Gibbs states do not exist in this case. Hence, we have that $\mathcal{P}(1) < 0$, independently of the comparison between $\mathcal{P}(1)$ and $-\frac{1}{2}\log\sigma$. There thus exists some unique t_0 such that $\mathcal{P}(t_0) = 0$.

Theorem B. *The Hausdorff dimension of any open piece of unstable leaf W^u intersected with Λ is t_0.*

One of the fundamental steps to construct equilibrium states associated to a potential φ is that

$$\omega(x,x') = \sum_{n=0}^{+\infty} \varphi \circ f^n(x) - \varphi \circ f^n(x')$$

converges as long as x and x' belong to the same stable leaf. In the uniformly hyperbolic setting, this happens as a consequence of the fact that φ is Hölder continuous.

In our case, for the map f and the potential $\varphi = -t \log J^u$, the presence of a critical orbit introduces a distortion that must be controlled in order to have the convergence of the series. In fact, the closer to the critical point we are, the worse hyperbolicity is, and the bigger the distortion bound must be. To get the necessary estimates, we have to work with an adapted Markov partition, with countably many "rectangles", whose sizes decrease to 0 close to the critical orbit.

1.3 Structure of the paper

In section 2, we prove the existence of equilibrium states. For that, we prove the upper semi-continuity of the metric entropy. We also study what are the possible accumulation points for sequences of unstable Lyapunov exponents $\lambda^u(\mu_n)$. We prove that any equilibrium state μ satisfies $\mu\{(0,0)\} = 0$ (in particular, $\delta_{(0,0)}$ cannot be an equilibrium state). This finally yields the existence of the t-conformal measures. Finally, we prove that the condition $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$ is optimal in the case $\sigma.\lambda \leq 1$.

In section 3, we construct a countable Markov partition in rectangles and compute the distortion we mentioned above for points in the same rectangle. We then prove uniqueness of the equilibrium state, in the case where $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. We also prove that $t \mapsto \mathcal{P}(t)$ is analytic when $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. This completes the proof of Theorem A.

In section 4, we compute the unstable Hausdorff dimension and prove Theorem B. For that we follow Mc Cluskey *et al.* [20], adapting the methods to our setting.

2 Existence of the equilibrium state

The goal of this section is to prove existence of equilibrium states associated to $-t\log J^u$. In the uniformly hyperbolic case, the result comes for Hölder-continuous potentials as follows. First, one chooses a sequence of measures μ_n whose pressures converge to the pressure of the potential. Due to the upper semi-continuity of the metric entropy, and to the fact that the potential is continuous, any accumulation point of μ_n is an equilibrium state. In this section we prove that, in our non uniformly hyperbolic context, the entropy is still upper semi-continuous. We also prove that the discontinuity of the potential is not an obstacle for the existence of the equilibrium state. That last fact comes from an estimate of a lower bound for $\mathcal{P}(t)$.

2.1 Semi-continuity for the entropy

In the uniformly hyperbolic case, the upper semi-continuity property for the entropy allows one to prove that the supremum in the variational principle which defines $\mathcal{P}(t)$ is a maximum. This upper semi-continuity is usually deduced from the expansiveness property of the map f. We proved in [18] that the map f does not satisfy the expansiveness property. Nevertheless, the upper semi-continuity holds.

Lemma 2.1. *The map $\mu \mapsto h_\mu(f)$ is upper semi-continuous.*

Proof. Following [18], consider the partition $\mathcal{G}_0^1 = \{R_0, R_1, R_2\}$, where $(0,0) \in R_0$, and define $\Theta\colon \Sigma_3 = \{0,1,2\}^{\mathbb{Z}} \to \Lambda$ by $\Theta([s_i]_{i\in\mathbb{Z}}) = \cap_{i\in\mathbb{Z}} f^i(R_{s_i})$. See [18] for a proof that Θ is a Hölder continuous semi-conjugacy between the shift map and f. It holds also that, for any μ in $\mathcal{M}_f$, μ-almost every point in Λ has only one preimage under the map Θ.

More precisely, only points of the critical orbit have 2 preimages by Θ, and every other point in Λ has a unique preimage. As usual, the map Θ is a homeomorphism when it is restricted to the preimages of points which have a single preimage (see [18]).

Note that this set of points has full measure for any invariant probability in Σ_3. Then, for any σ-invariant probability measure $\widetilde{\mu}$, Θ is $\widetilde{\mu}$-almost everywhere one-to-one, and for every μ in $\mathcal{M}_f$, Θ^{-1} is μ-a.e. well defined (and continuous on a set with full measure). Therefore, for every μ in $\mathcal{M}_f$, $h_\mu(f) = h_{\Theta^{-1}(\mu)}(\sigma)$; conversely, for any σ-invariant probability measure $\widetilde{\mu}$, $h_{\widetilde{\mu}}(\sigma) = h_{\Theta(\widetilde{\mu})}(f)$. As expansiveness holds in Σ_3, we deduce the upper semi-continuity property for the metric entropy. This thus also holds for the map f. $\square$

2.2 Convergence of $-t\int \log J^u \, d\mu_n$

Since the map J^u is not continuous, the convergence of a sequence of f-invariant probabilities $\nu_n \to \nu$ in the weak* topology does not imply $\int \log J^u d\nu_n \to \int \log J^u d\nu$. Hence, the main question is to know what are the possible accumulation points for $\int \log J^u \, d\nu_n$.

Lemma 2.2. *Let (ν_n) be a sequence of f-invariant ergodic probabilities converging to ν in the weak* topology. If $\nu\{(0,0)\} = 0$, then* $\displaystyle\lim_{n\to+\infty} \int \log J^u \, d\nu_n = \int \log J^u \, d\nu.$

Proof. We denote by $R_k(0)$, the element of the partition $\mathcal{G}_k^k$ which contains the point $(0,0)$. The sets $R_k(0)$ and $\Lambda \setminus R_k(0) = R_k(1)$ are compact. Consider a partition of the unity $\{\chi_{0,k}, \chi_{1,k}\}$ such that $\chi_{i,k|R_k(j)} \equiv \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Let $\overline{L} \in \mathbb{R}$ and $\underline{L} \in \mathbb{R}$ be, respectively, the lim sup and lim inf of $\displaystyle\int \log J^u \, d\nu_n$.

Consider a subsequence (ν_{n_j}) of ν_n such that $\displaystyle\int \log J^u \, d\nu_{n_j} \to \overline{L}$. For simplicity we assume that the convergence occurs for the entire sequence. Then, for any n and for any k we have

$$\int \log J^u \, d\nu_n = \int \chi_{1,k} \log J^u \, d\nu_n + \int \chi_{0,k} \log J^u \, d\nu_n. \tag{2}$$

Note that $\log J^u$ is continuous in $\Lambda \setminus \{(0,0)\}$, hence

$$\lim_{n\to+\infty} \int \chi_{1,k} \log J^u \, d\nu_n = \int \chi_{1,k} \log J^u \, d\nu.$$

By assumption, the left-hand side in (2) goes to $\overline{L}$ when n goes to $+\infty$. This implies that the last term in the right-hand side of (2) also converges when n goes to $+\infty$.

Now, by the dominated convergence theorem and the fact that $\nu(\{(0,0)\}) = 0$, we have that $\lim_{k\to+\infty} \int \chi_{1,k} \log J^u \, d\nu$ exists and is equal to $\int \log J^u \, d\nu$. Moreover, due to (1), we also get

$$\left| \lim_{n\to+\infty} \int \chi_{0,k} \log J^u \, d\nu_n \right| \leq \log(2\sigma)\nu(R_k(0)).$$

Thus, picking first the limit when n goes to $+\infty$ and then the limit when k goes to $+\infty$ in (2), we get $\overline{L} = \displaystyle\int \log J^u \, d\nu$. Doing the same with $\underline{L}$, we get the result. □

Lemma 2.3. *Let us assume that the sequence of ergodic invariant probabilities μ_n converges to $u.\delta_{(0,0)} + v.\nu$, where $\nu \neq \delta_{(0,0)}$ is an ergodic and f-invariant probability, and $v = 1 - u$ and $u \in]0,1[$. Then* $\liminf \displaystyle\int \log J^u \, d\mu_n \geq \frac{u}{2} \log \sigma + v \int \log J^u \, d\nu$.

Proof. Let $R_k(0)$, $R_k(1)$, $\chi_{0,k}$, and $\chi_{1,k}$ be as above. Continuity of the map $\chi_{1,k} \log J^u$ yields for every k:

$$\lim_{n\to+\infty} \int \chi_{1,k} \log J^u \, d\mu_n = v. \int \chi_{1,k} \log J^u \, d\nu. \tag{3}$$

We have now to consider the part $\chi_{0,k} \log J^u$. Let us pick some very small positive ε. We assume that k is fixed large enough such that $R_k(0)$ has ν-measure lower than $\frac{1}{2}u\varepsilon$ (remember $u > 0$). Hence, $\limsup_{n\to+\infty} \mu_n(R_k(0))$ belongs to $[u, u(1+\frac{1}{2}\varepsilon)]$. Now, for every $k' \geq k$, $\liminf_{n\to+\infty} \mu_n(R_{k'}(0)) \geq u$. Therefore, we can assume that n is large enough such that

$$(1-\varepsilon).u \leq \mu_n(R_k(0)) \leq (1+\varepsilon).u \text{ and } (1-\varepsilon).u \leq \mu_n(R_{k+k^2}(0)) \leq (1+\varepsilon).u. \tag{4}$$

The map $\chi_{0,k} \log J^u$ is in $L^1(\mu_n)$, for every n, thus the Birkhoff theorem holds. For μ_n-a.e. ξ,

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j<m} (\chi_{0,k} \log J^u) \circ f^j(\xi) = \int \chi_{0,k} \log J^u \, d\mu_n. \tag{5}$$

Later we make some assumptions on k that do not depend on the choice of n.

Let ξ be generic, and set $\xi_m = f^m(\xi)$. Let us consider a piece of the forward orbit of ξ between one arrival to $R_k(0)$ and the following departure from $R_k(0)$. Observe that before this piece of orbit, the corresponding point is close to the critical point $Q = (q, 0)$, then the iterates stay into the horizontal band R_1 and enter into $R_k(0)$. After this, they stay for some time in the vertical band R_1', and finally reach R_3 or R_4 or R_5 just before leaving R_1'.

Equation (4) means that the probability that a piece of orbit of ξ visits $R_k(0)$ without visiting $R_{k+k^2}(0)$ is smaller than $2u\varepsilon$. Hence, the influence of these pieces of orbits on the mean value in (5) is lower than $\widehat{C}.\varepsilon$ for some universal constant $\widehat{C}$ (remember $\log J^u$ is bounded). We thus only consider pieces of orbits which visit $R_{k+k^2}(0)$; namely this corresponds to pieces of orbits which stay for at least k^2 iterates in $R_k(0)$.

Let us now study such a piece of orbit. For simplicity we assume that the point ξ itself belongs to the horizontal band

$$\mathcal{B}^h_{2k+l} = \{(x, y) \in \mathcal{Q},\ \frac{1}{\sigma^{2k+l+1}} \leq y \leq \frac{1}{\sigma^{2k+l}}\}$$

for some integer $l \geq k^2$. Let $(x + q, y)$ be the coordinates of ξ. Let us assume for simplicity that $x \geq 0$. We recall (see [18]) that the unstable direction is close to the tangent at ξ to the local parabola which contains ξ. This means that we can choose some unstable vector e^u_0 (for ξ) on the form

$$e^u_0 = (1, 2Ccx),$$

where C belongs to $[\frac{1}{3}; 3]$ (see 1.1.3). We also recall that the equation of that parabola has the form $Y = c(X - q)^2 - C_{\xi,2k+l}$. All these estimates yield

$$e^u_0 = \left(1, 2C'.\sqrt{c}\sqrt{\frac{1}{\sigma^{2k+l}} + C_{\xi,2k+l}}\right),$$

where C' is some constant in $[\frac{1}{3\sqrt{\sigma}}, 3]$. Using the fact that only the part of the orbit in $R_k(0)$ influences the computation of the mean value of $\chi_{0,k} \log J^u$, we thus just compute the expansion between the k^{th} iterate and the $k + l^{th}$ iterate.

As the map is linear in the considered region,

$$Df^k(e^u_0) = \left(\lambda^k, 2C'.\sqrt{c}\sqrt{\frac{1}{\sigma^l} + \sigma^{2k} C_{\xi,2k+l}}\right).$$

As all the norms are equivalent, we compute the expansion using the norm of the maximum between the two components of the vector. After l more iterations, we get

$$Df^{l+k}(e^u_0) = \left(\lambda^{l+k}, 2C'.\sqrt{c}\sqrt{\sigma^l + \sigma^{2k+2l} C_{\xi,2k+l}}\right).$$

As $l \geq k^2$, the second component of that vector is larger than $2C'\sqrt{c}\sigma^{\frac{k^2}{2}}$, and for large k it is greater than the first component. Therefore, and whichever the greatest component of $Df^k(e_0^u)$ is, the expansion of the unstable vector between the k^{th} iterate and the $k+l^{th}$ iterate is greater than $\widetilde{C}\sigma^{\frac{l}{2}}.\lambda^k$, where $\widetilde{C}$ is some universal constant larger than $\frac{2\sqrt{c}}{3\sqrt{\sigma}}$.

Therefore, we get for the Birkhoff sum:

$$\sum_{j=0}^{2k+l} \chi_{0,k} \log J^u(\xi_j) \geq \log \widetilde{C} + l\frac{1}{2}\log\sigma + \sqrt{\frac{l}{2}}\log\lambda.$$

Computing $\displaystyle\frac{1}{N}\sum_{j=0}^{N}(\chi_{0,k}\log J^u)(\xi_j)$ for a very large N, we get

$$\frac{1}{N}\sum_{j=0}^{N}(\chi_{0,k}\log J^u)(\xi_j) + \widehat{C}.\varepsilon \geq \frac{L_N}{N}\log\widetilde{C} + \frac{\sum l_i}{N}\frac{1}{2}\log\sigma + \frac{\sum\sqrt{\frac{l_i}{2}}}{N}\log\lambda, \tag{6}$$

where L_N is the number of visits into $R_{k+k^2}(0)$ between the 0^{th} and the N^{th}-iterates, and l_i is the length of the i^{th} visit to $R_k(0)$, and $\widehat{C}.\varepsilon$ is the contribution of the pieces of orbits which visit $R_k(0)$ without visiting $R_{k+k^2}(0)$.

Note that $\frac{L_N}{N} \leq \frac{1}{k+k^2}$, and that $\widetilde{C}$ is a universal constant (independent of the choices of k and n). We can thus assume that at the beginning, k was chosen such that $\left|\frac{1}{k+k^2}\log\widetilde{C}\right| < \varepsilon$.

The integer N is bigger than the sum of the length of the visits into $R_k(0)$, and each visit we consider has a length l_i larger than k^2. Therefore, we have

$$\frac{\sum\sqrt{\frac{l_i}{2}}}{N} < \frac{1}{k},$$

and here again, k can be assumed to be big enough such that $0 > \dfrac{\sum\sqrt{\frac{l_i}{2}}}{N}\log\lambda > -\varepsilon$.

Note that if N goes to $+\infty$, the term $\frac{\sum l_i}{N}$ represents the proportion of time the chosen generic orbit stays in $R_k(0)$ (and visits $R_{k+k^2}(0)$). By (4), this is larger than $u(1-3\varepsilon)$.

Letting N go to $+\infty$ in (6), we get for every large enough k, that

$$\liminf \frac{1}{N}\sum_{j=0}^{N}(\chi_{0,k}\log J^u)(\xi_j) \geq u(1-3\varepsilon).\frac{1}{2}\log\sigma - \widehat{C}.\varepsilon - 2\varepsilon.$$

Hence, for every large enough k, and for every large enough n (in function of k), we get

$$\int \log J^u\, d\mu_n \geq \frac{u(1-3\varepsilon)}{2}\log\sigma - \widehat{C}.\varepsilon + v.\int \chi_{1,k}(0)\log J^u\, d\nu - 3\varepsilon.$$

Letting n go to $+\infty$ and then k to $+\infty$ and ε to 0, we get (remember $\nu(\{(0,0\})=0$)

$$\liminf_{n\to+\infty}\int \log J^u\, d\mu_n \geq \frac{u}{2}\log\sigma + v\int \log J^u\, d\nu.$$

□

2.3 Existence of equilibrium states

Now we prove the existence of equilibrium states for the potential $-t\log J^u$, for $t>0$ satisfying $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$.

Note that $\delta_{(0,0)}$ cannot be an equilibrium state. Indeed, the t-pressure for $\delta_{(0,0)}$ is equal to $-t\log\sigma$, which is strictly lower than $-t\frac{1}{2}\log\sigma < \mathcal{P}(t)$. The next Lemma says that we can obtain the existence of equilibrium states by the same method as in the uniformly hyperbolic case.

Lemma 2.4. *Let $t>0$ be such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$, and assume that μ_n is a sequence of f-invariant and ergodic probabilities which converges to μ and satisfies $h_{\mu_n} - t\int logJ^u d\mu_n > \mathcal{P}(t) - 1/n$. Then $\mu(\{(0,0)\}) = 0$ and μ is an equilibrium state.*

Proof. Let $\mu = u\delta_{(0,0)} + v\nu$ with $u+v=1$ and $\nu(\{(0,0)\})=0$. We claim that $v\neq 0$. In fact, if it is not true, we have $\mu=\delta_{(0,0)}$, and by Lemma 2.1 we have $\lim_{n\to+\infty} h_{\mu_n}(f) = 0$. Therefore,

$$-t\frac{1}{2}\log\sigma < \mathcal{P}(t) = \lim_{n\to+\infty} -t\int \log J^u\, d\mu_n \leq -t\inf_{\nu\in\mathcal{M}_f}\int \log J^u\, d\nu \leq -t\frac{1}{2}\log\sigma,$$

which is a contradiction.

Again, by the upper semi-continuity of the entropy, and by Lemma 2.3, we find

$$\begin{aligned}
\mathcal{P}(t) &= \lim_{n\to+\infty}[h_{\mu_n} - t\int \log J^u d\mu_n] \leq \limsup_{n\to+\infty} h_{\mu_n} - t\liminf_{n\to+\infty}\int \log J^u d\mu_n \\
&\leq h_\mu - t(\frac{u}{2}\log\sigma + v\int \log J^u d\nu) = v(h_\nu - t\int \log J^u d\nu) - \frac{u}{2}t\log\sigma \\
&\leq v(h_\nu - t\int \log J^u d\nu) + u\mathcal{P}(t),
\end{aligned}$$

where we use $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$ to get the last inequality. Note that equality holds in the last inequality only if $u=0$. Since $v\neq 0$, $u+v=1$ and $u,v\geq 0$, we have

$$v\mathcal{P}(t) \leq v.(h_\nu - t\int \log J^u d\nu).$$

Since $h_\nu - t\int \log J^u d\nu \leq \mathcal{P}(t)$, we have that ν is an equilibrium state, and $u=0$. This completes the proof of the lemma. □

2.4 Optimal lower bound

We now prove that the lower bound $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$ is optimal in the case $\sigma.\lambda \leq 1$. First, recall that the map $t \mapsto \mathcal{P}(t)$ is decreasing and convex; thus it is differentiable everywhere with the possible exception of a countable set. The slope of the map $\mathcal{P}(t)$ is increasing and non-positive. It converges to some limit value θ as t goes to $+\infty$. Again, convexity yields that the graph of the map $\mathcal{P}$ is above the line $t\theta$; it is also well-known that $-\theta$ is the lower bound for the unstable Lyapunov exponent λ^u. This lower bound is greater than or equal to $\frac{1}{2}\log\sigma$, by (1). We now construct a family of measures, such that their unstable Lyapunov exponents converge to $\frac{1}{2}\log\sigma$.

Lemma 2.5. *When* $\sigma.\lambda < 1$, $\displaystyle\inf_{\mu\in\mathcal{M}_f}\int \log J^u\,d\mu = \frac{1}{2}\log\sigma.$

Proof. The proof is very similar to the proof of lemma 2.3. We construct a sequence of measures ν_n, each one supported on a periodic orbit, converging to $\delta_{(0,0)}$, such that

$$\lim_{n\to+\infty}\int \log J^u\,d\nu_n = \frac{1}{2}\log\sigma.$$

Let $R_k(0)$, $R_k(1)$, $\chi_{0,k}$, and $\chi_{1,k}$ be as in Lemma 2.2. Since the sequence (ν_n) converges to $\delta_{(0,0)}$, then, $\nu_n(R_k(1))$ goes to 0 and we get that, if $\int \log J^u\,d\nu_{n_j}$ converges to L along the subsequence (n_j), then, for any k, $\int \chi_{0,k}\log J^u\,d\nu_{n_j}$ also converges to L. This implies that the part of the orbit close to $(0,0)$ determines the Lyapunov exponent.

We thus pick some integer k. For $l \in \mathbb{N}$, we define the periodic orbit $\mathcal{O}_{l+2k}$ as follows. First note that the horizontal band

$$\mathcal{B}^h_{2k+l} = \{(x,y)\in\mathcal{Q},\ \frac{1}{\sigma^{2k+l+1}} \leq y \leq \frac{1}{\sigma^{2k+l}}\}$$

has for image by f^{2k+l} the band

$$\mathcal{B}^v_{2k+l} = \{(x,y)\in\mathcal{Q},\ x \leq \lambda^{2k+l},\ \frac{1}{\sigma} \leq y \leq 1\}.$$

Then, the image by f of this last band intersects twice the horizontal band $\mathcal{B}^h_{2k+l}$ close to the critical point $Q = (q,0)$ (in a horseshoe-like way). In each connected component of the intersection we must have some $2k+l+1$-periodic orbit. We pick one of these two periodic orbits to be $\mathcal{O}_{2k+l}$.

To this orbit we associate the invariant measure whose support is the periodic orbit. This gives a sequence of measures, depending on l. We now check that this sequence satisfies the required property.

Let us now study the dynamic of $\mathcal{O}_{2k+l}$. We denote by ξ^{2k+l} the element of the orbit which is in $f(\mathcal{B}^v_{2k+l})\cap\mathcal{B}^h_{2k+l}$. This point needs k-iterations of f to reach $R_k(0)$. Then it stays into $R_k(0)$ for exactly l more iterations of f. After $k+1$ more iterates the orbit closes.

Let $(x+q, y)$ be the coordinates of ξ^{l+2k}. Let us assume for simplicity that $x \geq 0$. Remember that the unstable direction is close to the tangent at ξ^{2k+l} to the local parabola which contains ξ^{2k+l}. We also recall that the equation of that parabola has the form $Y = c(X-q)^2 - C_{2k+l}$. This means that we can choose the unstable vector e^u_{2k+l} (for ξ^{2k+l}) of the form

$$e^u_{2k+l} = (1, 2Ccx),$$

where C belongs to $[\frac{1}{3}; 3]$. The constant C_{2k+l} satisfies $\lambda^{2k+l+2} \leq C_{2k+l} \leq \lambda^{2k+l+1}$, and y is lower than $\frac{1}{\sigma^{2k+l}}$, and larger than $\frac{1}{\sigma^{2k+l+1}}$. All these estimates yield

$$e^u_{2k+l} = \left(1, 2C'.\sqrt{c}\sqrt{\frac{1}{\sigma^{2k+l}} + \lambda^{2k+l+1}}\right),$$

for some new universal (almost) constant C'. Using the fact that only the part of the orbit in $R_k(0)$ influences the computation of the Lyapunov exponent (at the limit when l goes to $+\infty$), we thus just compute the expansion between the k^{th} iterate and the $k+l^{th}$ iterate.

We have that

$$Df^k(e^u_{2k+l}) = \left(\lambda^k, 2C'.\sqrt{c}\sqrt{\frac{1}{\sigma^l} + \sigma^{2k}\lambda^{2k+l+1}}\right).$$

For very large l the first component is larger than the second one. As all the norms are equivalent, we compute the expansion using, again, the norm of the maximum between the two components of the vector. After l more iterations, we get

$$Df^{l+k}(e^u_{2k+l}) = \left(\lambda^{l+k}, 2C'.\sqrt{c}\sqrt{\sigma^l + \sigma^{2k+2l}\lambda^{2k+l+1}}\right).$$

For large l (remember that k is considered as a constant) the second component of that vector is larger than the first one. Using the assumption $\sigma.\lambda \leq 1$, we get that $\sigma^{2k+2l}\lambda^{2k+l+1}$ has order lower than σ^l. Hence, the expansion of the unstable vector between the k^{th} iterate and the $k+l^{th}$ iterate has order $\widetilde{C}\sigma^{\frac{l}{2}}$. Taking the logarithm and dividing by the length of the orbit $(2k+l+1)$, and then letting l go to $+\infty$, we get the result. $\square$

3 Uniqueness of the equilibrium state

The goal of this section is to prove the uniqueness of the equilibrium state associated to $-t\log J^u$, for values of $t \geq 0$ such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. To do this, we first study some ergodic properties of induced subsystems.

In the first subsection, we construct a countable Markov partition, choosing rectangles of $\mathcal{G} = \bigcup_{n,m\in\mathbb{N}} \mathcal{G}^n_m$. Each rectangle of the partition support an induced dynamics, and is endowed with an adapted metric. In the second subsection we prove that the potential $-\log J^u$ has bounded variation inside a rectangle of the partition. In the third subsection we give the spectral properties for a good family of transfer operators. We then construct, for the subsystem, a family of local equilibrium states (associated to some good potentials). In the fourth subsection we deduce uniqueness of the global equilibrium state for t such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. The main idea is to identify the global equilibrium state among the local equilibrium states. We also prove the analytic regularity of the map $t \mapsto \mathcal{P}(t)$.

The method we use here was first introduced in [16] for the uniformly hyperbolic case. In [17] the author also used this method, introducing another parameter in the operator. Some results about regularity were proved in [6]. Many of the results we present here were proved in the references above, for the uniformly hyperbolic case. We refer the reader to the book [1], for a broader view on transfer operators, specially [section 1.3, page 28].

3.1 Induced subsystem

3.1.1 Dynamical Markov partition

Let us fix a positive integer a, and consider the partition $\mathcal{G}_a^a$. We now construct a Markov partition with countably many rectangles, adapted to the dynamics of F, which refines $\mathcal{G}_a^a$.

Let $M = (x, y)$ be a point in the intersection of the domains of F and F^{-1} (in particular, we have $xy \neq 0$). Define $n^+(M)$ as the positive integer such that $F(M) = f^{n^+(M)}(M)$ and $n^-(M)$ as the positive integer such that $F^{-1}(M) = f^{-n^-(M)}(M)$. We set then

$$\mathcal{R}(M) = \bigcap_{k=-n_-(M)}^{n^+(M)} f^{-k}(\mathcal{G}_a^a(f^k(M))) = \bigvee_{k=-n^-(M)-a}^{k=n^+(M)+a} f^{-k}(\mathcal{Q})(M).$$

For $M = (0, y)$ in the domain of F, with $y > \frac{1}{\sigma}$ F^{-1} is not well defined, and we set $n^-(M) = +\infty$, and define

$$\mathcal{R}(M) = \bigcap_{k=-\infty}^{n^+(M)} f^{-k}(\mathcal{G}_a^a(f^k(M))) = \bigvee_{k=-\infty}^{k=n^+(M)+a} f^{-k}(\mathcal{Q})(M).$$

This defines a "one dimensional" rectangle, that is a vertical segment. Analogously, for $M = (x, 0)$ in the domain of F^{-1} with $x > \lambda$, $F(M)$ is not well defined, and we set $n^+(M) = +\infty$, and

$$\mathcal{R}(M) = \bigcap_{k=-n^-(M)}^{\infty} f^{-k}(\mathcal{G}_a^a(f^k(M))) = \bigvee_{k=-n^-(M)-a}^{\infty} f^{-k}(\mathcal{Q})(M).$$

Again, it is a "one dimensional" rectangle, that is a horizontal segment.

Lemma 3.1. *The family $\mathcal{R}$ of rectangles $\mathcal{R}(M)$ is a countable Markov partition for the map F.*

Proof. Note that, if $\mathcal{R}(M) = \mathcal{G}^{n+a}_{m+a}(M)$ and $\mathcal{R}(F(M)) = \mathcal{G}^{l+a}_{n+a}(F(M))$, then the rectangle $F(\mathcal{G}^{n+a}_{m+a}(M)) = \mathcal{G}^{a}_{n+m+a}(F(M))$ overlaps $\mathcal{R}(F(M))$ in the unstable direction, and is overlapped in the stable direction. $\square$

One of the properties of the partition above is that it is a dynamically coherent partition: for every element R of the partition $\mathcal{R}$, and for every pair of points $M, M' \in R$, $F(M) = f^n(M)$ if and only if $F(M') = f^n(M')$, and $F^{-1}(M) = f^m(M)$ if and only if $F^{-1}(M') = f^m(M')$. We also have that the rectangles are proper and disjoint.

Note that there exists some universal constant C such that, for every M and M' as above,

$$e^{-C} \leq \frac{l(M)}{l(M')} \leq e^{C}, \text{ and } e^{-C} \leq \frac{l(F^{-1}(M))}{l(F^{-1}(M'))} \leq e^{C}. \tag{7}$$

Indeed, note that the number $l(M)$ satisfies that $l^2(M)\sigma^{n^+(M)}$ is uniformly bounded away from zero and above. We set $l^2(M)\sigma^{n^+(M)} \in [1/d, d]$ where d is a positive universal constant (to see this, note that, for M in the critical zone, this is a consequence of the definition of $n^+(M)$: it is roughly the escape time for M. Outside the critical region, $l(M)$ is uniform and $n^+(M)$ is 1).

3.1.2 Notation for the induced map

Fix a rectangle $R \in \mathcal{R}$ and an unstable leaf $\mathcal{F}$, with $\mathcal{F} = W^u_{loc}(M) \cap R$ for some point M in R. Note that R does not contain any element of the critical orbit. The local product structure induced by the stable and unstable foliation allows us to define the projection $\pi_{\mathcal{F}}$ onto $\mathcal{F}$ along the stable leaves in R. We denote by g the first return map in R by iterations of f, and $g_{\mathcal{F}}$ is the map $\pi_{\mathcal{F}} \circ g$. This defines a new dynamical system $(\mathcal{F}, g_{\mathcal{F}})$. We denote by r_R the first return-time map. Namely $g(M) = f^{r_R(M)}(M)$. Using the inverse branches of $g_{\mathcal{F}}$, we define a family of *n-sets*, setting, for ξ in $\mathcal{F}$,

$$K_n(\xi) = f^{-r^n_R(\xi)}(W^u(g^n(\xi))),$$

where $r^n_R(\xi)$ denotes the n^{th} return time to R, and $r^1_R(\xi) = r_R(\xi)$. For a given point in $\mathcal{F}$ which returns infinitely many times to R, the n-sets are well-defined, for every n. Note that the set of the points which do not return infinitely many times to R has measure zero, for all invariant measures of the subsystem. Hence, every n-set is a compact set and the collection of the n-sets, for a fixed n, defines a partition of $\mathcal{F}$ (up to a zero-measure set) which refines the partition in $(n-1)$-sets (again, up to a zero measure set). Let $Pre_n(\xi)$ denote the set of preimages by $g^n_{\mathcal{F}}$ of ξ in $\mathcal{F}$.

Remark 1. An important property to be used later, is that every point in $\mathcal{F}$ has exactly one preimage by $g_{\mathcal{F}}^n$ in each set of the partition in n-sets. If ξ and ζ are in $\mathcal{F}$, every n-set contains exactly one ξ', such that $g_{\mathcal{F}}^n(\xi') = \pi_{\mathcal{F}} \circ f^{r_R^n(x')}(\xi') = \xi$, and exactly one ζ' such that $g_{\mathcal{F}}^n(\zeta') = \pi_{\mathcal{F}} \circ f^{r_R^n(\xi')}(\zeta') = \zeta$ with $r_R^n(\xi') = r_R^n(\zeta')$ (in particular, any two points in $\mathcal{F}$ have the same number of preimages by $g_{\mathcal{F}}^n$ in $\mathcal{F}$). Even if the direct map $g_{\mathcal{F}}$ is not well defined (some points do not return), the inverse branches are well defined. This is enough to get the definition of the transfer operator we are going to use.

One key point will be to use the Ionescu-Tulcea & Marinescu theorem for the transfer operator. For that we need two Banach spaces. The big one will be the set of continuous functions on $\mathcal{F}$. The small one will be the set of Hölder continuous functions on $\mathcal{F}$, but for a special metric, related to the dynamics. This metric on $\mathcal{F}$ is defined as follows. For M and M' in $\mathcal{F}$, we set

$$N(M, M') = max\{n \in \mathbb{N},\ \forall k,\ 0 \leq k \leq n\ \exists\ T_k \in \mathcal{G}_1^1, F^k(M), F^k(M') \in T_k\},$$

where the maximum is taken in $\overline{\mathbb{N}}$. Notice that $\mathcal{G}_1^1$ is a generating partition, hence $N(M, M')$ is well defined as long as $M \neq M'$. We then set

$$\eta(M, M') = \frac{1}{2^{N(M,M')}}.$$

Clearly, $\eta(M, M') = \eta(M', M)$, and $\eta(M, M') = 0$ if and only if $M' = M$. Moreover we have, for all $M, M', M'' \in \mathcal{F}$, that $\eta(M, M'') \leq \eta(M, M') + \eta(M', M'')$.

Remark 2. Due to the expansion in the unstable leaves, the topology defined by the Riemannian distance d^u and the topology defined by η are equivalent in $\mathcal{F}$.

Definition 3.2. *We denote by $C_\eta^\alpha(\mathcal{F})$ the set of functions $\varphi : \mathcal{F} \mapsto \mathbb{R}$ such that,*

$$K_\varphi = \sup_{M \neq M' \in \mathcal{F}} \frac{|\varphi(M) - \varphi(M')|}{(\eta(M, M'))^\alpha} < +\infty.$$

For φ in C_η^α, we set $||\varphi||_{\alpha,\eta} = ||\varphi||_\infty + K_\varphi$. It is obvious that any function in C_η^α is continuous. Also, $(C_\eta^\alpha, ||\ ||_{\alpha,\eta})$ is a Banach space, see [15] for a proof. Moreover, for any sequence (φ_n) of continuous functions such that $||\varphi_n||_{\alpha,\eta} \leq C$, for some constant C, if (φ_n) converges to φ for the norm $||\ ||_\infty$, then φ is in C_η^α and $||\varphi||_{\alpha,\eta} \leq C$.

3.2 Control of the variations

This is a very technical subsection. The main result is Proposition 3.5, which is one key point for using the Transfer Operator later. For M' in $\mathcal{F}$ and M in R such that $\pi_{\mathcal{F}}(M) = M'$, we set

$$\omega(M, M') = \sum_{k=0}^{+\infty} \log J^u \circ f^k(M) - \log J^u \circ f^k(M')\ .$$

We also set $\omega(M) = \omega(M, \pi_{\mathcal{F}}(M))$.

It is well known that the main ingredient to construct Gibbs measures associated to some potential φ is to control the values and the regularity of $\sum_{n\geq 0} \varphi\circ f^{\pm n}(M) - \varphi\circ f^{\pm n}(M')$ where, either M' belongs to $W^s(M)$ (and in that case we consider the sum with f^n) or M' belongs to $W^u(M)$ (and in that case we consider the sum with f^{-n}). In the uniformly hyperbolic case, associated with the uniform local product structure, these variations are easily controlled, for instance, when φ is Hölder continuous.

The goal of this section is to choose a good Markov partition in order to control the distortion. In [2] (see *e.g.* lemma 1.6), it is proved that any Hölder continuous potential is cohomologuous to some other Hölder continuous potential which only depends on backward iterates. This is strongly related to the bounded angles in the local product structure of the hyperbolic set. In our case the local product structure still holds, but in a degenerated way: the angle between stable and unstable vectors go to zero as the base point approaches Q. To overcome this difficulty, we consider smaller rectangles close to Q, such that the variation of the angles is small inside each rectangle. Also, since $\log J^u$ is not continuous, even though discontinuity occurs only in one point, it makes the classical control of distortion more difficult.

We also emphasize, that the Hölder regularity of the unstable direction is not sufficient to get the control on the distortion. The reason is unfortunately quite technical, and we explain this later in Remark 5. We can however give now some hint:

To control $\log E - \log E'$ we need to get control on $\dfrac{E-E'}{E}$. To control the distortion we have to control a sum of terms like $\log E - \log E'$ along a forward obit. The Hölder regularity of the unstable direction will give control on each term in $E - E'$, and the sum shall be summable; nevertheless, along the orbit, the term E can be very small, and the summability of the terms in $E - E'$ is not sufficient.

For M in A, we define $e^u(M)$ in the following way: the linear space $E^u(M)$ is a one-dimensional non-vertical space in $\mathbb{R}^2$. We then denote by $e^u(M)$ the unique element in $E^u(M)$ whose first component in the usual canonical base is equal to 1. Then we have:

Lemma 3.3. *There exist universal constants, $0 < C_1, C_2$ such that, for every M and M' in A satisfying*

1. *$F^{-1}(M)$ exists,*
2. *$M' \in W^s(M, \mathcal{R}(M))$,*

the following holds:

$$|e^u(M) - e^u(M')| \leq C_1|M - M'| + C_2\lambda^n\sqrt{|F^{-1}(M) - F^{-1}(M')|},$$

where n is the integer such that $F(F^{-1}(M)) = f^n(F^{-1}(M))$.

Proof. Let M and M' in A be such that $M' \in W^s_{loc}(M)$. We set $M = (x+q, y)$ and $M' = (x'+q, y')$, where $Q = (q, 0)$ is the critical point in A. Again we consider the norm defined by $|(x+q, y)| = \max(|x|, |y|)$. This norm is used for both points and vectors. Recall that $l(M) = x$.

Note that if $M_{-1} = f^{-1}(M)$ and $M'_{-1} = f^{-1}(M')$ belong to R'_3 or R'_5, then the result holds: indeed, $M_{-1} = F^{-1}(M)$ and $M'_{-1} = F^{-1}(M')$. Now,

E^u is $\frac{1}{2}$-Hölder continuous in that region and Df is C^1. Then we have

$$|e^u(M) - e^u(M')| \leq Const.\sqrt{|F^{-1}(M) - F^{-1}(M')|}.$$

We can thus assume that $f^{-1}(M)$ and $f^{-1}(M')$ belong to R'_1 (the case where only one of them is in $f^{-1}(M')$ does not happen, by the condition $M' \in W^s_{loc}(M)$).

The vector $(0, 1)$ is sent to a vector proportional to $(1, 2cx)$, by the map $Df(M_{-1})$. The expansion is greater than σ (see Subsection 1.1; item $e)-ii)$) and bounded above by 2σ. We define σ_1 by the equation $Df(M_{-1}).(0, 1) = \sigma_1(1, 2cx)$; analogously, the vector $(1, 0)$ is sent to a vector proportional to $(2cx, -1)$. We set $Df(M_{-1}).(1, 0) = \lambda_1(2cx, -1)$. This also holds for $Df(M'_{-1})$, and we set

$$Df(M'_{-1}).(0, 1) = \sigma'_1(1, 2cx')$$

and

$$Df(M'_{-1}).(1, 0) = \lambda'_1(2cx', -1).$$

Let (τ, υ) be a vector in $E^u(M_{-1})$, and (τ', υ') a vector in $E^u(M'_{-1})$. Then we have

$$e^u(M) - e^u(M') = \left(0, \frac{\sigma_1 2cx\upsilon - \lambda_1.\tau}{\sigma_1\upsilon + \lambda_1.2cx\tau} - \frac{\sigma'_1 2cx'\upsilon' - \lambda'_1.\tau'}{\sigma'_1\upsilon' + \lambda'_1.2cx'\tau'}\right). \tag{8}$$

We now compare the orders of the terms in the above expression. First recall that σ_1 and σ'_1 belong to $[\sigma, 2\sigma]$ and λ_1 and λ'_1 belong to $[\frac{\lambda}{2}, \lambda]$. We also have that f is $\mathcal{C}^2$. Therefore, if we exchange in the right hand term of (8) σ'_1 and λ'_1 by σ_1 and λ_1, we are just adding some term with order $\mathcal{O}(|M - M'|)$.

Now, recall that M_{-1} belongs to R'_1. Then we can set $M_{-1} = f^n(M_{-1-n})$, where M_{-1-n} belongs to A and n is the escape time for M_{-1-n}. In particular we have $M_{-1-n} = F^{-1}(M)$ and we set $M'_{-1-n} = F^{-1}(M') := f^{-1-n}(M')$.

For vector (τ, υ) we choose the vector

$$(\tau, \upsilon) := (\lambda^n, 2c\sigma^n C(M_{-1-n})) = Df^n(M_{-1-n}).e^u(M_{-1-n}),$$

where

$$e^u(M_{-1-n}) = (1, 2cC(M_{-1-n})),$$

and $C(.)$ is a $\frac{1}{2}$-Hölder continuous function satisfying (see 1.1.3)

$$\frac{1}{3}l(M_{-1-n}) \leq C(M_{-1-n}) \leq 3l(M_{-1-n}). \tag{9}$$

We also have $(\tau', \upsilon') = (\lambda^n, 2c\sigma^n C(M'_{-1-n}))$. Computing the iterations of the linear map f at the point M_{-1-n}, we get $6c\sigma^n.x_0 \geq |\upsilon| \geq \frac{2}{3}c\sigma^n.x_0$ where $l(M_{-n-1}) = x_0$. We also have

$$\sigma^n c x_0^2 \geq \frac{1}{3} \text{ and } \frac{\lambda^{n+1}}{x^2} \leq 2c. \tag{10}$$

The first inequality is obtained by saying that the point M_{-1-n} is below the local parabola of the critical point Q. The second is obtained by saying that M is above the x-axis. Therefore $\upsilon \gtrsim \frac{1}{x_0}$, with $x_0 \sim 0$, where $e \gtrsim f$ means $e \geq C.f$ for some universal constant C. We also have $\lambda.\tau \lesssim x^2$.

Computing the term in (8) we get for the numerator

$$\sigma_1^2 2c\upsilon\upsilon'(x - x') + \lambda_1^2 c\tau\tau'(x - x') + \lambda_1\sigma_1(\tau'\upsilon - \tau\upsilon') + \lambda_1\sigma_1 4c^2 xx'(\upsilon\tau' - \tau\upsilon'). \tag{11}$$

The second term in (11) is much smaller than the first one. In the same way the fourth term is much smaller than the third one. Dividing by $\sigma_1^2\upsilon\upsilon'$, which is the dominating term in the denominator in (8), we get a dominating term in $C(x - x')$ for the two first terms. Now $\tau = \tau'$, hence

$$\tau'\upsilon - \tau\upsilon' = \tau(\upsilon - \upsilon'),$$

and (9) yields

$$\frac{\tau|\upsilon - \upsilon'|}{\sigma_1^2\upsilon\upsilon'} \propto \frac{\lambda^n}{\sigma^n l(M_{-1-n}) l(M'_{-1-n})} |C(M_{-1-n}) - C(M'_{-1-n})|, \tag{12}$$

where $e \propto f$ means $e \lesssim f$ and $e \gtrsim f$.

Due to proposition 5.3 in [18], the directions E^i are $\frac{1}{2}$-Hölder continuous in A. Therefore

$$|C(M_{-1-n}) - C(M'_{-1-n})| \propto |e^u(M_{-1-n}) - e^u(M'_{-1-n})| \lesssim \sqrt{|M_{-1-n} - M'_{-1-n}|}.$$

Now (7) means that $l(M'_{-1-n}) \propto l(M_{-1-n}) = x_0$, hence, (12) and the first inequality in (10) give

$$\frac{\tau|\upsilon - \upsilon'|}{\sigma_1^2\upsilon\upsilon'} \leq C_2\lambda^n\sqrt{|M_{-1-n} - M'_{-1-n}|}.$$

this achieves the proof. □

Remark 3. Recall that if M' belongs to $W^u(M, R)$, then the upper-bound in Lemma 3.3 holds with $C_2 = 0$, because the unstable leaves are $C^{1+\varepsilon}$.

Remark 4. We have that all the upper-bounds are obtained with λ in the numerators (and σ in the denominators). Therefore, decreasing λ (and increasing σ) would improve the previous estimates. In the following, we will assume some conditions for σ and λ that will keep valid those estimates.

Proposition 3.4. *There exist positive constants C'_1, C''_1, C'_2 and C''_2 such that, for every M and M' in the same rectangle $T \in \mathcal{R}$, if $M' \in W^s(M,T)$ and $F^{-1}(M)$ is well defined, then*

$$\begin{aligned} |\sum_{k=0}^{+\infty} \log J^u \circ f^k(M) - \log J^u \circ f^k(M')| &\leq C'_1|M-M'| + \\ &\qquad (\frac{C''_2}{l(M)} + C'_2)\sqrt{|M-M'|}. \end{aligned} \tag{13}$$

Moreover, if M belongs to A, we also have

$$\begin{aligned} |\sum_{k=0}^{+\infty} \log J^u \circ f^k(M) - \log J^u \circ f^k(M')| &\leq (C'_1 + \frac{C''_1}{l(M)})|M-M'| + C'_2\sqrt{|M-M'|} \\ &\quad + \frac{C''_2\lambda^n}{l(M)}\sqrt{|F^{-1}(M) - F^{-1}(M')|}, \end{aligned} \tag{14}$$

where n is such that $f^n(F^{-1}(M)) = M$.

Proof. The main idea in the proof is to use uniform hyperbolicity of the map F (see 1.1.4). For that we split the forward f-orbit into pieces corresponding to forward F-iterates. The first step of the proof is to study the variations of $\log(J^u_F)$ for a piece of F-orbit. In the second step we finish the proof, gluing together all the pieces of F-orbits.

Step one: bounds for $\log J^u$ - Let M be in T, and $M' \in W^s(M,T)$. Note that for every $n \geq 0$, $F^n(M)$ and $F^n(M')$ belong to the same element of the partition $\mathcal{R}$. Due to proposition 5.3 in [18], the directions E^i are $\frac{1}{2}$-Hölder continuous and we can use this fact in the region where $F \equiv f$. The difficulty occurs when $M_k = f^k(M)$ belongs to A. In that case, $F^{-1}(M_k)$ is well defined. Let n_k be such that $F(F^{-1}(M_k)) = f^{n_k}(F^{-1}(M_k))$.

Let us set $M_k = (x+q, y)$ $M'_k = (x'+q, y')$, where $Q = (q, 0)$ is the critical point in A. Let n be the escape time for M_k, and M'_k; namely $f^{k+n}(M)$ is the first positive iterate of M_k which belongs to R_j with $j \geq 3$ (see 1.1.3). Here we assume that M_k does not belong to the segment $[0,1] \times \{0\}$. Again, we write

$$e^u(M_k) = (1, 2cC(M_k)),$$

where $C(.)$ is a $\frac{1}{2}$-Hölder continuous function satisfying (see 1.1.3)

$$\frac{1}{3}l(M_k) \leq C(M_k) \leq 3l(M_k). \tag{15}$$

Again, we consider the norm defined by $|(x,y)| = \max(|x|,|y|)$. By definition of f, we have

$$Df^n(M_k).e^u(M_k) = (\lambda^n, \sigma^n.2c.C(M_k))$$

and

$$Df^n(M'_k).e^u(M'_k) = (\lambda^n, \sigma^n.2c.C(M'_k)).$$

Here again we have $\sigma^n C(M_k) \gtrsim 1$ and $\sigma^n C(M'_k) \gtrsim 1$, by definition of the escape time. This means that for both vectors $Df^n(M_k).e^u(M_k)$ and $Df^n(M'_k).e^u(M'_k)$, the biggest component is the second one. Therefore we have

$$\begin{aligned}\sum_{i=k}^{k+n} \log J^u \circ f^i(M) - \log J^u \circ f^i(M') &= \log |Df^n(M_k).e^u(M_k)| - \\ &\log |Df^n(M'_k).e^u(M'k)| = \log \frac{C(M_k)}{C(M'_k)}.\end{aligned}$$

Note that $C(M_k)$ and $C(M'_k)$ have the same sign. Remember that $C(M_k) \propto l(M_k)$, $C(M'_k) \propto l(M'_k)$ (inequalities (15)) and $l(M_k) \propto l(M'_k)$ (see (7)). Therefore we get

$$|\sum_{i=k}^{k+n} \log J^u \circ f^i(M) - \log J^u \circ f^i(M')| \propto \frac{|e^u(M_k) - e^u(M'_k)|}{l(M_k)}. \tag{16}$$

Using Lemma 3.3 and , we get

$$|\sum_{i=k}^{k+n} \log J^u(M_i) - \log J^u(M'_i)| \lesssim C_1 \frac{|M_k - M'_k|}{l(M_k)} + \frac{C_2 \lambda^{n_k}}{l(M_k)} \sqrt{|F^{-1}(M_k) - F^{-1}(M'_k)|}. \tag{17}$$

Step two: bounds for the sum - We pick M in $T \in \mathcal{R}$ and M' in $W^s(M,T)$. If the forward orbit of M does not meet A anymore, then we simply use the $\frac{1}{2}$-Hölder regularity of E^u in the region where $F \equiv f$. This also holds if the forward orbit of M meets A only finitely many times.

Let us thus assume that the forward orbit of M meets A infinitely many times. Let n_i be the complete increasing sequence of times such that $f^{n_i}(M) \in A$. As the partition is dynamically coherent, and M_k and M'_k belong to the same element of the partition for $k \geq 0$, we have that M_k belongs to A if and only if M'_k belongs to A (with $k \geq 0$).

In order to compute $\sum_{i=0}^{+\infty} \log J^u \circ f^i(M) - \log J^u \circ f^i(M')$, we decompose this sum in blocks of length $n_{i+1} - n_i$. It may be that between two visits to A the orbit visits the uniformly hyperbolic region (the complement of $A \cup R'_1$) when it leaves R'_1, but this only improves the estimates. Hence, the worst case is when the orbit reaches A just after leaving R'_1.

For the i^{th}-visit and $i > 0$, the first term in the right-hand side of (17) satisfies

$$\frac{|M_{n_i} - M'_{n_i}|}{l(M_{n_i})} \leq \frac{\sqrt{\lambda}^{n_i - n_{i-1}}}{l(M_{n_i})} |M_{n_{i-1}} - M'_{n_{i-1}}| \leq C.|M_{n_{i-1}} - M'_{n_{i-1}}|.$$

To get this inequality, we first use the fact that M'_k belongs to $W^s_{loc}(M_k)$ and, thus, there is contraction with ratio smaller than $\sqrt{\lambda}$. Then, we use the second inequality in (10), which means that $\lambda^{n_i - n_{i-1}} \lesssim l^2(M_{n_i})$. Hence we have

$$\sum_{i \geq 1} \frac{|M_{n_i} - M'_{n_i}|}{l(M_{n_i})} \leq C. \sum_{i \geq 1} |M_{n_{i-1}} - M'_{n_{i-1}}| \leq C'_1.|M - M'|, \qquad (18)$$

because the map F is uniformly hyperbolic between 2 successive visits.

For the i^{th}-visit and $i > 0$, the second term in the right-hand side of (17) is

$$\frac{C_2 \lambda^{n_i - n_{i-1}}}{l(M_{n_i})} \sqrt{|M_{n_{i-1}} - M'_{n_{i-1}}|}.$$

Again we use the second inequality in (10) to get

$$\frac{C_2 \lambda^{n_i - n_{i-1}}}{l(M_{n_i})} \sqrt{|M_{n_{i-1}} - M'_{n_{i-1}}|} \quad \lesssim \quad \sqrt{\lambda}^{n_i - n_{i-1}} \sqrt{|M_{n_{i-1}} - M'_{n_{i-1}}|}.$$

Now, remember that F is uniformly hyperbolic, and we get

$$\sum_{i \geq 1} \frac{C_2 \lambda^{n_i - n_{i-1}}}{l(M_{n_i})} \sqrt{|M_{n_{i-1}} - M'_{n_{i-1}}|} \leq C'_2.\sqrt{|M - M'|}. \qquad (19)$$

We now deal with the term corresponding to $i = 0$. For this we use (16), always under the assumption that this part of orbit of M reaches A just after leaving R'_1. Again, estimates are better and simpler if this case does not hold. Hölder regularity and (16) yield:

$$|\sum_{i=0}^{n_1 - 1} \log J^u \circ f^i(M) - \log J^u \circ f^i(M')| \lesssim \frac{|e^u(M) - e^u(M')|}{l(M)} \lesssim \frac{\sqrt{|M - M'|}}{l(M_k)}. \quad (20)$$

Finally, expressions (18), (19), and (20) yield

$$\begin{aligned} |\sum_{k=0}^{+\infty} \log J^u \circ f^k(M) - \log J^u \circ f^k(M')| \quad &\leq \quad C'_1 |M - M'| + \\ &\qquad (\frac{C''_2}{l(M)} + C'_2) \sqrt{|M - M'|} \end{aligned}$$

for uniform constants C'_1, C_2 and C'_2.

If M belongs to A, we can also use Lemma 3.3 in (16), instead of (20) to get (14)

This completes the proof. □

We emphasize the argument here, because we will use it again. Losing one iteration of F, we get a better hyperbolic estimation. We then use that the countable partition is dynamically coherent and refines the partition $\mathcal{G}_1^1$.

Remark 5. As we said above, the Hölder regularity is not sufficient to get good estimates because of the denominator in (16) and (17). This denominator, namely $l(M_k)$, is not bounded from below away from 0 along the orbit.

Proposition 3.5. *There are positive constants C and α such that, for every ξ, ζ, ξ' and ζ' in $\mathcal{F}$ satisfying $g_{\mathcal{F}}(\xi') = \xi$ and $g_{\mathcal{F}}(\zeta') = \zeta$, with ξ' and ζ' in the same 1-set, we have*

$$|\omega(g(\xi')) - \omega(g(\zeta'))| \leq C.(\eta(\xi,\zeta))^{\alpha}.$$

Proof. For the uniformly hyperbolic case this result is usual. In our case the proof involves the same argument, that is a splitting of the forward orbits of $g(\xi')$, $g(\zeta')$, ξ and ζ: for small enough iterations of f, the images are close enough. We can thus control variations in function of the distance between the points $f^j \circ g(\xi')$ and $f^j \circ g(\zeta')$ in one hand and $f^j(\xi)$ and $f^j(\zeta)$ in the other hand. For large iterations, we can control variations in function of the distance between $f^j(\xi)$ and $f^j \circ g(\xi')$ in one hand and $f^j(\zeta)$ and $f^j \circ g(\zeta')$ in the other hand. The main difference with the uniformly hyperbolic case is that in our case we have to deal with iterations of the map F to get good hyperbolic estimates.

Let us set $n = N(\xi,\zeta)$. We thus have $\eta(\xi,\zeta) = \frac{1}{2^n}$. Inequality (13) in Proposition 3.4 and inequalities (7) (see page 18) mean that there exists a constant $\kappa = \kappa(\mathcal{F})$ such that $|\omega(g(\xi'))| \leq \kappa$ for every ξ in $\mathcal{F}$. Therefore, and as long as n is small (say $n < 100$), the result of Proposition 3.5 holds for every α, up to the fact that C is chosen big enough.

We now assume that n is large. Note that $g(\xi')$ and $g(\zeta')$ belong to the same local unstable leaf in R. This also holds for ξ and ζ. Moreover the direction E^u is Lipschitz continuous in the unstable leaves, in $A \cup R_3 \cup R_5 \cup R_4 \setminus Q$ (see Remark 3).

We consider the case where n is even. Let i be the integer such that $F^{n/2}(\xi) = f^i(\xi)$. Note that the analogous equation will be valid for the points ξ', ζ, ζ', due to the Markov property. Now we write

$$\sum_{k=0}^{i} \log J^u \circ f^k(\xi) - \log J^u \circ f^k(\zeta) = \sum_{k=0}^{n/2} \log\left|DF(F^k(\xi)).e^u\right| - \log\left|DF(F^k(\zeta)).e^u\right|,$$

and the analogous equation for $g(\xi')$ and $g(\zeta')$. Therefore, we have

$$|\sum_{k=0}^{i} \log J^u \circ f^k \circ g(\xi') - \log J^u \circ f^k \circ g(\zeta')| \leq C.|F^{n/2}(g(\xi')) - F^{n/2}(g(\zeta'))|,$$

and

$$|\sum_{k=0}^{i} \log J^u \circ f^k(\xi) - \log J^u \circ f^k(\zeta)| \leq C.|F^{n/2}(\xi) - F^{n/2}(\zeta)|,$$

where C is a uniform constant (which depends on $\sum_k \sqrt{\sigma}^{-k}$). Recall that the map F is uniformly expanding in the unstable leaves, with ratio larger than $\sqrt{\sigma}$. Since $F^j(\xi)$ and $F^j(\zeta)$ (and also $F^j(g(\xi'))$ and $F^j(g(\zeta'))$) are inside the same element of $\mathcal{G}_1^1$, for all $0 \leq j \leq n$, we have that

$$|F^j(\xi) - F^j(\zeta)| \leq C\sqrt{\sigma}^{j-n},$$

and the analogous expression for $F^j(g(\xi'))$ and $F^j(g(\zeta'))$, for all $0 \leq j \leq n$. This implies

$$|F^{n/2}(g(\xi')) - F^{n/2}(g(\zeta'))| \leq C.\sigma^{-\frac{n}{4}} \quad \text{and} \quad |F^{n/2}(\xi) - F^{n/2}(\zeta)| \leq C.\sigma^{-\frac{n}{4}}.$$

We now have to give upper bounds for $\displaystyle\sum_{k=0}^{+\infty} \log J^u \circ f^{k+i} \circ g(\xi') - \log J^u \circ f^{k+i}(\xi)$, and the same term with ζ' and ζ. Here again i satisfies $F^{n/2}(\xi) = f^i(\xi)$, and we only consider the case n even. We use (13) in Proposition 3.4:

if $F^{\frac{n}{2}}(\xi) \notin A$, then $l(F^{n/2}(\xi))$ is uniformly bounded away from zero. Therefore Proposition 3.4 directly gives:

$$\sum_{k=0}^{+\infty} \log J^u \circ f^{k+i} \circ g(\xi') - \log J^u \circ f^{k+i}(\xi) \leq C.\sqrt{|F^{n/2}(\xi) - F^{n/2} \circ g(\xi')|},$$

for some uniform constant C. But F is uniformly contracting in the stable leaves. For the same reason than above, we thus get

$$|F^{n/2}(\xi) - F^{n/2} \circ g(\xi')| \leq C.\lambda^{\frac{n}{4}},$$

for some constant C. The same holds for ζ and ζ' instead of ξ and ξ'. Note that, due to the Markov property, $F^{n/2}(\zeta)$ belongs to A if and only if $F^{n/2}(\xi)$ belongs to A. Therefore, the proposition is proved in the case where $F^{\frac{n}{2}}(\xi) \notin A$.

We now deal with the case where $F^{\frac{n}{2}}(\xi) \in A$. Now, we use (14) in Proposition 3.4 to get:

$$\begin{split}|\sum_{k=0}^{+\infty} \log J^u \circ f^{k+i} \circ g(\xi') - \log J^u \circ f^{k+i}(\xi)| \leq (\frac{C_1''}{l(F^{n/2}(\xi))} + C_1')|F^{n/2}(\xi) - F^{n/2} \circ g(\xi')| + \\ C_2'\sqrt{|F^{n/2}(\xi) - F^{n/2} \circ g(\xi')|} + \frac{C_2\lambda^p}{l(F^{n/2}(\xi))}\sqrt{|F^{n/2-1}(\xi) - F^{n/2-1} \circ g(\xi')|},\end{split} \tag{21}$$

where p is the integer such that $F(F^{n/2-1}(\xi)) = f^p(F^{n/2-1}(\xi))$.

In the right-hand side of the expression (21), we replace the terms $|F^{n/2}(\xi) - F^{n/2} \circ g(\xi')|$ by $\sqrt{\lambda}^p|F^{n/2-1}(\xi) - F^{n/2-1} \circ g(\xi')|$. We then use the second inequality in (10) for $F^{n/2}(\xi)$ to get

$$\begin{aligned}|\sum_{k=0}^{+\infty}\log J^u\circ f^{k+i}\circ g(\xi')-\log J^u\circ f^{k+i}(\xi)| &\leq C.|F^{n/2-1}(\xi)-F^{n/2-1}\circ g(\xi')| \\ &+C'\sqrt{|F^{n/2-1}(\xi)-F^{n/2-1}\circ g(\xi')|},\end{aligned}$$

for some uniform constants C and C'. This gives that the left-hand side of (21) is bounded from above by $C.\lambda^{\frac{n-2}{4}}$. Now, choosing a small positive α such that $max\{\sigma^{-n/4},\lambda^{n/4}\}<n^{\alpha}(\xi,\zeta)$, and $C>0$ big enough, we finish the proof in the case where n is even. The case where n is odd can be obtained in a similar way, except that the piece of F-orbit has to be split into two pieces, one from 0 to $[\frac{n-1}{2}]$, and the tail. □

Remark 6. Since W^u is $C^{1+\varepsilon}$, we also have

$$|\sum_{k=0}^{r_R(\xi)-1}\log J^u\circ f^k(\xi')-\log J^u\circ f^k(\zeta')|\leq C.(\eta(\xi,\zeta))^{\alpha}.$$

3.3 Local equilibrium states

Let $0<\alpha$ be such that Proposition 3.5 holds. For simplicity we say that φ is α-Hölder if it belongs to C^{α}_{η}. We fix some t such that $\mathcal{P}(t)>-t\frac{1}{2}\log\sigma$. Then we set for ξ in $\mathcal{F}$

$$\Phi(\xi)=\sum_{k=0}^{r_R(\xi)-1}-t.\log J^u\circ f^k(\xi)+t\omega\circ g(\xi)\ .$$

Note that, by Proposition 3.5 and the inequality in Remark 6, Φ belongs to C^{α}_{η}. We define the Ruelle-Perron-Frobenius operator $\mathcal{L}_S$ by

$$\mathcal{L}_S(\mathcal{T})(\xi)=\sum_{\xi'\in Pre_1(\xi)}e^{\Phi(\xi')-r_R(\xi')S}\mathcal{T}(\xi')\ ,$$

where S is a real parameter and $\mathcal{T}$ is a continuous function from $\mathcal{F}$ to $\mathbb{R}$. Recall that $Pre_1(\xi)$ is defined in section 3.1.2.

As any continuous function on the compact set $\mathcal{F}$ is bounded, the convergence of the series $\mathcal{L}_S(T)(\xi)$ is equivalent to the convergence for $\mathcal{L}_S(\mathbb{1}_{\mathcal{F}})(\xi)$. Now, the Hölder properties of ω and $\log J^u$ imply that the convergence for one ξ in $\mathcal{F}$ guarantees the convergence for any ξ in $\mathcal{F}$. For ξ in $\mathcal{F}$, write

$$\mathcal{L}_S(\mathbb{1}_{\mathcal{F}})(\xi)=\sum_{n=1}^{+\infty}\left(\sum_{\substack{\xi'\in Pre_1(\xi)\\ r_R(\xi')=n}}e^{\Phi(\xi')}\right)e^{-nS},$$

and define

$$S_c = \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\substack{\xi' \in Pre_1(\xi) \\ r_R(\xi')=n}} e^{\Phi(\xi')} \right). \tag{22}$$

Due to the Hölder regularity, S_c is independent of the choice of ξ. We assume for the moment, and prove later in Proposition 3.10, that $S_c < +\infty$. We have, for every $S > S_c$, for every $\xi \in \mathcal{F}$, and for every $T \in C^0(\mathcal{F}, \mathbb{R})$, that

$$\mathcal{L}_S(T)(\xi) < +\infty. \tag{23}$$

The real S_c is the smallest real number with this property. Following the steps in [16], we state the next important lemma, which is a consequence of the Markov property and the hyperbolic structure of F:

Lemma 3.6. *There exists a positive constant C_t, independent of S, such that, for all $\xi, \zeta \in \mathcal{F}$, $S > S_c$ and $n \in \mathbb{Z}$,*

$$e^{-C_t}\mathcal{L}_S^n(\mathbb{1}_{\mathcal{F}})(\xi) \le \mathcal{L}_S^n(\mathbb{1}_{\mathcal{F}})(\zeta) \le e^{C_t}\mathcal{L}_S^n(\mathbb{1}_{\mathcal{F}})(\xi).$$

We want to point out that the constant C_t depends only on Φ (hence on t). The independence of S for the constants C_t in lemma 3.6 is implied by Remark 1: for ξ and ζ in $\mathcal{F}$ we associate to each n-preimage of ξ a unique n-preimage of ζ in the same n-set. Two such preimages have the same n-return time r_R^n, and this removes the dependence on S. Then, we simply use the Hölder continuity of Φ to get the constant C_t. Now we can produce local equilibrium states:

Proposition 3.7. *There exist a measure m_S on $\mathcal{F}$, a positive real number λ_S and a positive Hölder-continuous function H_S on $\mathcal{F}$ such that*

1. $\mathcal{L}_S^*(m_S) = \lambda_S m_S$;
2. $\lambda_S = \int \mathcal{L}_S(\mathbb{1}_{\mathcal{F}})\, dm_S$;
3. $\mathcal{L}_S(H_S) = \lambda_S . H_S$.

Proof. See [16] for the complete proof: even if the map f is not uniformly hyperbolic, the situation for the dynamical system $(\mathcal{F}, g_{\mathcal{F}})$ in our case is exactly the same than in there. Note that Remarks 2 and 6 imply that the set of continuous functions is invariant by $\mathcal{L}_S$. Moreover $\mathcal{L}_S$ is a continuous operator on C^0. Now proposition 3.5 and again Remark 6 imply that C_η^α is also $\mathcal{L}_S$-invariant.

We first use the Schauder-Tychonoff theorem to construct m_S. The adjoint and normalized operator acts continuously on the compact set of probabilities. It thus has a fixed point m_S; and $\lambda_S = \int \mathcal{L}_S(\mathbb{1}_{\mathcal{F}})\, dm_S$.

To construct H_S, we use the Ionescu-Tulcea & Marinescu theorem (see [11]) with C^α_η and C^0; this also provides the spectral gap for $\mathcal{L}_S$. To use that theorem, we recall that the main requirement is satisfied, by Lemma 3.6. This lemma implies that the family of functions $\frac{1}{\lambda_s^n}\mathcal{L}_S^n(\mathbb{1}_{\mathcal{F}})$ is uniformly bounded for the $|| \, ||_\infty$-norm. Moreover it also implies the Lassota-York inequality :

$$||\frac{1}{\lambda_s^{n_0}}\mathcal{L}_S^{n_0}(\mathcal{T})||_{\alpha,\eta} \leq \tau.||\mathcal{T}||_{\alpha,\eta} + \upsilon.||\mathcal{T}||_\infty,$$

for any Hölder continuous function $\mathcal{T}$, and where $0 < \tau < 1$, $\upsilon \geq 0$ are real numbers independent of $\mathcal{T}$, and n_0 is a positive integer independent of $\mathcal{T}$. The fact that $\tau < 1$ follows from the contraction in the unstable direction when we iterate f^{-1} (remember $\mathcal{F}$ is a piece of unstable leaf). □

We recall that H_S is defined by the formula:

$$H_S = \lim_{n\to+\infty}\frac{1}{n}\sum_{0\leq k<n}\frac{1}{\lambda_S^n}\mathcal{L}_S^n(\mathbb{1}_{\mathcal{F}}).$$

We also have

$$e^{-C_t} \leq H_S(\xi) \leq e^{C_t}. \tag{24}$$

Let us set $d\nu_S = H_S dm_S$. The measure ν_S is $g_{\mathcal{F}}$-invariant. Moreover we have

Lemma 3.8. *The measure ν_S is ergodic, and λ_S is a simple single dominating eigenvalue for $\mathcal{L}_S$.*

Proof. Again, see [16] for the complete proof. Let $K \subset \mathcal{F}$ be some 1-set in $\mathcal{F}$. Its image by $g_{\mathcal{F}}$ is exactly $\mathcal{F}$. The function $\mathbb{1}_K$ is Hölder-continuous, since Λ is totally disconnected and the measure ν_S is conformal in the sense that

$$\begin{aligned}\nu_S(K) &= \int_{\mathcal{F}} H_S(\xi)\mathbb{1}_K(\xi)\,dm_S(\xi) = \frac{1}{\lambda_S}\int_{\mathcal{F}}\mathcal{L}_S(H_S\mathbb{1}_K)\,dm_S \\ &= \int_{\mathcal{F}}\frac{H_S(\xi')}{H_S(\xi)}\,e^{\Phi(\xi')-r_R(\xi')S-\log\lambda_S}d\nu_S(\xi),\end{aligned} \tag{25}$$

where ξ' is the preimage of ξ in the considered 1-set K. Therefore, the density theorem proves that ν_S is exact, hence mixing and ergodic. Ergodicity implies that λ_S is simple. Mixing also implies that λ_S is a single dominating eigenvalue. □

The measure ν_S is the unique equilibrium state associated to $\Phi(\cdot) - S.r_R(\cdot)$ for the system $(\mathcal{F}, g_{\mathcal{F}})$. The natural extension of ν_S, denoted by ν'_S, is the unique equilibrium state associated to the potential

$$\sum_{k=0}^{r_R(\xi)-1} -t\log J^u\circ f^k(\xi) - S.r_R(\xi)$$

for the system (R, g).

Remark 7. Due to the Gibbs property, every open set in $\mathcal{F}$ has positive ν_S measure. In fact, every n-set has positive measure, and every open set contains some n-set.

Now, relation (22) proves that for every $S > S_c$, $\mathbb{E}_{\nu'_S}[r_R] < +\infty$. This is a simple consequence of Lemma 3.6 (with $n = 1$) and of the fact that $\lambda_S = \int \mathcal{L}_S(\mathbb{I}_{\mathcal{F}})\, d\nu_S$. Therefore the measure ν'_S can be opened out: there exists a σ-finite invariant and ergodic probability measure $\widehat{\mu}_S$ on Ω such that

$$\frac{\widehat{\mu}_S(.\cap R)}{\widehat{\mu}_S(R)} = \nu'_S(.).$$

Let P_S denote the $-t\log J^u$-pressure of this measure $\widehat{\mu}_S$. Then we have

$$\begin{aligned} P_S &= h_{\widehat{\mu}_S}(f) - t\int \log J^u\, d\widehat{\mu}_S = \widehat{\mu}_S(R)\left(h_{\nu'_S}(g) + \int \sum_{j=0}^{r_R(.)-1} -t\log J^u\circ f^j\, d\nu'_S\right) \\ &= \widehat{\mu}_S(R)\left(S\int r_R\, d\nu'_S + \log\lambda_S\right) = S + \widehat{\mu}_S(R)\log(\lambda_S). \end{aligned} \tag{26}$$

Lemma 3.9. *The map $\psi : S \mapsto \log(\lambda_S)$ is convex and analytic on a complex neighborhood of $]S_c, +\infty[$.*

Proof. First, λ_S is a simple single dominating eigenvalue. We thus have

$$\mathcal{L}_S(\mathbb{I}_{\mathcal{F}})^n(\xi) = \lambda_S^n H_S(\xi) + \lambda_S^n \Psi^n(\mathbb{I}_{\mathcal{F}})(\xi),$$

where Ψ is some operator with spectral radius strictly smaller than 1. This implies that

$$\forall \xi\in\mathcal{F},\quad \log\lambda_S = \lim_{n\to+\infty}\frac{1}{n}\log\left(\mathcal{L}_S^n(\mathbb{I}_{\mathcal{F}})(\xi)\right). \tag{27}$$

This, together with the Hölder's inequality, implies that ψ is convex. Moreover, $\mathcal{L}_S$ is a quasi-compact operator with a simple isolated dominating eigenvalue; analyticity (in some complex neighborhood of $]S_c, +\infty[$) is thus obtained via the perturbation Theorem from [10] (see Th III.8). □

Using (27), we have (see [6] for a proof)

$$\psi'(S) = -\frac{1}{\widehat{\mu}_S(R)}. \tag{28}$$

In particular, this implies that ψ is a decreasing and continuous one-to-one map from $]S_c, +\infty[$ onto its image.

3.4 Uniqueness of the global equilibrium state

In this subsection, we first prove that S_c belongs to $\mathbb{R}$, and give an upper-bound for its value. Then we deduce uniqueness of the global equilibrium state associated to $-t\log J^u$, and the regularity of $t \mapsto \mathcal{P}(t)$. As we saw above, S_c is defined by the relation (22). The key result is the following:

Proposition 3.10. *With the previous notations, $S_c \leq \mathcal{P}(t)$.*

Proof. The main idea is to copy the proof of lemma 20.2.3 in [13], p.624. We fix a point ξ in $\mathcal{F}$, and set

$$\upsilon_n = \left(\sum_{\substack{\xi' \in Pre_1(\xi) \\ r_R(\xi')=n}} e^{-t.S_n(\log J^u)(\xi')} \right)^{-1} \sum_{\substack{\xi' \in Pre_1(\xi) \\ r_R(\xi')=n}} e^{-t.S_n(\log J^u)(\xi')} \delta_{\xi'},$$

and $\tau_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} f_*^k \upsilon_n$. Let τ be a weak* accumulation point of the sequence τ_n. For convenience we denote by $Pre_1(\xi, n)$ the set of preimages ξ' of ξ whose return time equals n. Notice that all the terms $\omega(g(\xi'))$ are uniformly bounded, which implies that

$$S_c = \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\xi' \in Pre_1(\xi,n)} e^{S_n(-t\log J^u)(\xi')} \right).$$

By construction, the rectangle R is an element of the partition $\mathcal{G}_{n^-}^{n^+}$, for some positive integers n^- and n^+.

We now claim that for any n, $\bigvee_{k\leq n-1} f^{-k}\mathcal{G}_{n^-}^{n^+}$ separates $Pre_1(\xi, n)$.

To prove the claim, let ξ' and ξ'' be two different points in $Pre_1(\xi, n)$. By construction, they are two different points in $\mathcal{F}$. Hence, the Markov property implies that, for every $m \geq 0$, $f^{-m}(\xi')$ and $f^{-m}(\xi'')$ belong to the same element of $\mathcal{G}_{n^-}^{n^+}$. By definition of $Pre_1(\xi, n)$, $f^n(\xi')$ and $f^n(\xi'')$ belong to $W^s(\xi, R)$. Again, the Markov property implies that, for every $m \geq 0$, $f^{m+n}(\xi')$ and $f^{m+n}(\xi'')$ belong to the same element of $\mathcal{G}_{n^-}^{n^+}$. As the partition $\mathcal{G}_{n^-}^{n^+}$ separates orbits, ξ' and ξ'' must belong to two different elements of $\bigvee_{k\leq n-1} f^{-k}\mathcal{G}_{n^-}^{n^+}$.

Copying the proof of lemma 20.2.3 in [13] p.624 we get, for every fixed q:

$$S_c \leq \limsup_{n\to+\infty} \frac{1}{q} H_{\tau_n}[\bigvee_{k\leq q} f^{-k}\mathcal{G}_{n^-}^{n^+}] + \int -t\log J^u \, d\tau_n.$$

Note that in [13], the result holds when the system is expansive. However, the proof only uses that each element of the partition $\bigvee_{k\leq n-1} f^{-k}\mathcal{G}_{n^-}^{n^+}$ contains at most one point of $Pre_1(\xi, n)$. This is true in our case. The only other argument is that the limit measure τ does not weigh the boundary of the partition $\mathcal{G}_{n^-}^{n^+}$. This also occurs in our case, except if $\tau(\{(0,0)\}) > 0$. We discuss this possibility in the rest of the proof.

We use Lemmas 2.1 2.2 and 2.3 to conclude that:

- either $\tau(\{(0,0)\}) = 0$, and S_c is lower than the $-t\log J^u$-pressure of τ, which is by definition lower than $\mathcal{P}(t)$,
- or $\tau = u.\delta_{(0,0)} + v.\nu$, where $u \neq 0$, $u+v=1$ and $\nu \neq \delta_{(0,0)}$ is ergodic. In that case,
$$\limsup_{n\to+\infty} H_{\tau_n}[\bigvee_{k\le q} f^{-k}\mathcal{G}_m] \le v.H_\nu[\bigvee_{k\le q} f^{-k}\mathcal{G}_m];$$
therefore
$$S_c \le v.h_\nu(f) + \limsup_{n\to+\infty} -t\int \log J^u\, d\tau_n \le v.\mathcal{P}(t) - \frac{ut}{2}\log\sigma < \mathcal{P}(t).$$

In both cases we get $S_c \le \mathcal{P}(t)$. $\square$

We can now prove uniqueness of the equilibrium state associated to $-t\log J^u$. Let μ be any such equilibrium state (existence has been proved above).

- The measure μ cannot be Dirac measure $\delta_{(0,0)}$ (see page 14).
- Therefore μ must weight at least one rectangle of the Markov partition $\mathcal{R}$. We call R one of these rectangles. Since $\mu(R) > 0$, we can induce μ on R, and then on $\mathcal{F}$. We call μ' this $g_{\mathcal{F}}$-invariant measure.
- The relation $h_\mu(f) - t\int \log J^u\, d\mu = \mathcal{P}(t)$, yields
$$\mu(R)\left(h_{\mu'}(g_{\mathcal{F}}) + \int \Phi(\xi) - \mathcal{P}(t).r_R(\xi)\, d\mu'(\xi)\right) = 0, \tag{29}$$
which means that the $(\Phi - \mathcal{P}(t).r_R)$-pressure of $(\mathcal{F}, g_{\mathcal{F}})$ is non-negative.
- On the other hand, relation (26), for $S > \mathcal{P}(t) \ge S_c$, implies that $\lambda_S \le 1$ as long as $S > \mathcal{P}(t)$.
- Therefore lemma 3.6 and (*2*) in proposition 3.7 imply that for any ξ in $\mathcal{F}$ the series of positive terms $\mathcal{L}_S(\mathbb{1}_{\mathcal{F}})(\xi)$ is bounded, thus converges for $S = \mathcal{P}(t)$. This implies that there exists a unique equilibrium state associated to $\Phi(.) - \mathcal{P}(t).r_R(.)$ in $\mathcal{F}$.
- The theorem of monotone convergence applied for each integer n and for each point $\xi \in \mathcal{F}$, plus lemma 3.6, imply that for every ξ and for every n
$$\mathcal{L}^n_{\mathcal{P}(t)}(\mathbb{1}_{\mathcal{F}})(\xi) \le e^{C_t},$$
which means that $\log\lambda_{\mathcal{P}(t)} \le 0$.

- Hence, the pressure on $\mathcal{F}$ for $\Phi - \mathcal{P}(t).r_R$ is non-positive and non-negative. Therefore it is equal to 0 and the unique equilibrium state in $\mathcal{F}$ associated to $\Phi - \mathcal{P}(t).r_R$ is the unique measure with zero pressure. As the pressure of μ' is zero, this equilibrium state must be μ'.

- As μ' is obtained from the measure μ in $\mathcal{Q}$, it means that $\mathbb{E}_{\nu'_{\mathcal{P}(t)}}[r_R] < +\infty$; for $S = \mathcal{P}(t)$, the measure ν'_S can be opened-out in Λ.

- Therefore, the mixing property of f and the fact that $\nu_{\mathcal{P}_t}$ weighs any open set in $\mathcal{F}$ (see Remark 7), imply that *every* rectangle of $\mathcal{R}$ must have positive μ-measure, and μ is uniquely defined on these rectangles.

This completes the proof of the uniqueness of the equilibrium state in the case $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. From now on, μ_t will denote this equilibrium state.

Using uniqueness, we can get analyticity for $t \mapsto \mathcal{P}(t)$. The first step is to give a better bound for S_c.

Lemma 3.11. *Let $t \geq 0$ be such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$. With the previous notation, we have*

$$S_c(t) < \mathcal{P}(t).$$

Proof. We use notation of the proof of proposition 3.10. Let φ be a continuous function supported in R, with image in $[0,1]$. Then we have

$$0 \leq \int \varphi \, d\tau_n \leq \frac{1}{n}.$$

This immediately yields $\int \varphi \, d\tau = 0$; this occurs for any continuous function φ with support in R, and thus $\tau(\overset{\circ}{R}) = 0$. Therefore, τ cannot be the global equilibrium state, because the mixing-property of f and the fact that μ' is conformal imply that any open set in $\Lambda \setminus \{xy = 0\}$ has positive μ_t-measure. Hence, the t-pressure of τ, $P_\tau(t)$, satisfies $P_\tau(t) < \mathcal{P}(t)$. In the proof of proposition 3.10 we get

$$S_c \leq P_\tau(t) < \mathcal{P}(t),$$

if $\tau(\{(0,0)\}) = 0$, and we get $S_c \leq v.\mathcal{P}(t) - u.t\frac{1}{2}\log\sigma < \mathcal{P}(t)$ if $\tau(\{(0,0)\}) > 0$. In both cases we have $S_c < \mathcal{P}(t)$. □

We then use this gap to prove analyticity for $\mathcal{P}(t)$. For that we just copy the arguments from [17], section 3.2.

For $S = \mathcal{P}(t)$, the measure $\nu'_S = \mu_t$ is the unique equilibrium state associated to $-t\log J^u$. Therefore (26) is valid for $S = \mathcal{P}(t)$ and $\widehat{\mu}_S = \mu_t$. This yields $\lambda_{\mathcal{P}(t)} = 1$.

Define the operator $\mathcal{L}_{S,t}$ by

$$\mathcal{L}_{S,t}(T)(\xi) = \sum_{\xi' \in Pre_1(\xi)} e^{-tS_{r_R(\xi')}(\log J^u)(\xi') - r_R(\xi')S + t.\omega(g(\xi'))} T(\xi').$$

All the work we have done yields

- For every S and t, $\mathcal{L}_{S,t}(\mathbb{1}_{\mathcal{F}})$ converges as long as $S > S_c(t)$.
- The map $t \mapsto S_c(t)$ is convex (directly from definition), and satisfies $S_c(t) < \mathcal{P}(t)$.
- For every t such that $\mathcal{P}(t) > -t\frac{1}{2}\log\sigma$, we have $\lambda_{\mathcal{P}(t),t} = 1$.
- The function $S \mapsto \log\lambda_{S,t}$ is analytic in a complex neighborhood of $]S_c(t), +\infty[$.
- Equality (28) gives $\dfrac{\partial \log\lambda_{S,t}}{\partial S}|_{S=\mathcal{P}(t)} = \dfrac{-1}{\mu_t(R)} \neq 0$.

Now, straightforward arguments prove that the map $t \mapsto \mathcal{L}_{S,t}$ is analytic, in the open set $\{(S,t),\ S \geq S_c(t),\ \mathcal{P}(t) > -t\frac{1}{2}\log\sigma\}$ (see [6]); we can again use the perturbation theorem (see [10]) to conclude that the map $t \mapsto \lambda_{S,t}$ is analytic in some complex neighborhood of its real interval of definition. This also holds for $t \mapsto \log\lambda_{S,t}$, since $\lambda_{S,t}$ belongs to $\mathbb{R}_+^*$ for $t \in \mathbb{R}_+$.

Moreover, $\mathcal{P}(t)$ satisfies $\lambda_{\mathcal{P}(t),t} = 1$, or equivalently $\log\lambda_{\mathcal{P}(t),t} = 0$. We have just seen above that $\frac{\partial \log\lambda_{S,t}}{\partial S}|_{S=\mathcal{P}(t)} \neq 0$; thus, the implicit mapping theorem for holomorphic functions in several complex variables (see [24]) proves that the function $t \mapsto \mathcal{P}(t)$ is analytic.

4 Unstable Hausdorff dimension

The proof of Theorem B follows the method in [20], with some modifications to adapt it to our setting (in particular, since the map $\log J^u$ is not continuous, the generic set $\mathcal{G}_\mu$ for any ergodic measure μ does not give any information for the convergence of the unstable Lyapunov exponent).

Recall that the Hausdorff dimension of any set X is given by the following process. Set

$$m_t(X) = \lim_{\varepsilon\downarrow 0}\inf\{\sum(\operatorname{diam} U_i)^t,\ \bigcup U_i \supset X,\ \operatorname{diam} U_i \leq \varepsilon\},$$

where U_i are open sets. The quantity $m_t(X)$ is the t-Hausdorff measure of X. Then there exists a unique δ such that

$$\delta = \sup\{t, m_t(X) = +\infty\} = \inf\{t,\ m_t(X) = 0\}.$$

This real number δ is the Hausdorff dimension of X.

Now, to prove Theorem B, it is sufficient to prove that, with the previous notations, the Hausdorff dimension of $\mathcal{F}$ is equal to t_0. Indeed, any local piece of unstable leaf which does not contains any point of the critical orbit can be decomposed as the countable union of unstable leaves intersected with rectangles of the Markov partition.

4.1 Lower bound for the Hausdorff dimension

Recall that t_0 is uniquely determined by

$$\mathcal{P}(t_0) = 0.$$

and that $\mathcal{P}(1) < 0$, and $t \mapsto \mathcal{P}(t)$ decreases. Let $t < t_0$ (and hence $t < 1$). Recall that $\lambda_{\mathcal{P}(t),t} = 1$, therefore the Transfer-Operator has 1 for spectral radius. This operator is defined by

$$\mathcal{L}_{\mathcal{P}(t),t}(T) = \sum_{\xi' \in Pre_1(\xi)} e^{-tS_{r_R(\xi')}(\log J^u)(\xi') + t\omega \circ f^{r_R(\xi')}(\xi') - r_R(\xi')\mathcal{P}(t)}.$$

For the rest of the proof we omit the dependence in t in the notation. Let us set

$$\widetilde{\mathcal{L}}_j(T) = \sum_{\xi' \in Pre_1(\xi,j)} e^{-tS_{r_R(\xi')}(\log J^u)(\xi') + t\omega \circ f^{r_R(\xi')}(\xi') - r_R(\xi')\mathcal{P}(t)},$$

where $Pre_1(\xi, j)$ denote the set of preimages by $g_{\mathcal{F}}$, such that $r_R(\xi) \leq j$. For each j, $\widetilde{\mathcal{L}}_j$ is a quasi-compact operator with a single dominating simple eigenvalue, ρ_j. Moreover the family of operators converge in the generalized sense to $\mathcal{L}$ (see [12] chapter IV). Section 5 in [12] chapter IV, p. 213 yields that the spectral radius of $\mathcal{L}_j$ (namely ρ_j) converges to 1 as j goes to $+\infty$. We thus choose ε and j such that $\rho_j \in [1-\varepsilon, 1+\varepsilon]$, with

$$1 - \varepsilon > e^{-\mathcal{P}(t)}. \tag{30}$$

Notice that lemma 3.6 holds also for $\widetilde{\mathcal{L}}_j$. This yields

$$\forall \xi \in \mathcal{F}\ , \rho_j^{-n} \widetilde{\mathcal{L}}_j^n(\mathbb{1}_{\mathcal{F}})(\xi) \in [e^{-C_t}, e^{C_t}]. \tag{31}$$

To $\widetilde{\mathcal{L}}_j^n(\mathbb{1}_{\mathcal{F}})$, we associate its "covering" in n-sets, which are all the n-sets in $\mathcal{F}$ such that their successive k^{th}-return times ($k \leq n$) are all lower than $k.j$. We denote this covering by $\mathcal{V}_{j,n}$. This is a finite union of disjoint compact sets. Recall that R does not contain any point of the critical orbit.

Let $\delta > 0$ be smaller than half the size of any of the gaps between two consecutive n-set of $\mathcal{V}_{j,n}$. We also assume that δ is lower than the radius of any of these n-sets. For a fixed j, denote by $\mathcal{V}_{j,+\infty}$, the intersection of the sets in $\mathcal{V}_{j,n}$.

If K is one of the n-sets in $\mathcal{V}_{j,n}$, the Markov property and the Lipschitz regularity of J^u in the unstable leaves, yield that for any ξ in K,

$$e^{-C'_t}(\mathrm{diam}\mathcal{F})^t \leq (\mathrm{diam}K)^t e^{tS_{r^n_R(\xi)}(\log J^u)(\xi)} \leq e^{C'_t}(\mathrm{diam}\mathcal{F})^t, \tag{32}$$

where C'_t only depends on t.

Let us now consider any countable covering $\mathcal{U}_i$ of $\mathcal{V}_{j,+\infty}$ with radius lower than δ. Recall that, for any positive a and b, we have that $(a+b)^t \leq a^t + b^t$ (recall that

$0 < t < 1$). Then, rearranging the elements of $\mathcal{U}_i$ in the same n-sets of $\mathcal{V}_{j,n}$, and using (31) and (32), we get:

$$\begin{aligned}
\sum_{U\in\mathcal{U}_i} (\operatorname{diam} U)^t &\geq \sum_{K\in\mathcal{V}_{j,n}} (\operatorname{diam} K)^t \qquad (33)\\
&\geq e^{-C_t - C'_t} (\operatorname{diam}\mathcal{F})^t \sum_{\xi'\in Pre_1(\xi,n)} e^{-t.S_{r^n_R(\xi')}(\log J^u)(\xi')}\\
&\geq C(t)\rho_j^n \sum_{\xi'\in Pre_1(\xi,n)} e^{r^n_R(\xi')\mathcal{P}(t)}\\
&\geq C(t).\rho_j^n.e^{n\mathcal{P}(t)},
\end{aligned}$$

where $C(t)$ is a constant which only depends on t. Taking the lim inf in δ (the diameter of the cover $\mathcal{U}_i$), we get

$$\forall n,\ m_t(\mathcal{V}_{j,+\infty}) \geq C(t).(1-\varepsilon)^n e^{n\mathcal{P}(t)} > 0.$$

Then, (30) immediately yields that the Hausdorff dimension of $\mathcal{F}$ is greater than or equal to t_0.

4.2 Upper bound for the Hausdorff dimension

We pick $1 > t > t_0$ with $-t\frac{1}{2}\log\sigma < \mathcal{P}(t) < 0$. As the map $\mathcal{P}(.)$ is continuous and decreasing, we can pick such a t.

Let n be a positive integer, and consider the cover of $\mathcal{F}$ by the n-sets. No n-set is an open set, but we can extend them in the following sense: We pick some small open neighborhood of R, U; if K is a n-set, then $f^{r^n_R(K)}(K)$ is (by definition) an unstable leaf in R. Notice that $r^n_R(K)$ is the n^{th} return-time associated to K; namely, $r^n_R(K) = r^n_R(\xi)$ for any ξ in $\overset{\circ}{K}\subset \mathcal{F}$.

Let ξ be in K. We then consider $K' = f^{-r^n_R(K)}(W^u_{loc}(f^{r^n_R(K)}(\xi))\cap U)$. This is our *extended* n-set.

Considering n as large as wanted, and the extended n-sets, we have an open cover of $\mathcal{F}$ with diameter as small as wanted. Note that (32) is still valid when K is an extended n-set. We call $\mathcal{V}_n$ this cover with extended n-sets. Therefore we get

$$\begin{aligned}
\sum_{K\in\mathcal{V}_n} (\operatorname{diam} K)^t &\leq C(t) \sum_{\xi'\in Pre_n(\xi)} e^{-tS_{r^n_R(\xi')}(\log J^u)(\xi') + t\omega(g^n(\xi')) - r^n_R(\xi')\mathcal{P}(t)} e^{r^n_R(\xi')\mathcal{P}(t)}\\
&\leq C(t) \sum_{\xi'\in Pre_n(\xi)} e^{-tS_{r^n_R(\xi')}(\log J^u)(\xi') + t\omega(g^n(\xi')) - r^n_R(\xi')\mathcal{P}(t)} e^{n\mathcal{P}(t)}\\
&\leq C(t)\mathcal{L}^n_{\mathcal{P}(t),t}(\mathbb{1}_{\mathcal{F}})(\xi)e^{n\mathcal{P}(t)}\\
&\leq C(t)e^{n\mathcal{P}(t)}. \qquad (34)
\end{aligned}$$

Hence, (34) yields $m_t(\mathcal{F}) \le C(t)e^{n\mathcal{P}(t)}$, for any n, and $\mathcal{P}(t) < 0$ yields $m_t(\mathcal{F}) = 0$. This holds for any $t > t_0$, thus the Hausdorff dimension of $\mathcal{F}$ is smaller than or equal to t_0. This completes the proof.

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