Existence of SRB-measures for some topologically hyperbolic diffeomorphisms

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Abstract

A diffeomorphism f of a compact manifold M is called *Almost Axiom-A* if it is hyperbolic in a neighborhood of some compact f-invariant set Ω, except in some singular set of neutral points. We prove that, if there exists some f-invariant set of hyperbolic points, with positive *unstable*-Lebesgue measure, and such that for every point in this set the stable and unstable leaves are "long enough", then, f admits either a probability SRB-measure or a σ-finite SRB-measure.

1 Introduction and statement of results

Let us consider a smooth dynamical system (M, f), where M is a compact smooth Riemannian manifold (of dimension N), f a C^2 diffeomorphism acting on M. Let μ be a f-invariant ergodic probability measure on M. By the Ergodic Theorem, the set G_μ of generic points, *i.e.* the set of points x such that for every continuous function ϕ ,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k(x)=\int\phi\,d\mu \tag{1}$$

has full μ-measure. From the point of view of physics, this convergence can be actually observed only when this set G_μ has strictly positive measure for the measure of volume of the manifold, also called the Lebesgue measure on M and denoted by Leb_M.

Definition 1.1. *We say that a f-invariant ergodic probability measure is a physical measure, if, and only if,* $Leb_M(G_\mu) > 0$.

If f is Axiom-A, results about existence of a physical measure are known: one can construct special Gibbs States which are the so called Sinai-Ruelle-Bowen measures, and which are special physical measures (see e.g. [5]). In the non-uniformly hyperbolic case, some results are known but there is no general theory, maybe because there are a lot of possibilities to make uniform hyperbolicity fail. However several works give some conditions on a dynamical system (M, f) yielding existence of a physical measure: in [3], it is proved that for the special case of Hénon's maps, many maps of this type admit a unique SRB measure. In [8], it is proved that every non degenerate Almost Anosov Diffeomorphism on a surface admits a finite or σ-finite SRB measure. In [4] or [2] it is proved that a partially hyperbolic diffeomorphism such that the central direction is mostly contracting or mostly expanding admits a SRB measure.
In this paper we consider dynamical systems, in dimension ≥ 2, for which hyperbolicity comes from a point-wise behavior but fails on a globally f-invariant set S of indifferent points; this case is new in dimension higher than 3, and new problems arise. For instance, even the question of integrability of the hyperbolic splitting is not obvious, because the derivative has no uniform spectral gap near to the set of indifferent points; therefore the classical method of the graph transform does not work. Hence, in this situation, the notion of SRB-measure is not obvious.

1.1 Statement of Results

We first define the class of dynamical systems that we shall consider in the sequel:

Definition 1.2. *Let f be in Diff$^2(M)$. It is said to be Almost-Axiom-A if there exists an open set U which contains a non-empty f-invariant compact set $\Omega \subset U$ such that:*
(i) For every $x \in U$ there is a df-invariant splitting (invariant where it makes sense) of the tangent space $T_xM = E^u(x) \oplus E^s(x)$ with $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ two Hölder continuous maps (with uniformly bounded Hölder constant).
(ii) There exist two continuous and non negative functions $x \mapsto k^u(x)$ and $x \mapsto k^s(x)$ such that

$$\forall x \in U, \qquad \begin{array}{ll} \forall v \in E^s(x) & ||df(x).v||_{f(x)} \leq e^{-k^s(x)}||v||_x \\ \forall v \in E^u(x) & ||df(x).v||_{f(x)} \geq e^{k^u(x)}||v||_x, \end{array}$$

where $|| \, ||_x$ denotes the Riemannian norm on T_xM.
(iii) the exceptional set, $S = \{x \in U, \ k^u(x) = k^s(x) = 0\}$ satisfies $f(S) = S$, and for every x in $U \setminus S$, $k^u(x)$ and $k^s(x)$ are positive real numbers.

Remark: In this definition the set U can be considered as hyperbolic in a weak point of view. We probably do not really need hyperbolicity in U, and we should obtain the same results without this assumption. However, we think it makes the proof easier and particularly less technical than it would be without this assumption.

From now on, f will be some Almost-Axiom-A, that is, f obeys the conditions of definition 1.2. The sets S and Ω and the functions k^u and k^s are fixed as in the definition. The splitting $T_xM = E^u(x) \oplus E^s(x)$ will be called the hyperbolic splitting.

Definition 1.3. *Let $\lambda \in]0, +\infty[$. A point x in Ω is said to be λ-hyperbolic if*

1. $\forall v \in E^u(x) \setminus \{0\}$, $\limsup_{n\to+\infty} \frac{1}{n} \log ||df^{-n}(x).v|| < -\lambda$;

2. $\forall v \in E^s(x) \setminus \{0\}$, $\limsup_{n\to+\infty} \frac{1}{n} \log ||df^{n}(x).v|| < -\lambda$.

A f-invariant set Λ_λ such that every point in Λ_λ is λ-hyperbolic will be called a λ-hyperbolic set.

The notion of (non-uniformly) hyperbolicity is not completely standard, and several definitions exist in the literature. If hyperbolic currently means the existence of some splitting of the tangent space in one expanding direction and one contracting direction, the notion of "expansion" is not clear. In our case, the definition of hyperbolicity is directly taken from the definition of Axiom-A, where expansion means contraction in the negative time.

Definition 1.4. *A point x in Ω is said to be a point of integration of the hyperbolic splitting if there exist $\varepsilon > 0$ and two $\mathcal{C}^1$-disks $D^u_\varepsilon(x)$ and $D^s_\varepsilon(x)$ of size ε (and centered in x) such that for every y in $D^i_\varepsilon(x)$ $(i = u, s)$ $T_yD^i_\varepsilon(x) = E^i(y)$.*

The set of points of integration is invariant by f. As usual when we have two families of local unstable and stable manifolds we define $\mathcal{F}^u(x) = \bigcup_{n\geq 0} f^n D^u_{\varepsilon(-n)}(f^{-n}(x))$ and $\mathcal{F}^s(x) = \bigcup_{n\geq 0} f^{-n} D^s_{\varepsilon(n)}(f^n(x))$, where $\varepsilon(k)$ is the size of the disks associated to $f^k(x)$. They are the global unstable and stable manifolds.
As we said before, it is not obvious, in our case and in dimension higher then 3, that there exist some points of integration of the hyperbolic splitting. Our first result is to prove their existence.

Theorem A
Let $\lambda > 0$. Any λ-hyperbolic point is a point of integration of the hyperbolic splitting.

Remark: This result is well known when the hyperbolic splitting is dominated, because this condition yields the existence of a uniform spectral gap for the derivative df. A very close result is also well known in the Pesin Theory, but only on a set of full measure. Therefore, the precise topological characterization of the set of points of integration given by Pesin Theory depends on the choice of the invariant measure.

In our case the improvement consists in the fact that we prove integrability even in the presence of indifferent points for a set of points whose precise characterization does not depend on the ergodic properties of some invariant probability measure which would be given *a priori*.

Let x be some point of integration; the two manifolds $\mathcal{F}^u(x)$ and $\mathcal{F}^s(x)$ are also Riemannian manifolds. We denote by d^u and d^s their Riemannian metrics, and by Leb^u_x and Leb^s_x their Riemannian measures. If a measurable partition is subordinated to the unstable foliation $\mathcal{F}^u$ (see [14] and [11]), any f-invariant measure admits a unique system of conditional measures with respect to the given partition.

Definition 1.5. *In this article we will refer to an invariant measure having absolutely continuous conditional measures on unstable leaves $\mathcal{F}^u(x)$ with respect to Leb^u_x as a Sinai-Ruelle-Bowen measure or a SRB-measure (in abridged version).*

Definition 1.6. *Let two real numbers $\lambda > 0$ and $\varepsilon_0 > 0$ be fixed. A point x in Ω is said to be (ε_0, λ)-regular if the following conditions are satisfied:*
(i) x is λ-hyperbolic.
(ii) for $i = u, s$, $\mathcal{F}^i(x)$ contains a disk $D^i_{\varepsilon_0}(x)$ (centered in x) of size ε_0.
Let Λ be a f-invariant compact set in Ω. It is said to be (ε_0, λ)-regular if all points of Λ are (ε_0, λ)-regular points.

Markov partitions (see [5]) play an important role in the theory of uniformly hyperbolic diffeomorphism. Unfortunately there is no general way to build some Markov partition in the non-uniformly hyperbolic case. In our case we are able to construct some Markov partition and, under some additional assumption, to produce SRB measures:

Theorem B
Let f be an Almost-Axiom-A diffeomorphism. Let Λ be a (ε_0, λ)-regular set. Then there exists a countable Markov partition of Λ.
Moreover, if there exists some x_0 in Λ such that $Leb^u_{x_0}(D^u_{\varepsilon_0}(x_0) \cap \Lambda) > 0$, then there exists a SRB measure for f which is finite or σ-finite.

It is important to notice that hypothesis in Theorem B is very weak. If there exists some probability SRB-measure, μ, then, there exists some (ε_0, λ)-regular set Λ of full μ-measure such that for μ-a.e. x in Λ, $Leb^u_x(D^u_{\varepsilon_0}(x) \cap \Lambda) > 0$. However, a work due to M. Herman ([7]) proves that there exist some dynamical systems on the circle such that Lebesgue-almost-every point is "hyperbolic" but there is no SRB-measure (even σ-finite).

The rest of this paper proceeds as follows: in Section 2 we prove Theorem A. The aim of Section 3 is to construct some special reduced dynamical system (R, g). This is a very technical part and certainly the key point in the proof of Theorem B. We first state and prove a shadowing lemma. Several shadowing lemmas already exist in literature but they introduce some invariant measure (which would be given *a priori*), or they hold for the so called Pesin's set (the ergodic way). As far as we know, this is the first shadowing lemma for non-uniformly "topologically" hyperbolic dynamical systems. Then, we use our shadowing lemma to construct several Markov covers of Λ. Finally we produce the special reduced system. Section 4 is devoted to the rest of the proof of Theorem B: we prove the existence of some special measure for the reduced system and we extend it to the global system (M, f). This extended measure will be an SRB-measure.

The prooves are all based on the same key point: the estimates are all uniformly hyperbolic outside some fixed bad neighborhood $B(S, \varepsilon_1)$ of S. Moreover a λ-hyperbolic point cannot stay too long in the fixed neighborhood. The fact that f is an Almost-Axiom-A implies that an incursion in $B(S, \varepsilon_1)$ cannot spoil too much the (uniformly) hyperbolic estimates of contractions or expansions.
Obviously all the constants appearing are strongly correlated, and special care is taken in Section 3 in choosing them in the right order.

2 Proof of Theorem A

Let $\lambda > 0$ be fixed. For $\varepsilon_1 > 0$ we set $B(S, \varepsilon_1) \stackrel{def}{=} \{y \in M, d(S, y) < \varepsilon_1\}$. We pick some small $\varepsilon_1 > 0$: precise conditions are stated along the way. We first assume that ε_1 is small enough such that $B(S, \varepsilon_1) \subset U$. By continuity, we can choose ε_1 small enough such that, for every $x \in B(S, \varepsilon_1)$, $1 \leq ||df(x)_{|E^u(x)}|| < e^{\frac{\lambda}{100}}$ and $1 \leq ||df^{-1}(x)_{|E^s(x)}|| < e^{\frac{\lambda}{100}}$.
There exists some $\lambda^u > 0$ (namely $\min\{k^u(x), x \in \Omega \setminus B(S, \varepsilon_1)\}$) and $\lambda^s > 0$ (namely $\min\{k^s(x), x \in \Omega \setminus B(S, \varepsilon_1)\}$) such that for every $x \in \Omega \setminus B(S, \varepsilon_1)$

- $\forall v \in E^u(x), v \neq 0, \begin{cases} ||df(x).v||_{f(x)} > e^{\lambda^u}||v||_x, \\ ||df^{-1}(x).v||_{f^{-1}(x)} < e^{-\lambda^u}||v||_x. \end{cases}$

- $\forall v \in E^s(x), v \neq 0, \begin{cases} ||df^{-1}(x).v||_{f^{-1}(x)} > e^{\lambda^s}||v||_x, \\ ||df(x).v||_{f(x)} < e^{-\lambda^s}||v||_x. \end{cases}$

We denote by $|\,.\,|$ the euclidean norm on $\mathbb{R}^N$. By continuity we always can assume that the map $x \mapsto \dim E^i(x)$ are constant $(i = u, s)$. From now on, we will denote by $\mathbb{R}^u$ the space $\mathbb{R}^{\dim E^u} \times \{0\}^{\dim E^s}$ In the same way $B^u(0, \rho)$, $B^s(0, \rho)$,and $\mathbb{R}^s$ will denote the spaces $B^{\dim E^u}(0, \rho) \times \{0\}^{\dim E^s}$, $\{0\}^{\dim E^u} \times B^{\dim E^s}(0, \rho)$, $\{0\}^{\dim E^u} \times \mathbb{R}^{\dim E^s}$).

Proposition 2.1. *Let ε be small compared to λ, λ^u or λ^s. There exist constants $\rho_1 > 0$, $0 < K_1 < K_2$, a positive function $\bar{\rho}$, and a family of embeddings $\phi_x : B_x(0, \rho_1) \subset \mathbb{R}^N \to M$ such that*
(1) $\phi_x(0) = x$ and $d\phi_x(0)$ maps respectively $\mathbb{R}^u$ and $\mathbb{R}^s$ onto $E^u(x)$ and $E^s(x)$.
(2) Set $\widehat{f}_x = \phi_{f(x)}^{-1} \circ f \circ \phi_x$ and $\widehat{f}_x^{-1} = \phi_{f^{-1}(x)}^{-1} \circ f^{-1} \circ \phi_x$; then

(2.1) if x is in $\Omega \setminus B(S, \varepsilon_1)$, then

$$- \forall v \in \mathbb{R}^u, v \neq 0, \begin{cases} |d\widehat{f}_x(0).v| > e^{\lambda^u}|v|, \\ |d\widehat{f}_x^{-1}(0).v| < e^{-\lambda^u}|v|. \end{cases}$$

$$- \forall v \in \mathbb{R}^s, v \neq 0, \begin{cases} |d\widehat{f}_x^{-1}(0).v| > e^{\lambda^s}|v|, \\ |d\widehat{f}_x(0).v| < e^{-\lambda^s}|v|. \end{cases}$$

(2.2) if x is in $B(S, \varepsilon_1)$, then

$$- \forall v \in \mathbb{R}^u, v \neq 0, \begin{cases} |v| \leq |d\widehat{f}_x(0).v| < e^{\frac{\lambda}{100}}|v|, \\ |v| \geq |d\widehat{f}_x^{-1}(0).v| > e^{-\frac{\lambda}{100}}|v|. \end{cases}$$

$$- \forall v \in \mathbb{R}^s, v \neq 0, \begin{cases} |v| \leq |d\widehat{f}_x^{-1}(0).v| < e^{\frac{\lambda}{100}}|v|, \\ |v| \geq |d\widehat{f}_x(0).v| > e^{-\frac{\lambda}{100}}|v|. \end{cases}$$

(3) For every x in $B(S, \varepsilon_1)$, $0 < \bar{\rho}(x) \leq \rho_1$, and for every x in $\Omega \backslash B(S, \varepsilon_1)$, $\bar{\rho}(x) = \rho_1$.
(4) On the ball $B_x(0, \bar{\rho}(x))$ we have $Lip(\widehat{f}_x - d\widehat{f}_x(0)) < \varepsilon$ and $Lip(\widehat{f}_x^{-1} - d\widehat{f}_x^{-1}(0)) < \varepsilon$.
(5)For every x and for every $z, z' \in B_x(0, \rho_1)$,

$$K_1|z - z'| \leq d(\phi_x(z), \phi_x(z')) \leq K_2|z - z'|.$$

This is a simple consequence of the fibered map *exp*. However, it is important for the rest of the paper to understand that the two constants K_1 and K_2 do not depend on ε_1. They result from the distortion due to the angle between the two sub-spaces E^u and E^s, plus the injectivity radius. This quantities are uniformly bounded.

Moreover Ω is a compact set in U and thus, we can choose $\rho_1 > 0$ such that $B(\Omega, \rho_1) \subset U$. As the maps $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ are continuous, we also can assume that ρ_1 is small enough such that for every x in Ω and y in $\phi_x(B(0, \rho_1))$, the slope of $d\phi_x^{-1}(y)(E^u(y))$ in $\mathbb{R}^N = \mathbb{R}^u \oplus \mathbb{R}^s$ is smaller than 1/2.
For the rest of the paper, we set $\Omega_0 \stackrel{def}{=} \Omega \setminus B(S, \varepsilon_1)$, $\Omega_1 = \Omega_0 \cap f(\Omega_0) \cap f^{-1}(\Omega_0)$ and $\Omega_2 = \Omega_0 \setminus \Omega_1$.

2.1 First step: the graph transform theory

The idea to prove Theorem A is to use the graph transform theory (see [13]). We first give some general result:

Proposition 2.2. *Let E be some Banach space and $T : E \to E$ be some linear map such that there exists a T-invariant splitting $E = E_1 \oplus E_2$. We set $T_i \stackrel{def}{=} T_{|E_i}$ and we assume that the norm on E is adapted to the splitting, i.e. $||.||_E = \max(||.||_{E_1}, ||.||_{E_2})$. We also assume that there exist two numbers $\lambda_2 < 0 < \lambda_1$ such that:*

for every v in E_1, $||T_1^{-1}v||_{E_1} \le e^{-\lambda_1}||v||_{E_1}$,
for every v in E_2, $||T_2 v||_{E_2} \le e^{\lambda_2}||v||_{E_2}$.
Let ρ be a positive number, ε a positive number very small compared to λ_1 and $-\lambda_2$ and F be some C^1-map from E to E such that:
(i) $F(0) = 0$,
(ii) on the ball $B(0,\rho)$, $Lip(F - T) < \varepsilon$ and $Lip(F^{-1} - T^{-1}) < \varepsilon$.
Then

1. *The image by F of the graph of any map $g : B_1(0,\rho) \to B_2(0,\rho)$ satisfying $g(0) = 0$ is a graph of some map $\Gamma(g) : B_1(0, \rho e^{\lambda_1 - 2\varepsilon}) \to B_2(0, \rho e^{-\lambda_2 + 2\varepsilon})$.*

2. *This induce some operator Γ on the set Lip_1 of 1-Lipschitz-continuous maps $g : B_1(0,\rho) \to B_2(0,\rho)$ satisfying $g(0) = 0$.*

3. *On Lip_1 (with the standard norm on the set of Lipschitz-continuous maps) Γ is a contraction. Thus it admits some unique fixed point.*

For the proof of Theorem A, it is important to keep in mind the 2 key points of the proof of proposition 2.2. In one hand, the fact that Γ is a contraction on Lip_1 is essentially due to the spectral gap of dF. This is obtained by the properties of the T_i's and Lipschitz proximity of F and T. On the other hand, the fact that the image by F of any graph (from $B_1(0,\rho)$ to $B_2(0,\rho)$) extends beyond the boundary of the ball $B(0,\rho)$ is essentially due to expansion on E_1.

2.2 Second step: estimations in our case

We want to use proposition 2.2 in the fibered case of $\widehat{f}_x$. If x is in $\Omega_0 \cap f^{-1}(\Omega_0)$ the spectral gap of $d\widehat{f}_x(0)$ is uniformly bounded from below in $B_x(0,\rho_1)$, and so we can apply the proposition with $d\widehat{f}_x(0)$, $\widehat{f}_x$ and $\rho = \rho_1$. In $B(S,\varepsilon_1)$, the value of $\bar{\rho}(x)$ has to decrease to 0 when x tends to S: the spectral gap of df tends to 0 as x tends to S because $k^u(x) + k^s(x)$ tends to 0. The idea is to apply proposition 2.2 for $d\widehat{f}_x^n(0)$ and $\widehat{f}_x^n$ for some good n. First, we have to check that hypothesis of the proposition hold.

Proposition 2.3. *Let $x \in \Omega_0$ and $n \ge 2$ such that :*
(1) $f^n(x) \in \Omega_0$,
(2) $\forall 0 < k < n$, $f^k(x) \in B(S,\varepsilon_1)$.
Then, there exists some constant $C > 0$ such that for every $0 < r \le 1$, proposition 2.2 holds for $F = \widehat{f}_x^n$, $T = dF(0)$ and $\rho = C.r.e^{-\frac{9n\lambda}{200}}$

Proof. Let us fix some r. We consider that the norm on $\mathbb{R}^N = \mathbb{R}^u \oplus \mathbb{R}^s$ is the adapted norm (with respect to the splitting); this norm is equivalent to the Euclidean norm.

We first control what happens in the bad area $B(S,\varepsilon_1)$ (for $f^k(x)$, with $1 \leq k \leq n-1$). For y in $B_{f(x)}(0,\rho_1)$ we have

$$\begin{aligned} |(d\widehat{f}^k_{f(x)}(0) - d\widehat{f}^k_{f(x)}(y)).v| &\leq \|d^2\widehat{f}\|.|\widehat{f}^{k-1}_{f(x)}(y)|.|d\widehat{f}^k_{f(x)}(y).v| + \\ &\quad \|d\widehat{f}_{f(x)}(0)\|.|(d\widehat{f}^{k-1}_{f(x)}(0) - d\widehat{f}^{k-1}_{f(x)}(y)).v| \end{aligned} \tag{2}$$

as soon as it makes sense to define $\widehat{f}^k_{f(x)}(y)$.
Let us assume that for every y in $B_{f(x)}(0,l)$ (with $l \leq \rho_1$) and for every $0 \leq k \leq p < n-1$ we have

(i) $|\widehat{f}^k_{f(x)}(y)| \leq l.e^{\frac{3}{2}k\frac{\lambda}{100}} \leq \rho_1$,

(ii) $\|d\widehat{f}^k_{f(x)}(y)\| \leq e^{\frac{3}{2}k\frac{\lambda}{100}}$,

and let us pick some y in $B_{f(x)}(0,l)$. Then we can define $\widehat{f}^{p+1}_{f(x)}(y)$ and (2) gives

$$\|(d\widehat{f}^{p+1}_{f(x)}(0) - d\widehat{f}^{p+1}_{f(x)}(y))\| \leq \|d^2\widehat{f}\|.e^{3p\frac{\lambda}{100}}l + e^{\frac{\lambda}{100}}\|(d\widehat{f}^{p}_{f(x)}(0) - d\widehat{f}^{p}_{f(x)}(y))\|. \tag{3}$$

As this formula also holds for every $k \leq p$, we obtain by induction

$$\|(d\widehat{f}^{p+1}_{f(x)}(0) - d\widehat{f}^{p+1}_{f(x)}(y))\| \leq \|d^2\widehat{f}\|.l.\frac{e^{3p\frac{\lambda}{100}}}{1-e^{-2\frac{\lambda}{100}}}.$$

We have now to compute how l must be, to be sure that conditions (i) and (ii) from above still hold for $p+1$.
We first see that

$$\|d\widehat{f}^{p+1}_{f(x)}(y)\| \leq \|d\widehat{f}^{p+1}_{f(x)}(y) - d\widehat{f}^{p+1}_{f(x)}(0)\| + \|d\widehat{f}^{p+1}_{f(x)}(0)\|,$$

and so, in view to satisfy (ii), it is sufficient to have

$$\|d^2\widehat{f}\|.l.\frac{e^{3p\frac{\lambda}{100}}}{1-e^{-2\frac{\lambda}{100}}} + e^{(p+1)\frac{\lambda}{100}} \leq e^{\frac{3}{2}(p+1)\frac{\lambda}{100}},$$

which is true for $l \leq (e^{\frac{\lambda}{100}}-1).e^{\frac{\lambda}{100}}.\frac{1-e^{-2\frac{\lambda}{100}}}{\|d^2\widehat{f}\|}e^{-2p\frac{\lambda}{100}}$.
Let set $C_1 \stackrel{def}{=} (e^{\frac{\lambda}{100}}-1).e^{\frac{\lambda}{100}}.\frac{1-e^{-2\frac{\lambda}{100}}}{\max(\|d^2\widehat{f}\|,\|d^2\widehat{f}^{-1}\|)}$, and pick some $l = \rho_1.l'.C_1.e^{-2p\frac{\lambda}{100}}$, with $l' \leq 1$. Then (ii) holds for every point in the convex ball $B_{f(x)}(0,l)$ with $k = p+1$. Hence we have for every y in this ball

$$\begin{aligned} |\widehat{f}^k_{f(x)}(y)| &= \left|\int_0^1 d\widehat{f}^k_{f(x)}(ty).y\,dt\right| \\ &\leq e^{\frac{3}{2}p\frac{\lambda}{100}}.l. \end{aligned}$$

and (i) holds for $k = p+1$.
Therefore, if $C \stackrel{def}{=} \rho_1.C_2.C_1.e^{-\lambda^u - 2\varepsilon}$ (with $C_2 \leq 1$), for every y in $B_x(0, C.l''.r.e^{-3n\frac{\lambda}{100}})$ (with $l'' \leq 1$), we can define $\widehat{f}_x^n(y)$, and (3) gives

$$\|d\widehat{f}_x^n(0) - d\widehat{f}_x^n(y)\| \leq C_2.l''.r.\rho_1.$$

This formula yields

$$Lip(\widehat{f}_x^n - d\widehat{f}_x^n(0)) \leq C_2.l''.r.\rho_1. \qquad (4)$$

Moreover, the previous discussion proves that $\widehat{f}_x^n(y)$ belongs to $B_{f^n(x)}(0, C_2.l''.r.\rho_1.e^{-\frac{3}{2}n\frac{\lambda}{100}})$. On the other hand, we have $|d\widehat{f}_x^n(0).v| \geq e^{\lambda^u}.|v|$ for every v in $\mathbb{R}^u$ and $|d\widehat{f}_x^n(0).v| \leq e^{-\lambda^s}.|v|$ for every v in $\mathbb{R}^s$. For the same reasons we have $|d\widehat{f}_{f^n(x)}^{-n}(0).v| \geq e^{\lambda^s}.|v|$ for every v in $\mathbb{R}^s$ and $|d\widehat{f}_{f^n(x)}^{-n}(0).v| \leq e^{-\lambda^u}.|v|$ for every v in $\mathbb{R}^u$. Hence, the previous discussion gives some formula which is equivalent to (4) but for $\widehat{f}_{f^n(x)}^{-n}$ as soon as $l'' \leq e^{-\frac{3}{2}n\frac{\lambda}{100}}$:

$$Lip(\widehat{f}_{f^n(x)}^{-n} - d\widehat{f}_{f^n(x)}^{-n}(0)) \leq C_2.r.\rho_1. \qquad (5)$$

Therefore (4) and (5) prove that for $C_2 \leq \varepsilon$we can apply proposition 2.2 with $F = \widehat{f}_x^n$, $T = d\widehat{f}_x^n(0)$ and $\rho = C.r.e^{-9n\frac{\lambda}{200}}$, with $C = C_3C_2.C_1$, where C_3 is a constant that is introduced because the norm on $\mathbb{R}^N$ is not the adapted norm. □

We remark that the proposition 2.2 also holds with $F = \widehat{f}_{f^n(x)}^{-n}$, $T = d\widehat{f}_{f^n(x)}^{-n}(0)$ and $\rho = C.r.e^{-\frac{9}{2}n\frac{\lambda}{100}}$.

Definition 2.4. *Let x in Ω_0.*
If $f(x)$ is in $B(S,\varepsilon_1)$ we call forward length of stay of x the integer (in $\overline{\mathbb{N}}$)

$$n^+(x) \stackrel{def}{=} \sup\{n \ / \ \forall 0 < k < n, \ f^k(x) \in B(S,\varepsilon_1)\}.$$

If $f^{-1}(x)$ is in $B(S,\varepsilon_1)$ we call backward length of stay of x the value

$$n^-(x) \stackrel{def}{=} \sup\{n \ / \ \forall 0 < k < n, \ f^{-k}(x) \in B(S,\varepsilon_1)\}.$$

Let x in Ω_0.

- If $f(x)$ is in $B(S,\varepsilon_1)$ and $n := n^+(x)(< +\infty)$, then we set $l^f(x) \stackrel{def}{=} C.e^{-\frac{9n\lambda}{200}}$,
- If $f^{-1}(x)$ is in $B(S,\varepsilon_1)$ and $n := n^-(x)(< +\infty)$, then we set $l^b(x) \stackrel{def}{=} C.e^{-\frac{9n\lambda}{200}}$.

For the rest of the proof, if x is in $\Omega_0 \cap f^{-1}(\Omega_0)$ we will call the *one-step graph transform* the graph transform due to proposition 2.2 with $F = \widehat{f}_x$, $T = d\widehat{f}_x(0)$ and $\rho = \rho_1$. If x belongs to $\Omega_0 \cap f^{-1}(B(S,\varepsilon_1))$, the graph transform that results from proposition 2.3 will be refer as the *$n^+(x)$-steps graph transform.* Both will be denoted by Γ_x.

2.3 Third step: end of the proof

Let x in Ω_0 be some λ-hyperbolic point. We are going to construct a piece of unstable leaf as some set $\phi_x(graph(g_x))$, where g_x would be some special map from $B^u_x(0, l(x))$ to $B^s_x(0, l(x))$ satisfying $g_x(0) = 0$ (for some $l(x) \leq \rho_1$). For that purpose we will use the graph transform along the backward orbit of x.
For any ξ in Ω_0, we set $\widehat{0}_\xi$ then null-map from $B^u_\xi(0, \rho_1)$ to $B^s_\xi(0, \rho_1)$. We also set $\xi_n \stackrel{def}{=} f^{-n}(\xi)$.

Firstly we check that every λ-hyperbolic point must return infinitely many times in Ω_0 (in the future and in the past):

Lemma 2.5. *Let ξ in Ω_0 be some λ-hyperbolic point such that $f(\xi) \notin \Omega_0$ (resp. $f^{-1}(\xi) \notin \Omega_0$). Then $n^+(\xi) < +\infty$ (resp. $n^-(\xi) < +\infty$).*

Proof. Let us assume that $f(\xi) \notin \Omega_0$ and $n^+(\xi) = +\infty$, *i.e.* $\forall n \geq 1,\ f^n(\xi) \in B(S, \varepsilon_1)$. This yields

$$\forall n \geq 1,\ \forall v \in E^s(f^n(\xi)),\ ||df(f^n(\xi)).v|| \geq e^{-\frac{n\lambda}{100}}||v||. \tag{6}$$

Hence we get

$$\liminf_{n \to +\infty} \frac{1}{n} \log ||df^n_{|E^s}(\xi)|| \geq -\frac{\lambda}{100},$$

which contradicts the fact that ξ is λ-hyperbolic.
The proof is of the same kind for the other case. □

Therefore, there exist two sequences $0 \leq q_0 < p_0 \leq q_1 < p_1 \leq \ldots$ of integers such that

- For $0 \leq k \leq q_0$, $x_k \in \Omega_0$,
- For every $i \geq 0$, for $q_i < k < p_i$, $x_k \in B(S, \varepsilon_1)$,
- For every $i > 1$, for every $p_{i-1} \leq k \leq q_i$, $x_k \in \Omega_0$.

We set $y_i \stackrel{def}{=} x_{q_i} = f^{-q_i}(x)$, $z_i \stackrel{def}{=} x_{p_i} = f^{-p_i}(x)$ and $m_i := p_i - q_i$.
Let n be any integer such that x_n belongs to Ω_0. Then, we define Γ^n_x as the composition of the graph transforms along the piece of orbit $x_n, x_{n-1}, \ldots, x_1$, where we take the one-step graph transform Γ_{x_k} if x_k and x_{k-1} are in Ω_0 and the m_i-steps graph transform if x_k is one of the z_i's.
The goal is to prove that the sequence of maps $\Gamma^n_x(\widehat{0}_{x_n})$ converges to some map. This is well-known for uniformly hyperbolic dynamical systems, but here, the critical set S influences the graph transform: by construction, Γ^n_x is a contraction (of ratio smaller than $e^{-(n-\sum_{i,p_i \leq n}(m_i-1))(\lambda^u+\lambda^s-4\varepsilon)}$), but it is not clear that the length of the graph associated to $\Gamma^n_x(\widehat{0}_{x_n})$ is uniformly (in n) bounded away from 0. Now, we explain how S influences the graph transform, and prove that the length of the graphs associated to the $\Gamma^n_x(\widehat{0}_{x_n})$'s are uniformly bounded away from 0.

Let n be any integer such that x_n belongs to Ω_0. There exists some integer k such that $p_k \leq n \leq q_{k+1}$.
For $p_k + 1 \leq i \leq n$ we just apply the one-step graph transform, and so we get in $B_{z_k}(0, \rho_1)$ some graph. At this moment we apply the m_k-steps graph transform to the graph restricted to the ball $B_{z_k}(0, l^f(z_k))$, and we get some graph in $B_{y_k}(0, l^p(y_k))$. We call this phenomenon a $\ll$ truncation$\gg$. For $p_{k-1} \leq i \leq q_k$, we can again apply the one-step graph transform, but to the small piece of graph (with length $2.l^p(y_k)$).

However, the length increases along this piece of orbit (as long as it is smaller than $2.\rho_1$) because, at each (one-)step, we can take the whole part of the image-graph. For $i := p_{k-1}$ three cases may occur:

(i) The length of the graph is $2.\rho_1$ and so, there is a new truncation. Thus there is a "past-stabilization" (a notion due to S. Newhouse).

(ii) The length of the graph is strictly smaller than $2.\rho_1$, but is bigger than $2.l^f(z_{k-1})$. Again, there is a new truncation and again there is a "past-stabilization".

(iii) The length of the graph, $2.l$, is strictly smaller than $2.l^f(z_{k-1})$. We apply the m_{k-1}-steps graph transform with $\rho = l$, and we obtain some graph in $B_{y_{k-1}}(0, l)$. From x,we can see the truncation due to Γ_{z_k},

and so on, along the piece of orbit $y_{k-1}, \ldots, x_{-1}$.
Hence, the length of the graph $\Gamma_x^n(\widehat{0}_{x_n})$ is (at least) equal to $2.l^f(z_j)$, where j is such that p_j is the smallest integer where a truncation occurs. We set $i(n) := j$ and we want now to give a more precise estimate for this length.

Lemma 2.6. *For every λ-hyperbolic point ξ and for every v in $E^u(\xi)$,*

$$\lim_{j \to +\infty} e^{\frac{9m_j\lambda}{200}} ||df^{-q_j}(\xi).v|| = 0.$$

Proof. There exists $n = n(\varepsilon_1, \xi)$ such that $\forall p \geq n$, $||df^{-p}_{|E^u}(\xi)|| < e^{-p\frac{99\lambda}{100}}$.
Pick j such that $q_j \geq n$. Then,

$$||df^{-(q_j+m_j)}_{|E^u}(\xi)|| < e^{-(q_j+m_j)\frac{99\lambda}{100}}. \tag{7}$$

Moreover, by definition of ε_1 we have

$$\forall v \in E^u(f^{-(m_j+q_j)}(\xi)) \quad ||df^{m_j}(f^{-q_j-m_j}(\xi)).v|| \leq e^{\frac{m_j\lambda}{100}}||v||. \tag{8}$$

Hence inequalities (7) and (8) give for every v in $E^u(\xi)$

$$||df^{-q_j}(\xi).v|| \leq e^{-m_j\frac{98\lambda}{100}} e^{-q_j\frac{99\lambda}{100}} \|v\|.$$

□

With the previous notations, the length l_n of the graph associated to $\Gamma_x^n(\widehat{0}_{x_n})$ is smaller than $2\rho_1$ but at least

$$2.l^f(z_{i(n)})\frac{1}{\|df_{|E^u}^{-q_{i(n)}}(x)\|}$$

(if this quantity is smaller than $2.\rho_1$). Lemma 2.6 proves that there exists some integer p such that if $i(n)\geq p$, then $l^f(z_{i(n)})\dfrac{1}{\|df_{|E^u}^{-q_{i(n)}}(x)\|}>\rho_1$. Therefore, x sees only the truncations due to the p_j's with $j\leq p$, which proves that the sequences of lengths (l_n) is bounded away from 0. The family of maps $\Gamma_x^n(\widehat{0}_{x_n})$ converges to some map $g_x: B_x^u(0,l(x))\to B_x^s(0,l(x))$, with $g_x(0)=0$, and where $l(x)$ is such that for every n, $0<l(x)\leq l_n$.

For the rest of the paper we set $\mathcal{F}_{loc}^u(x)\stackrel{def}{=}\phi_x(graph(g_x))$.
Lemma 2.5 also proves that the backward orbit of x returns infinitely many often in Ω_0. Therefore, if $f^{-k}(x)$ belongs to $B(S,\varepsilon_1)$ we set

$$\mathcal{F}_{loc}^u(f^{-k}(x))\stackrel{def}{=}f^n(\mathcal{F}_{loc}^u(f^{-(n+k)}(x))),$$

where n is the smallest positive integer such that $f^{-(n+k)}(x)$ belongs to Ω_0. Then, we set

$$\mathcal{F}^u(x)=\bigcup_{n,\ f^{-n}(x)\in\Omega_0}f^n(\mathcal{F}_{loc}^u(f^{-n}(x))).$$

The uniqueness of the map g_x and its construction prove that $\mathcal{F}^u(x)$ is an immersed manifold.
To complete the proof of Theorem A, we must check that the $\mathcal{C}^1$-disks that have been constructed are tangent to the correct spaces. Let y be in $\mathcal{F}_{loc}^u(x)$. By construction of $\mathcal{F}_{loc}^u(x)$ we have for every n such that $f^{-n}(x)\in\Omega_0$, $f^{-n}(y)\in\mathcal{F}_{loc}^u(f^{-n}(x))$. For such an integer n, we pick some map $g_{n,y}:B^u(0,\rho_1)\to B^s(0,\rho_1)$ such that $f^{-n}(y)\in\phi_{f^{-n}(x)}(graph(g_{n,y}))$ and $T_{f^{-n}(y)}\phi_{f^{-n}(x)}(graph(g_{n,y}))=E^u(f^{-n}(y))$. As the map g_x is obtained as some unique fixed-point for the graph transform, the sequence $\Gamma_x^n(g_{n,y})$ converges to g_x. By df-invariance of E^u (until the orbit of y leaves U) we must have $T_y\mathcal{F}^u(x)=E^u(y)$.
We also can do the same construction with f^{-1} to obtain some immersed manifolds $\mathcal{F}^s(x)$. Then, $x\in\mathcal{F}^u(x)\cap\mathcal{F}^s(x)$.

However, it can be important to have an estimate for the length of $\mathcal{F}_{loc}^u(x)$ or $\mathcal{F}^u(x)$. Actually we are not able to give a lower bound for such estimates.

3 Shadowing lemma and Markov rectangles

Let Λ be some fixed (ε_0,λ)-regular set satisfying hypothesis of Theorem B. The goal of this section is to construct some Markov rectangle to use Young's method (see

for instance [9]). The principal steps of this section are the following :
In the first subsection we define the pseudo-orbits and we state (and prove) a shadowing lemma.
In the second subsection we prove that the dynamical system (Λ, f) is conjugate to some symbolic dynamical system (Σ_0, σ).
In the third subsection we define two generations of rectangles.
In the fourth subsection we fix the constants and construct a third generation of rectangles.
In the last subsection we define the special reduced system (R, g), where R is some rectangle satisfying the Markov property and g the return map in R.

We first assume that ε_1 has been chosen so small that for every x in Ω_2, $1 \leq ||df(x)_{|E^u(x)}|| < e^{2\frac{\lambda}{3}}$ and $1 \leq ||df^{-1}(x)_{|E^s(x)}|| < e^{2\frac{\lambda}{3}}$.

3.1 Shadowing lemma

For convenience, we briefly recall how Bowen constructed a Markov partition in [5] (for Axiom-A).
A pseudo-orbit is a sequence of points (x_n) such that for every n, $f(x_n)$ is "close enough" to x_{n+1}. Therefore, every sequence $(y_{n+1,k})_k$ of points on the local piece of unstable manifold $W^u_\varepsilon(x_{n+1})$ gives a new sequence $(z_{n+1,k})_k$ on the local piece of unstable manifold $W^u_\varepsilon(f(x_n))$ by sliding along the stable leaves. By contraction, this gives a sequence $(y_{n,k})_k$ of points on the local piece of unstable manifold $W^u_\varepsilon(x_n)$.
One of the key points in this proof is to control the distortion due to this stable holonomy: with the notations from above, it is untrue that the "unstable distance" along the unstable leaves, $d^u(f(x_n), z_{n+1,k})$ is exactly $d^u(f(x_n), x_{n+1}) + d^u(x_{n+1}, y_{n+1,k})$. However we have ,

$$d^u(f(x_n), z_{n+1,k}) \leq \kappa d(x_n, x_{n+1}) + R d^u(x_{n+1}, y_{n+1,k}), \tag{9}$$

where κ and R are some distortion constants. Therefore, to be able to use the contraction (in negative time) along the unstable leaves, Bowen considers pseudo-orbits for some f^k, where k is such that $Re^{-k\lambda^u} < 1$ (see [5] p. 74-75).

3.1.1 Control on the distortion in our case

Here we cannot control the distortion of the system of local coordinates by the dynamic because we do not have any control on how long a point stays in Ω_0. Hence we must compute this distortion and control it at each step.
Moreover we want to use the family of charts $(\phi_x, B_x(0, \rho_1))$, which introduce another distortion (that appears in κ and R in (9)). This distortion is not really correlated to the dynamic, firstly because it occurs uniformly and secondly because its effects do not add up along the orbits. Hence, for convenience, **we will assume that these charts introduce no distortion**, namely $K_1 = K_2 = 1$.

For x in Λ we set $D^i_\rho(x)$ $(i=u,s)$ the disk in $\mathcal{F}^i(x)$ of radius ρ (with $\rho \le \varepsilon_0$). We also set $D^i_\rho(y)$ for every y in $\mathcal{F}^i(x)$ when it makes sense. Let $0<\alpha<\frac{1}{8}$. For x in Ω_0 we define the unstable and the stable cones of angle α as

$$\begin{aligned} C^u_\alpha(x) &= \{v=v^u+v^s \in \mathbb{R}^u \oplus \mathbb{R}^s,\ \frac{|v^u|}{|v^s|} \ge \frac{1}{\alpha}\}, \\ C^s_\alpha(x) &= \{v=v^u+v^s \in \mathbb{R}^u \oplus \mathbb{R}^s,\ \frac{|v^u|}{|v^s|} \le \alpha\}. \end{aligned}$$

By continuity of E^u and E^s, if ρ is small enough, we have for every $x \in \Omega_0 \cap \Lambda$ and for every $y \in B(x,10\rho)$

$$\begin{cases} d\phi_x^{-1}(y).E^s(y) \subset C^s_\alpha(x), \\ d\phi_x^{-1}(y).E^u(y) \subset C^u_\alpha(x). \end{cases}$$

Therefore, for sufficiently small ρ (namely $\rho < \frac{\varepsilon_0}{100}$ and $\rho < \frac{\rho_1}{100}$), for every x in Ω_0 and for every y and z in $B(x,\rho)$ such that $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ exist, we can define

$$[z,y] \stackrel{def}{=} D^s_{2\rho}(z) \cap D^u_{2\rho}(y),$$

and we get a system of local coordinates. However, this system introduces some distortion, because the holonomies (stable and unstable) are not isometries. Thus we have formulas like (9): for x in Λ, the following holds:

$$\begin{aligned} d^u(x,[z,x]) &\le d(x,y) + r_1 d^u(y,[z,y]), &(10)\\ d^s(x,[x,y]) &\le d(x,z) + r_2 d^s(z,[z,y]), &(11)\\ d^u(x,[z,x]) &\le r_3 d(x,z), &(12)\\ d^s(x,[x,y]) &\le r_4 d(x,y). &(13) \end{aligned}$$

By definition of ρ the disks $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ must stay in some cones, and a standard computation in $\mathbb{R}^N$ proves that the distortion's constants r_i $(i=1,2,3,4)$ depends continuously on $d(x,y)$ and $d(x,z)$. Obviously we have $r_1=1$ if $x=y$ and $r_2=1$ if $x=z$. Moreover, the continuity of E^u and E^s proves that r_3 and r_4 tend to 1 when y and z go to x. On the other hand, if ρ is small enough (namely $\rho<\varepsilon_1$), there exist constants $\gamma^u>0$ and $\gamma^s>0$ (which depend on ρ) such that for every $x \in \Omega_0 \cap \Lambda$ and for every $y \in B(x,\rho)$

$$\forall v \in E^s(y), \quad \begin{cases} |d\widehat{f}_x(y).v| \le e^{-\gamma^s}|v|, \\ |d\widehat{f}_x^{-1}(y).v| \ge e^{\gamma^s}|v|, \end{cases}$$
$$\forall v \in E^u(y), \quad \begin{cases} |d\widehat{f}_x(y).v| \ge e^{\gamma^u}|v|, \\ |d\widehat{f}_x^{-1}(y).v| \le e^{-\gamma^u}|v|. \end{cases}$$

We also have $\gamma^i \to \lambda^i$ when ρ goes to 0. Hence, we assume that ρ is sufficiently small such that for every $x \in \Omega_0 \cap \Lambda$ and for every y and z in $B(x,\rho)$ such that $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ exist, the distortion's constants in (10), (11), (12) and (13) satisfy for every $i=1,2,3,4$:

$$r_i e^{-\frac{1}{2}\gamma^u} < 1 \text{ and } r_i e^{-\frac{1}{2}\gamma^s} < 1. \qquad (14)$$

3.1.2 Definition of a pseudo-orbit

We can now define more precisely what a pseudo-orbit is. Let us pick some $\rho_2 \leq \rho$ sufficiently small. We set

$$\begin{aligned}\Lambda_2^+(n) &= \{x \in \Omega_2 \cap \Lambda,\ f(x) \notin \Omega_0 \text{ and } n^+(x) = n\},\\ \Lambda_2^-(n) &= \{x \in \Omega_2 \cap \Lambda,\ f^{-1}(x) \notin \Omega_0 \text{ and } n^-(x) = n\}.\end{aligned}$$

By definitions of $n^+(x)$ (resp. $n^-(x)$) we have $\Lambda_2^i(m) \cap \Lambda_2^i(n) = \emptyset$ if $n \neq m$ with $i = +, -$.

Definition 3.1. *Let $\delta > 0$ be such that*

1. $\delta < \frac{\rho_2}{16}(1 - e^{-\frac{1}{2}\gamma^u})$,
2. $\delta < \frac{\rho_2}{16}(1 - e^{-\frac{1}{2}\gamma^s})$,
3. $\delta < \frac{\varepsilon_0}{4}(1 - e^{-\frac{\gamma^u}{2}})e^{-\frac{1}{2}\gamma^u}$,
4. $\delta < \frac{\varepsilon_0}{4}(1 - e^{-\frac{\gamma^s}{2}})e^{-\frac{1}{2}\gamma^s}$,
5. $\delta < \frac{\varepsilon_0}{4}\frac{1}{e^{\frac{1}{2}\gamma^s} + \frac{1}{1-e^{-\frac{\gamma^u}{2}}}}$,
6. $\delta < \frac{\varepsilon_0}{4}\frac{1}{e^{\frac{1}{2}\gamma^u} + \frac{1}{1-e^{-\frac{\gamma^s}{2}}}}$.

A sequence $(x_n)_{n\in\mathbb{Z}}$ of points in Λ is called a δ-pseudo-orbit if

(1) $\forall n \in \mathbb{Z}$, $d(f(x_n), x_{n+1}) < \delta$ and $d(x_n, f^{-1}(x_{n+1})) < \delta$.

(2) $\forall n \in \mathbb{Z}$, if $x_n \in \Lambda_2^+(m)$ then $x_{n+m} \in \Lambda_2^-(m)$ and

either $\forall 0 < k < m$, $x_{n+k} = f^k(x_n)$,
or $\forall 0 < k < m$, $x_{n+k} = f^{-m+k}(x_{n+m})$.

In both cases we have $d(x_n, f^{-k}(x_{n+k})) < \delta$ and $d(x_{n+k}, f^k(x_n)) < \delta$.

We say that the point x β-tracks the δ-pseudo-orbit (x_n) if and only if $\forall n \in \mathbb{Z}$, $d(f^n(x), x_n) < \beta$.

Definition 3.2. *A point x in Ω is said to be weakly hyperbolic if:*
(i) It is a point of integration of the hyperbolic splitting.
(ii) $\limsup_{n\to+\infty} \frac{1}{n}\log||df^n_{|E^s}(x)|| \leq 0$ and $\limsup_{n\to+\infty} \frac{1}{n}\log||df^{-n}_{|E^u}(x)|| \leq 0$.

Proposition 3.3 (shadowing lemma). *Let $\delta > 0$ be such as in definition 3.1 and (x_n) be some δ-pseudo-orbit. Let*

$$\beta(\delta) = \max\left(\frac{2\delta}{(1-e^{-\frac{1}{2}\gamma^u})}, \frac{2\delta}{(1-e^{-\frac{1}{2}\gamma^s})}\right).$$

Then, there exists a unique point $x \in U$ such that x $\beta(\delta)$-tracks the δ-pseudo-orbit (x_n).
Moreover, x is weakly hyperbolic.

Proof. The positive part $(x_n)_{n\geq 0}$ and the negative part $(x_n)_{n\leq 0}$ play symmetric roles. Therefore, we are just going to study the positive part. The goal is to "repatriate" by f^{-n}, for each n, a piece of $D^s_{\varepsilon_0}(x_n)$ close to x_0. We have to control that such a piece of stable leaf is long enough to be sure that it intersects $D^u_{\varepsilon_0}(x_0)$ in one point. This will allow us to define close to x_0 a family of pieces of stable leaves which will accumulate themselves on some limit piece of stable leaf. Doing the same in the negative time, we will also obtain a piece of unstable leaf close to x_0. The two pieces of leaves must intersect themselves in exactly one point, which must track the pseudo-orbit.
To be sure that this point tracks (x_n) the repatriation has to be done by induction.

Let us assume that x_0 is in Ω_0 and pick $n > 0$ such that $x_n \in \Omega_0$.

step 1 We first study how we can repatriate one piece of stable leaf close to the "previous" point.
Case one: $x_{n-1} \in \Omega_0$. Because δ is sufficiently small (conditions with ε_0) and the x_i's are (ε_0, λ)-regular we can define $[x_n, f(x_{n-1})]$ and (12) gives

$$d^u(f(x_{n-1}), [x_n, f(x_{n-1})]) \leq r_3\delta.$$

Moreover, (13) and (14) give $d^s(x_n, [x_n, f(x_{n-1})]) \leq r_4\delta \leq e^{\frac{1}{2}\gamma^s}\delta$.
Thus we can define $D^s_{2\beta(\delta)}([x_n, f(x_{n-1})])$, because $[x_n, f(x_{n-1})]$ is far away from the boundary of $\mathcal{F}^s(x_n)$. Then, by expansion along the stable leaves in negative times, we can define $D^s_{2\beta(\delta)}(f^{-1}([x_n, f(x_{n-1})]))$ and by contraction along the unstable leaves in negative times (14) yields

$$d(x_{n-1}, f^{-1}([x_n, f(x_{n-1})])) \leq e^{-\frac{1}{2}\gamma^u}\delta.$$

Case two: $x_{n-1} \notin \Omega_0$. Let m be such that $x_m \in \Omega_0$ and $\forall k$, $0 < k < n-m$, $x_{m+k} \notin \Omega_0$. By definition of a δ-pseudo-orbit there are two sub-cases

either $\forall 0 < k < n-m$ $x_{m+k} = f^k(x_m)$ (*case (2.1)*),

or $\forall 0 < k < n-m$ $x_{n+k} = f^{-n+k}(x_n)$ (*case (2.2)*).

In both cases we can define $[x_n, f^{n-m}(x_m)]$ and as in the first case we can define $D^s_{2\beta(\delta)}([x_n, f^{n-m}(x_m)])$. We set $\xi_{m,n} \stackrel{def}{=} f^{-(n-m)}\left([x_n, f^{n-m}(x_m)]\right)$. The unstable foliation is contracted and the stable foliation is expanded in negative time and both points x_m and x_n are in Ω_0 (where there is strong contractions). Thus :
(i) $d(x_m, \xi_{m,n}) \leq r_3 e^{-2\gamma^u}\delta \leq e^{-\frac{3}{2}\gamma^u}\delta$,
(ii) $f^k(\xi_{m,n}) \in B(x_{m+k}, \frac{1}{2}\beta(\delta))$ for every $0 < k < n-m$,
(iii) we can define $D^s_{2\beta(\delta)}(\xi_{m,n})$.

step 2 We now study how we can induce the construction.
On $D^u_{\varepsilon_0}(x_n) \cap B(x_n, \frac{1}{2}\beta(\delta))$ we have a family of points $(\xi_{n,j})_j$. For each j we can define $D^s_{2\beta(\delta)}(\xi_{n,j})$.
In the case one, we set $\xi_{n-1,j} \stackrel{def}{=} f^{-1}([\xi_{n,j} f(x_{n-1})])$. Then (10) and (14) give

$$d^u(x_{n-1}, \xi_{n-1,j}) \leq e^{-\gamma^u}(\delta + r\frac{1}{2}\beta(\delta)) \leq \delta + e^{-\frac{1}{2}\gamma^u}\frac{1}{2}\beta(\delta) \leq \frac{1}{2}\beta(\delta).$$

Moreover $d^s(\xi_{n,j}, [f(\xi_{n-1,j})]) \leq e^{\frac{1}{2}\gamma^s}\delta$. Therefore we can define $D^s_{2\beta(\delta)-e^{\frac{1}{2}\gamma^s}\delta}(f(\xi_{n-1,j}))$ and by expansion along the stable leaves in the negative times we can define $D^s_{2\beta(\delta)}(\xi_{n-1,j})$ (there is “strong” expansion close to Ω_0).
In the cases *(2.1)* and *(2.2)*, we set $\xi_{m,j} \stackrel{def}{=} f^{-(n-m)}\left([\xi_{n,j}, f^{n-m}(x_m)]\right)$ and for $0 < k < n-m$, $\xi_{m+k,j} = f^k(\xi_{m,j})$.
In the both cases, we obtain

$$d^u(x_m, \xi_{m,j}) \leq e^{-2\gamma^u}(\delta + r\frac{1}{2}\beta(\delta)).$$

Thus we get

$$d^u(x_m, \xi_{m,j}) \leq \frac{1}{2}\beta(\delta).$$

Just like in the first case, we can define $D^s_{2\beta(\delta)}(\xi_{n-1,j})$ because of the expansion in the stable leaves in negative times (that occurs at each step and at least two times “strongly” because both points x_n and x_m are in Ω_0). We also set $\xi_{m,j} = \xi_{m,n}$ for $m+1 < j < n$.

step 3 We can iterate this construction. This gives by induction a sequence of points $(\xi_{0,j})_{j\geq 1}$ in $D^u_{\varepsilon_0}(x_0) \cap B(x_0, \frac{1}{2}\beta(\delta))$, such that for each j we can define $D^s_{2\beta(\delta)}(\xi_{0,j})$. Such a s-disk can be viewed in $B_{x0(0,\rho_2)}$ as the graph of some map $g_{0,j} : B^s(0, \frac{1}{2}\beta(\delta)) \to B^u(0, \beta(\delta))$. The family of maps $(g_{0,j})_j$ is equicontinuous in the $\mathcal{C}^1$-topology because E^s is Hölder continuous. Therefore, by Ascoli's theorem, it converges (up to some sub-sequence) to some map $g^s : B^s(0, \frac{1}{2}\beta(\delta)) \to B^u(0, \beta(\delta))$. This proves that $\phi_{x_0}(graph(g^s))$ is a $\mathcal{C}^1$-disk of size $\frac{1}{2}\beta(\delta)$. This disk must contain a point in $B(x_0, \frac{1}{2}\beta(\delta))$ (obtained as an accumulation point of the sequence $(\xi_{0,j})$). Moreover for every point y in $\phi_{x_0}(graph(g^s))$ we have

$$T_y\phi_{x_0}(graph(g^s)) = E^s(y).$$

Doing the same with the negative part of the pseudo-orbit, we get some $\mathcal{C}^1$-u-disk of size $\frac{1}{2}\beta(\delta)$, which contains a point in $B(x_0, \frac{1}{2}\beta(\delta))$. The two disks must intersect themselves in exactly one point x. By construction of x we have for every integer n,

$$f^n(x) \in B(x_n, \beta(\delta)).$$

Moreover x is a point of integration of the hyperbolic splitting and at least weakly hyperbolic because it returns infinitely many times (in the past and in the future) in $B(\Omega_0, \rho_2)$ (where there are uniform expansions and contractions). □

3.2 Symbolic dynamic

We have now to prove the converse of proposition 3.3 : let δ be fixed as above. We want some countable set, $\Gamma \subset \Lambda$, such that every point in Λ is $\beta(\delta)$-tracked by some sequence (that should be some δ-pseudo-orbit) in $\Gamma^{\mathbb{Z}}$.

Let κ be

$$\max(\sup_{x\in M} ||df(x)||, \sup_{x\in M} ||df^{-1}(x)||).$$

We build a cover of Λ in the following way:
1– We take a finite cover of $\Lambda \cap \Omega_0$ by balls $B(p_i, \frac{\delta}{3\kappa})$, where p_i is in $\Lambda \cap \Omega_0$ and i describes $\{0, \dots, P_0\} \subset \mathbb{N}$. This can be done because $\Lambda \cap \Omega_1$ is totally bounded.
2– For each $n > 0$, we take a finite cover of $\Lambda_2^+(n)$ by balls $B_n(q_{i,n}, \frac{\delta}{3\kappa})$, where $q_{i,n}$ is in $\Lambda_2^+(n)$, $B_n(y, \epsilon)$ denotes $\bigcap_{0\leq j<n} f^{-j}(B(f^j(y), \epsilon))$ and i describes $\{0, \dots, P_n\} \subset \mathbb{N}$.
We can find such a cover because each dynamical ball $B_n(y, \frac{\delta}{3\kappa})$ contains the ball $B(y, \frac{\delta}{3\kappa^{n+1}})$.
3– For each $n > 0$, we take a finite cover of $\Lambda_2^-(n)$ by balls $B_{-n}(r_{i,n}, \frac{\delta}{3\kappa})$, where $r_{i,n}$ is in $\Lambda_2^+(n)$, $B_{-n}(y, \epsilon)$ denotes $\bigcap_{0\leq j<n} f^{j}(B(f^{-j}(y), \epsilon))$ and i describes $\{0, \dots, P_{-n}\} \subset \mathbb{N}$.

For $m \leq 0$, let us set $\mathbb{N}_m \stackrel{def}{=} \{0, \dots, m\}$; then we define

$$\Gamma \stackrel{def}{=} \{p_i, i \in \mathbb{N}_{P_0}\} \bigcup_{n>0, i\in\mathbb{N}_{P_n}} \{q_{i,n}, f(q_{i,n}), \dots, f^{n-1}(q_{i,n})\}$$
$$\bigcup_{n>0, i\in\mathbb{N}_{P_{-n}}} \{r_{i,n}, f^{-1}(r_{i,n}), \dots, f^{-n+1}(r_{i,n})\}.$$

The set Γ is a countable set. For convenience we will set $\Gamma = \{\xi_k, k \in \mathbb{N}\}$.

Proposition 3.4. *For every x in Λ, there exists some sequence $(x_n)_{n\in\mathbb{Z}}$ in $\Gamma^{\mathbb{Z}}$ such that:*
(i) $(x_n)_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit;
(ii) x $\beta(\delta)$-tracks $(x_n)_{n\in\mathbb{Z}}$.

Proof. Pick x in Λ. We always may assume that x is in Ω_0 (every point in Λ comes in Ω_0 along its orbit). The main idea of the proof is to construct the sequence by a recursive way. It will be constructed by "blocks".

step one. We first define the family of times, $(\tau_n)_{n\in\mathbb{Z}}$, such that the τ_n^{th}-iterate of x are in Ω_0:
$\tau_0 = 0$, and if we have built $\tau_{-n}, \dots, \tau_0, \dots, \tau_n$, then we define $\tau_{n+1} = \tau_n + \min\{k,\ 0 < k, f^{\tau_n+k}(x) \in \Omega_0\}$ and $\tau_{-(n+1)} = \tau_{-n} - \min\{k,\ 0 < k, f^{\tau_{-n}-k}(x) \in \Omega_0\}$.
The τ_n's are called the stopping times.

step two. We construct the first block of point(s) x_k. Each block will have a length $\tau_k - \tau_{k-1}$. There are several cases.

(c1) If x is in Ω_1, and there is some n_0 such that $x \in B(\xi_{n_0}, \frac{\delta}{3\kappa})$ (ξ_{n_0} is one of the p_i's). Then we set $x_0 = \xi_{n_0}$. The first block is $\{x_0\}$.

(c2) If x is in $\Lambda_2^+(m)$, then $\tau_1 = m$. Hence, there exists some n_0 such that $x \in B_m(\xi_{n_0}, \frac{\delta}{3\kappa})$ (ξ_{n_0} is one of the $q_{i,m}$'s). We set $x_k = f^k(\xi_{n_0})$, for every $0 \le k < m$. The first block is $\{x_0, \dots, x_{m-1}\}$.

(c3) If x is in Ω_2 but does not belong to any $\Lambda_2^+(j)$, then $\tau_{-1} = -m$. Hence, there exists some n_0 such that $x \in B_{-m}(\xi_{n_0}, \frac{\delta}{3\kappa})$ (ξ_{n_0} is one of the $r_{i,m}$'s). We set $x_k = f^k(\xi_{n_0})$, for every $-m < k \le 0$. The first block is $\{x_{-m+1}, \dots, x_0\}$.

step three. We construct new blocks by a recursive way. Let us assume that we have constructed $(x_i, \dots, x_j)$ with $i \le j$. We study how we can construct a block $(x_{j+1}, \dots, ?)$ (called an upper block) and a block $(?, \dots, x_{i-1})$ (called a lower block).

Upper block.

- If j is not a stopping time, then $j+1$ must be some stopping time τ_k and $f^{j+1}(x) \in \Lambda_2^-(\tau_k - \tau_{k-1})$.
  - * If $\tau_{k+1} = \tau_k + 1 = j + 2$, then we just construct x_{j+1}; $f^{j+1}(x)$ belongs to Ω_0 and there exists some $p = p_{m(j+1)}$ such that $f^{j+1}(x) \in B(p, \frac{\delta}{3\kappa})$. We set $x_{j+1} = p$.
  - * If $\tau_{k+1} > \tau_k + 1$, then $f^{j+1}(x) \in \Lambda_2^+(\tau_{k+1} - \tau_k)$, and there exists some $q = q_{m(j+1),\tau_{k+1}-\tau_k}$ such that $f^{j+1}(x) \in B_{\tau_{k+1}-\tau_k}(q, \frac{\delta}{3\kappa})$. Hence we set $x_{j+l} = f^l(q)$, for $1 \le l \le \tau_{k+1} - \tau_k$.
- If $j = \tau_k$, then necessarily $j + 1 = \tau_{k+1}$.
  - * If $f^{j+1}(x)$ belongs to Ω_1, then we construct x_{j+1} by choosing any $p = p_{m(j+1)}$ such that $f^{j+1}(x) \in B(p, \frac{\delta}{3\kappa})$. We set $x_{j+1} = p$.
  - * If $f^{j+1}(x)$ does not belong to Ω_1, then $f^{j+1}(x) \in \Lambda_2^+(\tau_{k+2} - \tau_{k+1})$ and there exists some $q = q_{m(j+1),\tau_{k+2}-\tau_{k+1}}$ such that $f^{j+1}(x) \in B_{\tau_{k+2}-\tau_{k+1}}(q, \frac{\delta}{3\kappa})$. Hence we set $x_{j+l} = f^l(q)$, for $1 \le l \le \tau_{k+2} - \tau_{k+1}$.

Lower block.

- If i is not a stopping time, then $i-1$ must be some stopping time τ_k.
    * If $\tau_{k-1} = \tau_k - 1$, then we just construct x_{i-1}. There exists some $p = p_{m(i-1)}$ such that $f^{i-1}(x) \in B(p, \frac{\delta}{3\kappa})$. We set $x_{i-1} = p$.
    * If $\tau_{k-1} < \tau_k - 1$, then $f^{i-1}(x) \in \Lambda_2^-(|\tau_{k-1} - \tau_k|)$, and there exists some $r = r_{m(i-1),|\tau_{k-1}-\tau_k|}$ such that $f^{i-1}(x) \in B_{\tau_{k-1}-\tau_k}(r, \frac{\delta}{3\kappa})$. Hence we set $x_{i-l} = f^l(r)$, for $1 \le l \le |\tau_{k-1} - \tau_k|$.
- If $i = \tau_k$,
    * If $f^{i-1}(x) \notin \Omega_0$, then $\tau_{k-1} < \tau_k - 1$, and $f^{\tau_{k-1}}(x)$ belongs to $\Lambda_2^+(\tau_k - \tau_{k-1})$. Hence, there exists some $q = q_{m(i-1),\tau_k-\tau_{k-1}}$ such that $f^{\tau_{k-1}}(x) \in B_{\tau_k-\tau_{k-1}}(q, \frac{\delta}{3\kappa})$. Hence we set $x_{\tau_{k-1}+l} = f^l(q)$, for $0 \le l < \tau_k - \tau_{k-1}$.
    * If $f^{i-1}(x) \in \Omega_1$, then we just construct x_{i-1} by choosing any $p = p_{m(i-1)}$ such that $f^{i-1}(x) \in B(p, \frac{\delta}{3\kappa})$. We set $x_{i-1} = p$.
    * If $f^{i-1}(x) \in \Omega_2$, then $f^{i-1}(x) \in \Lambda_2^-(|\tau_{k-2} - \tau_{k-1}|)$ but does not belong to any $\Lambda_2^+(m)$. Then, there exists some $r = r_{m(i-1),|\tau_{k-2}-\tau_{k-1}|}$ such that $f^{i-1}(x) \in B_{\tau_{k-2}-\tau_{k-1}}(q, \frac{\delta}{3\kappa})$. Hence we set $x_{i-l} = f^l(q)$, for $1 \le l \le \tau_{k-1} - \tau_{k-2}$.

We have now constructed $x_{i'}, \ldots, x_i, \ldots, x_j, \ldots, x_{j'}$.

We let the reader check that if i' (resp. j') is not a stopping time, then $i'-1$ (resp.$j'+1$) must be a stopping time.

step four. Hence, by construction, we obtain for every n,

$$d(f^n(x), x_n) < \frac{\delta}{3\kappa} < \beta(\delta).$$

The sequence $(x_n)_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit in $\Gamma^{\mathbb{Z}}$, and x $\beta(\delta)$-tracks $(x_n)_{n\in\mathbb{Z}}$. □

We denotes by Σ_0 the set of δ-pseudo-orbit in $\Gamma^{\mathbb{Z}}$. Proposition 3.3 proves that there is a canonical map $\Theta : \Gamma^{\mathbb{Z}} \to M$ that maps every δ-pseudo-orbit $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ to the unique x that $\beta(\delta)$-tracks $\underline{x}$. Conversely, Proposition 3.4 proves that Λ is contained into the image of Θ. Hence, there is a conjugacy of the dynamic on M with some symbolic dynamical system :

$$\begin{array}{cccc} \sigma: & \Sigma_0 & \to & \Sigma_0 \\ & \downarrow & & \downarrow\Theta \\ f: & M & \to & M \end{array}$$

where σ is the shift on Σ_0.

3.3 Rectangles

From now on, δ is some fixed constant as above (relatively to the choice of ε_1). Moreover, its exact magnitude will also depend on several (but finite) conditions that will follow. In Σ_0, there is the canonical partition into 1-cylinder: for ξ_n in Γ we set

$$[\xi_n] \stackrel{def}{=} \{\underline{x} \in \Sigma_0,\ x_0 = \xi_n\}.$$

There is a canonical map, $[\,.\,,\,.\,]$, from $\bigcup_{n\in\mathbb{N}} [\xi_n]^2$ to Σ_0 defined in the following way :

If $\underline{y} = (y_n)_{n\in\mathbb{Z}}$ and $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ are in $[\xi_p]$, then $[\underline{x}, \underline{y}]$ is the sequence $\underline{z} = (z_n)_{n\in\mathbb{Z}}$ defined by $z_n = x_n$ for every $n \geq 0$ and $z_n = y_n$ for every $n \leq 0$. The first goal of this subsection is to study the push forward of the 1-cylinders on M by Θ. In particular, we prove that the map $\Theta \circ [\,.\,,\,.\,]$ is the map $[\Theta(\,.\,), \Theta(\,.\,)]$ where the second $[\,.\,,\,.\,]$ is the system of local coordinates on the manifold. Then, we will produce several generations of rectangles.

Lemma 3.5. *Let $\underline{y} = (y_n)_{n\in\mathbb{Z}}$ and $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ be two sequences in Σ_0 such that for every $n \geq 0$, $x_n = y_n$. Then, if δ is small enough such that $\beta(\delta) < \dfrac{\rho_2}{8\kappa}$, we have $\Theta(\underline{x}) \in D^s_{2\beta(\delta)}(\Theta(\underline{y}))$.*

Proof. For convenience we will set $x = \Theta(\underline{x})$ and $y = \Theta(\underline{y})$. By construction there exist two points z_1 and z_2 such that $\{z_1\} = D^u_{2\beta(\delta)}(x) \cap D^s_{2\beta(\delta)}(y)$ and $\{z_2\} = D^u_{2\beta(\delta)}(y) \cap D^s_{2\beta(\delta)}(x)$. Let us assume that $x \neq z_1$. We are going to find some contradiction.
Because $x \neq z_1$, then we must have $y \neq z_2$. For $n \geq 0$ we have $d(f^n(x), x_n) \leq \beta(\delta)$, $d(f^n(y), x_n) \leq \beta(\delta)$ and $\lim_{n\to+\infty} d(f^n(x), f^n(z_2)) = 0$. Hence, we may assume that for every $n \geq 0$, $d(f^n(x), f^n(z_2)) \leq \beta(\delta)$, and so

$$\forall n \geq 0,\ f^n(z_2) \in B(x_n, 2\beta(\delta)) \text{ and } f^n(y) \in B(x_n, \beta(\delta)) \tag{15}$$

Moreover, there are infinitely many n_k in $\mathbb{N}$ such that $x_{n_k} \in \Omega_0$. Therefore, property (15) implies that, if n_k is such that for every $j \leq n_k$, $f^j(z_2)$ belongs to the connected component of $D^u_{2\beta(\delta)}(f^j(y)) \cap B(x_j, 2\beta(\delta))$ that contains $f^j(y)$, then we must have

$$d^u(f^{n_k}(y), f^{n_k}(z_2)) \geq e^{k\gamma^u} d^u(z_2, y). \tag{16}$$

Therefore, there exists some n such that for every $0 \leq k \leq n$, $f^k(z_2)$ belongs to the connected component of $\mathcal{F}^u(f^k(y)) \cap B(x_k, 2\beta(\delta))$ that contains $f^k(y)$ and $f^{n+1}(z_2)$ does not belong to the connected component of $\mathcal{F}^u(f^{n+1}(y)) \cap B(x_{n+1}, 2\beta(\delta))$ that contains $f^{n+1}(y)$.
Because δ has been chosen small enough, $f^{n+1}(z_2)$ belongs to the connected component of $\mathcal{F}^u(f^{n+1}(y)) \cap B(x_{n+1}, \frac{\rho_2}{2})$ that contains $f^{n+1}(y)$ and so $d(f^{n+1}(z_2), x_{n+1})$ must be greater than $2\beta(\delta)$, which is absurd. Therefore $x = z_1$ and $y = z_2$. □

3.3.1 First generation of rectangles

For n in $\mathbb{N}$ we set $T_n \stackrel{def}{=} \Theta([\xi_n])$. If x is in T_n, we set $W^u(x,T_n) \stackrel{def}{=} D^u_{2\beta(\delta)}(x) \cap T_n$ and $W^s(x,T_n) \stackrel{def}{=} D^s_{2\beta(\delta)}(x) \cap T_n$. As a direct consequence of lemma 3.5 we obtain

Proposition 3.6. *Let n be an integer and x and y be in T_n. Let $\underline{x}$ and $\underline{y}$ be in $[\xi_n]$ such that $\Theta(\underline{x}) = x$ and $\Theta(\underline{y}) = y$. Then,*
$W^u(x,T_n)$ (resp. $W^s(x,T_n)$) is exactly the image by Θ of $\{[\underline{z},\underline{x}],\ \underline{z} \in [\xi_n]\}$ (resp. $\{[\underline{x},\underline{z}],\ \underline{z} \in [\xi_n]\}$).
Moreover, $W^u(x,T_n) \cap W^s(y,T_n)$ exists and, $[y,x] = \Theta([\underline{y},\underline{x}]) = W^u(x,T_n) \cap W^s(y,T_n)$.

This proposition allows us to define T_n as a rectangle in the sense of Bowen (see [5]). The T_n's are called rectangles of first generation and their set is denoted by $\mathcal{T}$. They also satisfied some restricted Markov property:

Proposition 3.7. *Let T_i and T_j be in $\mathcal{T}$ such that some point x belongs to $T_j \cap f^{-1}(T_i)$. If there exists some $\underline{x}$ in Σ_0 such that :*
(i) $\Theta(\underline{x}) = x$,
(ii) $x_0 = \xi_j$,
(iii) $x_1 = \xi_i$,
then, $W^u(f(x),T_i) \subset f(W^u(x,T_j))$ and $W^s(f(x),T_i) \supset f(W^s(x,T_j))$.

Proof. Let y be in $W^s(x,T_j)$. There exists some $\underline{y}$ in $\Theta^{-1}(y) \cap [\xi_j]$. Moreover proposition 3.6 proves that $y = [x,y]$, and so, $y = \Theta([\underline{x},\underline{y}])$. If $\underline{z}$ denotes $[\underline{x},\underline{y}]$, then $z_0 = \xi_j$ and $z_1 = \xi_i$. Hence, $f(y) = \Theta \circ \sigma(\underline{z})$ belongs to T_i.
The other part of the proof is similar. □

Remark: It is important to notice that proposition 3.7 does not prove that all the rectangles satisfy a complete Markov property: If x is in $T_j \cap f^{-1}(T_i)$ and if there is no $\underline{x}$ in $\Theta^{-1}(x) \cap [\xi_j] \cap \sigma^{-1}[\xi_i]$, the result of the proposition would probably be false. In view to obtain the global Markov property we have to cut these rectangles. Before, we need to make them thinner.

3.3.2 Second generation of rectangles

The main idea of this part is to restrict the f-invariant set $\bigcup_{i\in\mathbb{N}} T_i$ $(= \Theta(\Sigma_0))$ to some subset satisfying good properties.

Proposition 3.8. *There exists some λ-hyperbolic set $\Delta \subset \Lambda$ such that*
(1) $Leb^u(\Delta) > 0$;
(2) every point x in Δ is a density point of Δ for Leb^u_x;
(3) there exist some $\zeta > 0$ such that for every x in Δ,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log ||df^{-1}_{|E^u(f^j(x))}(f^j(x)|| < -\zeta, \tag{17}$$

Proof. We first notice that, if (17) holds for some x in Λ (and for some fixed ζ in $\mathbb{R}_+^*$), then, it also holds for every $f^n(x)$, when n describes $\mathbb{Z}$.
The first part of the proof is to show that for every x in Λ,

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log||df^{-1}_{|E^u(f^j(x))}(f^j(x)||<0 \tag{18}$$

Let x be in Λ, ε_2 be some fixed real number in $]0,1[$ and $\Omega_3=\Omega_3(\varepsilon_2)$ be the set of point x in Ω such that

$$\min(\log||df_{|E^u(x)}||,-\log||df^{-1}_{|E^u(x)}||,-\log||df_{E^s(x)}||,\log||df^{-1}_{|E^s(x)}||)\geq\varepsilon_2\lambda.$$

We set $\delta_{\varepsilon_2}\stackrel{def}{=}\liminf\frac{1}{n}\#\{0\leq k<n,\ f^k(x)\in\Omega_3\}$ (where $\#A$ denotes the cardinal of the finite set A). Because x is λ-hyperbolic, it must spend enough time in Ω_3, and we have

$$\delta_{\varepsilon_2}\geq\frac{1-\varepsilon_2}{\frac{\log\kappa}{\lambda}-\varepsilon_2}>0. \tag{19}$$

Hence, inequality (19) yields

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log||df^{-1}_{|E^u(f^j(x))}(f^j(x)||<-\delta_{\varepsilon_2}\varepsilon_2\lambda,$$

which proves that (18) holds for every x in Λ.
Let us set

$$\Lambda_{|\zeta}\stackrel{def}{=}\left\{x\in\Lambda,\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log||df^{-1}_{|E^u(f^j(x))}(f^j(x)||<-\zeta\right\}.$$

Then, Λ is the increasing union of all $\Lambda_{|\zeta}$ (when ζ decreases to 0), and so, there exists some ζ such that $Leb^u(\Lambda_{|\zeta})>0$. We pick such a ζ.

Let us set $\Delta_0\stackrel{def}{=}\bigcup_{i\in\mathbb{N}}\{x\in T_i,\ \exists(y,z)\in\Lambda^2_{|\zeta}\ /x\in\mathcal{F}^u(y)\cap\mathcal{F}^s(z)\}$. Then, Δ_0 is a f-invariant set of λ-hyperbolic points in $\bigcup_{i\in\mathbb{N}}R_i$ such that (17) holds for every x in Δ_0. Moreover, $Leb^u(\Delta_0)>0$.

Now, if x is a density point of Δ_0 for Leb^u_x we check that every y in $\mathcal{F}^s(x)\cap\Delta_0$ is also a density point of Δ_0 for Leb^u_y:
This is a well-known fact in Pesin Theory that the stable holonomy is absolutely continuous with respect to Lebesgue measure on the unstable manifolds. This again holds in our case by definition of $\mathcal{F}^s(x)$: there exists some n such that for every $k\geq n$, $f^k(y)$ belongs to $\mathcal{F}^s_{loc}(f^k(x))$. Then we can use Mañé's method (see [12]).
Therefore, we denote by Δ the set of density points in Δ_0 for Leb^u. This set is f-invariant and stable by intersections of stable and unstable leaves. Hence, every point in Δ is also a density point of Δ for Leb^u. $\square$

For $i \in \mathbb{N}$, we denote by S_i the restriction of the rectangle T_i to Δ, *i.e.* $S_i = T_i \cap \Delta$. If x is in S_i we set $W^u(x, S_i) \stackrel{def}{=} W^u(x, T_i) \cap \Delta$ and $W^s(x, S_i) \stackrel{def}{=} W^s(x, T_i) \cap \Delta$. By construction of Δ, if x and y are in S_i, then $[x, y]$ is also in S_i. As before we have

$$\{[x,y]\} = W^s(x, S_i) \cap W^u(y, S_i) = D^s_{2\beta(\delta)}(x) \cap D^u_{2\beta(\delta)}(y).$$

The sets S_i are called rectangles of second generation, and their set is denoted by $\mathcal{S}$.

3.4 Third generation of rectangles

3.4.1 Constants

Since the beginning we have introduced several constants. It is time now to fix some of them. For that purpose we want first to summarize how do these constants depend each from others.

The constant κ is fixed by f. The two constants K_1 and K_2 are also fixed.
We fix the set Λ, and so, the two constants ε_0 and λ are also fixed. Then, we fix ζ as above (note that this is done before fixing ε_1). We can pick some ε, very small compared to λ and ζ (namely $\varepsilon < \zeta/10$).

Now, as soon as ε_1 is fixed (sufficiently small as it is asked in Theorem A's proof but only in relation with f and Λ), the constant ρ_1 can be adjusted. Therefore, we can choose the constants ρ, ρ_2, γ^u, and γ^s. After we can choose δ such that δ and $\beta(\delta)$ are sufficiently small to satisfy the several asked conditions. Several new conditions will also be stated later.

3.4.2 Choice of ε_i

We first fix ε_2 sufficiently small such that every x in Δ such that $\log ||df^{-1}_{|E^u(x)}|| < -\dfrac{\zeta}{3}$ belongs to Ω_3 (this set has been defined in proposition 3.8). This is possible because k^u and k^s vanishes at the same time.
Hence, we fix ε_1 sufficiently small such that Ω_3 is a closed subset of Ω which does not intersect $\overline{\Omega_2 \cup B(S, \varepsilon_1)}$. Therefore, there exist some $\rho_3 > 0$ such that $d(\Omega_3, \overline{\Omega_2 \cup B(S, \varepsilon_1)}) > \rho_3$, and we assume that δ satisfies

$$\beta(\delta) < \frac{\rho_3}{10}.$$

3.4.3 Third generation of rectangles

If S_i is in $\mathcal{S}$, we say that it has (or it is of) order 0 if ξ_i is in Ω_1.
We say that it has order n if either $\xi_i = f^k(\xi_j)$ with ξ_j in $\Lambda_2^+(n)$ and $0 \leq k < n$, or $\xi_i = f^{-k}(\xi_j)$ with ξ_j in $\Lambda_2^-(n)$ and $0 \leq k < n$. Because $\beta(\delta)$ is small enough,

none of the rectangles of second generation and of order 0 that intersect Ω_3 can intersect with some rectangle of order $n > 0$. We say that a rectangle of order 0 that intersects Ω_3 is of order 00. Then, each rectangle of order 00 intersects only with a finite number of other rectangles. Hence, we can cut them as in [5]: let S_i be a rectangle of order 00 and S_j be any other rectangle such that $S_i \cap S_j \neq \emptyset$. We set

$$\begin{aligned}
S_{ij}^1 &= \{x \in S_i, W^u(x, S_i) \cap S_j = \emptyset \text{ and } W^s(x, S_i) \cap S_j = \emptyset\},\\
S_{ij}^2 &= \{x \in S_i, W^u(x, S_i) \cap S_j \neq \emptyset \text{ and } W^s(x, S_i) \cap S_j = \emptyset\},\\
S_{ij}^3 &= \{x \in S_i, W^u(x, S_i) \cap S_j \neq \emptyset \text{ and } W^s(x, S_i) \cap S_j \neq \emptyset\},\\
S_{ij}^4 &= \{x \in S_i, W^u(x, S_i) \cap S_j = \emptyset \text{ and } W^s(x, S_i) \cap S_j \neq \emptyset\}.
\end{aligned}$$

Let us set $\mathcal{S}_0 = \{x \in M,\ \exists S_i \ni x \text{ of order } 00\}$. Then, for x in $\mathcal{S}_0$ we set

$$\mathcal{R}(x) \stackrel{def}{=} \{y \in M, \forall i, j \in \mathbb{N}, x \in S_{ij}^k \Rightarrow y \in S_{ij}^k\}.$$

This defines a partition $\mathcal{R}$ of $\mathcal{S}_0$. By construction it is finite, and each of its elements, R_i, is stable by the map $[\,.\,,\,.\,]$. Hence, each R_i is a rectangle; the sets R_i's are called rectangles of third generation. If x is in R_i, we set $W^u(x, R_i) \stackrel{def}{=} D^u_{2\beta(\delta)}(x) \cap R_i$ and $W^s(x, R_i) \stackrel{def}{=} D^s_{2\beta(\delta)}(x) \cap R_i$. It is easy to check that if x is in the rectangles of second generation $S_{i_1}, \dots S_{i_p}$, then $W^{u,s}(x, R_i) = W^{u,s}(x, S_{i_k}) \cap R_i$ for $1 \leq k \leq p$. Moreover, we let the reader check that proposition 3.7 yields the following result (as in [5]):

Proposition 3.9. *Let R_i and R_j be two rectangles of third generation, n be some integer in $\mathbb{N}$, and x be in $R_i \cap f^{-n}(R_j)$. Then,*

$$\begin{aligned}
f^n(W^u(x, R_i)) &\supset W^u(f^n(x), R_j),\\
\textit{and } f^n(W^s(x, R_i)) &\subset W^s(f^n(x), R_j).
\end{aligned}$$

This means that $\mathcal{R}$ is a Markov partition of $\mathcal{S}_0$.

Remark: this construction can also be done with the first generation of rectangles because every T_i that intersects with Ω_3 intersects only a finite number of T_j's.

3.5 Reduced dynamical systems

3.5.1 Hyperbolic times

Let x be in $\mathcal{S}_0$. It is a λ-hyperbolic point and so, it must return infinitely many often in Ω_3. Hence, every point in $\mathcal{S}_0$ returns infinitely many often in $\mathcal{S}_0$.
Let x be in $\mathcal{S}_0$ and n be in $\mathbb{N}^*$ such that $f^n(x)$ is in $\mathcal{S}_0$. There exist two rectangles of third generation R_l and R_k such that x is in R_l and $f^n(x)$ is in R_k. Then, the Markov property (proposition 3.9) implies that

$$f^{-n}(W^u(f^n(x), R_k)) \subset W^u(x, R_l).$$

We will need some uniform distortion bound, *i.e.* some uniform control on

$$\prod_{k=0}^{n-1} \frac{J^u(f^k(x))}{J^u(f^k(y))},$$

where y belongs to $f^{-n}(W^u(f^n(x), R_k))$ and $J^u(z)$ is the unstable Jacobian $\det df_{|E^u(z)}(z)$.

If the orbit of x comes into $B(S, \varepsilon_1)$ between x and $f^n(x)$, then we do not have good control on $\prod_{k=0}^{n-1} J^u(f(x))$, and in particular on the distortion ratio

$$\prod_{k=0}^{n-1} \frac{J^u(f^k(x))}{J^u(f^k(y))}.$$

To obtain good estimates we have to use the notion of hyperbolic times that was introduced in [1]

Definition 3.10. *Given $0 < r < 1$, we say that n is a r-hyperbolic time for x if for every $1 \leq k \leq n$*

$$\prod_{i=n-k+1}^{n} ||df^{-1}_{|E^u(f^i(x))}|| \leq r^k.$$

Moreover, it is known that, if

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log ||df^{-1}_{|E^u(f^j(x))}(f^j(x))|| < 2\log r,$$

then, there exist infinitely many r-hyperbolic times for x (see section 3, corollary 3.2 in [2]). Therefore, by construction of Δ, for $\rho_4 = e^{-\frac{1}{3}\zeta}$ and for every x in $\mathcal{S}_0$, there exist infinitely many ρ_4-hyperbolic times for x.

Lemma 3.11. *There exists $\delta' > 0$ such that, if $\delta < \delta'$, then , for every x in $\mathcal{S}_0$, for every ρ_4-hyperbolic times for x, n, and for every y in $B_{n+1}(x, 4\beta(\delta))$, the integer n is also a $\sqrt{\rho_4}$-hyperbolic time for y.*

Proof. By continuity of df and E^u, there exists some δ'' such that for every x in U and for every y in $B(x, \delta'')$, then

$$e^{-\varepsilon} < \frac{||df^{-1}_{|E^u(x)}(x)||}{||df^{-1}_{|E^u(y)}(y)||} < e^{\varepsilon}.$$

The definition of $\beta(\delta)$ proves that if δ is sufficiently small (namely $\delta < \delta'$ for some δ'), then $4\beta(\delta)$ is smaller than δ''. Let us assume that $\delta < \delta'$; let x be some point in $\mathcal{S}_0$ and n be some ρ_4-hyperbolic times for x. Let y be in $B_{n+1}(x, 4\beta(\delta))$. Then for every $0 \leq k \leq n$, y belongs to $B(x, 4\beta(\delta)) \subset B(x, \delta'')$, which means that for every $0 \leq k \leq n$

$$e^{-\varepsilon} < \frac{||df^{-1}_{|E^u(f^k(x))}(f^k(x))||}{||df^{-1}_{|E^u(f^k(y))}(f^k(y))||} < e^{\varepsilon}. \tag{20}$$

The real number ε is very small compared to ζ, thus (20) proves that for every $0 \leq k \leq n$,

$$||df^{k-n}_{E^u(f^k(y))}|| < (\sqrt{\rho_4})^{n-k}.$$

This proves that n is a $\sqrt{\rho_4}$-hyperbolic time for every y. □

For the rest of the proof, we assume that δ is smaller than δ'

Lemma 3.12. *Let x be in $\mathcal{S}_0$ and n be some ρ_4-hyperbolic time for x. Then, $f^n(x)$ is in $\mathcal{S}_0$.*

Proof. By definition of the ρ_4-hyperbolic times, $||df^{-1}_{|E^u(f^n(x))}|| < \rho_4 = e^{-\frac{1}{3}\zeta}$, which implies that $f^n(x)$ is in Ω_3 (by definition of ε_2). Hence, $f^n(x)$ is in $\mathcal{S}_0$ □

Conversely, if x and $f(x)$ belong to $\mathcal{S}_0$, then 1 is a ρ_4-hyperbolic time. Therefore, we say that a ρ_4-hyperbolic time for x is a *hyperbolic return time in $\mathcal{S}_0$*. Moreover, the Markov property of $\mathcal{R}$ proves that, if n is a ρ_4-hyperbolic time for x, then there exists $R_k \in \mathcal{R}$ such that $f^n(x)$ is in R_k and we have

$$f^{-n}(W^u(f^n(x), R_k)) \subset \mathcal{S}_0.$$

3.5.2 First return time map and special rectangle R

For x in $\mathcal{S}_0$, we define some return time $\tau(x)$ by

$$\tau(x) = \inf\left\{n \in \mathbb{N}^*, \exists y \in f^{-n}(W^u(f^n(x), \mathcal{R}(f^n(x)))) \ / \ n \text{ is a hyperbolic return time for } y\right\},$$

where $\mathcal{R}(f^n(x))$ denotes the rectangle of third generation that contains $f^n(x)$. This defines a map g_0 from $\mathcal{S}_0$ to itself by setting $g_0(x) \stackrel{def}{=} f^{\tau(x)}(x)$.

Remark: because of the Markov property, the map g_0 is well defined. It is an injection but not necessary a bijection.

Proposition 3.13. *There exists at least one rectangle of third generation R_k in $\mathcal{R}$ such that:*

1. *$Leb^u(R_k) > 0$,*
2. *for each x in R_k, the set of points y in $W^u(x, R_k)$ that return infinitely many often in R_k by iteration of g_0, has positive Leb^u_x-measure.*

Proof. We first notice that Δ has positive Leb^u-measure, thus, $Leb^u(\mathcal{S}_0) > 0$. Moreover, $\mathcal{S}_0$ is covered by a finite number of rectangles of the third generation, every point in $\mathcal{S}_0$ is a density point for Δ and every point in $\mathcal{S}_0$ returns infinitely many times in $\mathcal{S}_0$.
Therefore, there exists one rectangle R_k with positive Leb^u-measure, and such that for one x in R_k, the set of points y in $W^u(x, R_k)$ that return infinitely many often in R_k by iteration of g_0, has positive Leb^u_x-measure. Now, the Markov property and the absolutely continuous property of the foliations yield *2* for every z in R_k. □

Let R_k be one rectangle of third generation as in proposition 3.13. We denote by R the set of points y in R_k that return infinitely many often in R_k (by iteration of g_0). We also define $W^u(y, R) \stackrel{def}{=} W^u(y, R_k) \cap R$ and $W^s(y, R) \stackrel{def}{=} W^s(y, R_k) \cap R = W^s(y, R_k)$. Because of proposition 3.9, R is a rectangle, *i.e.*,

$$\forall x, y \in R,\ \{[x, y]\} = W^s(x, R) \cap W^u(y, R). \tag{21}$$

Moreover,

$$\forall x \in R,\ Leb^u_x(R) > 0, \tag{22}$$

and $\forall x \in R$, $\forall n \in \mathbb{N}$ such that $f^n(x) \in R$,

$$W^u(f^n(x), R) \subset f^n(W^u(x, R)),\ \text{and}\ W^s(f^n(x), R) \supset f^n(W^s(x, R)). \tag{23}$$

We denote by g the first-return-map in R (by iteration of g_0). By construction, there exists some first-return-time map r such that for every x in R, $g(x) = f^{r(x)}(x)$. As usually **we denote by $r^n(x)$ the n^{th}-return time in R by iteration of g, namely** $g^n(x) = f^{r^n(x)}(x)$ (this time can be different from the n^{th} return time of x in R by iteration of f).

4 Proof of Theorem B

4.1 *SRB*-measure for (R, g)

We copy the method of [9]. Let x be some point in R. We denote by μ_n the measure

$$\mu_n \stackrel{def}{=} \frac{1}{n} \sum_{i=0}^{n-1} g^i_* Leb^u_x,$$

and pick μ some accumulation point of the family (μ_n). To prove that μ is a *SRB*-measure, it is sufficient to prove that there exists some constant χ, such that for every integer n, for every y in $g^n(W^u(x, R))$ and for every z in $W^u(y, R)$

$$e^{-\chi} \le \frac{\prod_{i=0}^{r^n-1} J^u(f^{-i}(y))}{\prod_{i=0}^{r^n-1} J^u(f^{-i}(z))} \le e^{\chi}. \tag{24}$$

We have chosen the map g in relation with hyperbolic times. Hence we have some distortion bounds.

Lemma 4.1. *There exists* $0 < \omega_0 < 1$ *such that for every* z *in* $\mathcal{S}_0$ *and* y *in* $g_0^{-1}(W^u(g_0(z), \mathcal{R}(g_0(z))))$*, and for every* $0 \leq k \leq \tau(z)$*,*

$$d^u(f^k(z), f^k(y)) \leq (\omega_0)^{\tau(z)-k} d^u(g_0(z), g_0(y)).$$

Proof. If z is in $\mathcal{S}_0 \cap f^{-1}(\mathcal{S}_0)$, then $g_0(z) = f(z)$ and z is in Ω_3 (far away from S). If $g_0(z) = f^{\tau(z)}(z)$ with $\tau(z) > 1$, then by definition of g_0, there exists z' such that
(i) $\tau(z)$ is a ρ_4-hyperbolic time for z'.
(ii) z' is in $\mathcal{C}(z) \stackrel{def}{=} f^{-\tau(z)}(W^u(f^{\tau(z)}(z), \mathcal{R}(f^{\tau(z)}(z))))$.
The diameter of $\mathcal{R}(f^{\tau(z)}(z))$ is smaller than $2\beta(\delta)$; hence, by construction of the third generation of rectangles, $\mathcal{C}(z)$ is included into $B_{\tau(z)+1}(z, 4\beta(\delta))$, which yields to the fact that $\tau(z)$ is a $\sqrt{\rho_4}$-hyperbolic time for every point in $\mathcal{C}(z)$.
Therefore, the map $f^{k-\tau(z)} : W^u(f^{\tau(z)}(z), \mathcal{R}(f^{\tau(z)}(z))) \to f^k(\mathcal{C}(z))$ is a contraction, and satisfies $|||df^{k-\tau(z)}_{|E^u}||| \leq (\sqrt{\rho_4})^{\tau(z)-k}$. Thus,

$$d^u(f^k(z), f^k(y)) \leq (\rho_4)^{\frac{\tau(z)-k}{2}} d^u(g_0(z); g_0(y)).$$

□

Lemma 4.2. *There exist some constants* $\chi_1 > 0$ *and* $0 < \omega < 1$ *such that for every* $n \geq 1$*, for every* y *in* $W^u(x, R)$*, for every* z *in* $g^{-n}(W^u(g^n(y), R))$ *and for every* $m \leq n$*, we obtain*

$$\sum_{j=0}^{r^m(z)-1} \left| \log(J^u(f^j(z))) - \log(J^u(f^j(y))) \right| \leq \chi_1 \, \omega^{n-m}.$$

Proof. The map $x \mapsto J^u(x)$ is Hölder-continuous because the map E^u is Hölder-continuous; moreover it has its values into $[1, +\infty[$. The map $t \mapsto \log(t)$ is Lipschitz-continuous on $]1, +\infty[$. Thus, there exists some constants χ_2 and α, such that

$$\sum_{j=0}^{r^m(z)-1} \left| \log(J^u(f^j(z))) - \log(J^u(f^j(y))) \right| \leq \chi_2 \sum_{j=0}^{r^m(z)-1} (d^u(f^j(z), f^j(y)))^\alpha. \tag{25}$$

Hence, lemma 4.1 and (25) give

$$\sum_{j=0}^{r^m(z)-1} \left| \log(J^u(f^j(z))) - \log(J^u(f^j(y))) \right| \leq \chi_2 \left[\sum_{j=0}^{+\infty} \rho_4^{j\alpha/2} \right] (d^u(g^m(z), g^m(y)))^\alpha.$$

Lemma 4.1 also gives $d^u(g^m(z), g^m(y)) \leq \rho_4^{(n-m)/2} diam(R)$. □

Lemma 4.2 proves that (24) holds for every n, for every y in $g^n(W^u(x, R))$ and for every z in $W^u(y, R)$, with $\chi := \chi_1 + 2 \log \kappa$. Then the measure μ is a SRB-probability measure for (R, g).

4.2 *SRB*-measure for (M, f)

The map g satisfies $g(z) = f^{r(z)}(z)$ for every z in R. Let $R(i) = \{z \in R, r(z) = i\}$ be the set of points in R such that the first return time equals i. We set

$$m = \sum_{i=1}^{+\infty} \sum_{j=0}^{i-1} f_*^j(\mu|R(i)).$$

Then m is at least σ-finite SRB-measure. It is finite if and only if $\int_F r(x)dLeb_F < \infty$. In this case, we may normalize the measure to obtain some probability measure.

4.3 Markov partitions

There are several possibilities to construct a Markov partition from our cover. The simplest at that time is to use the iterates of R: if x is in $R(n)$, then for every $0 < k < n$ we set

$$C(f^k(x)) = f^k([f^{-n}(W^u(f^n(x)), R), W^s(x, R)]).$$

This defines a countable family of rectangles which satisfy the Markov property. Each rectangle has a diameter smaller than $2\beta(\delta)$. This is a Markov partition of Δ but Δ is strictly included in Λ.
However we could do the same with the partition $\mathcal{R}$ (instead of one of its elements) and the map g_0. This also defines a partition of Δ.
In fact, we can do it with our first generation of rectangles: we have made the rectangles of the first generation thinner because we needed some control on the distortion of the unstable-Jacobian, but the key point is that, rectangles that intersect Ω_3 must intersect only a finite number of other rectangles. This holds for the rectangles of the first generation. Therefore we can cut all the rectangles which intersect with Ω_3 like Bowen did. This gives a Markov partition of $\Omega_3 \cap \Lambda$ in rectangles (T_i'). For x in an element of this partition (T_j'), there exists an integer $n > 0$ such that $f^n(x)$ is again in such an element (T_i'). If n is the smallest positive integer which satisfies this condition, then we set for $0 < k < n$.

$$C(f^k(x)) = f^k([f^{-n}(W^u(f^n)(x), T_i'), W^s(x, T_j')]).$$

This defines a Markov partition of Λ with diameter smaller than the diameter of $\mathcal{T}$.

References

[1] J.F. Alves. Srb measures for nonhyperbolic systems with multidimensional expansion. *Ann. Sci. ENS*, iv(1), 2000.

[2] J.F. Alves, C. Bonatti, and M. Viana. Srb measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones Math.*, 140:351–298, 2000.

[3] M. Benedicks and L.-S. Young. Sinai-bowen-ruelle measures for certain hénon maps. *,Invent. Math.*, 112(3):541–576, 1993.

[4] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel Journal of Math.*, 1999.

[5] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, volume 470 of *Lecture notes in Math.* Springer-Verlag, 1975.

[6] M. Brin and Ya. Pesin. Partially hyperbolic dynamical systems. *Izv. Acad. Nauk. SSSR*, (1):177–212, 1974.

[7] M. Herman. Construction de difféomorphismes ergodiques. notes non publiées.

[8] Huyi Hu. Conditions for the existence of sbr measures for 'Almost Anosov Diffeomorphisms'. *Transaction of AMS*, 1999.

[9] Huyi. Hu and Lai-Sang Young. Nonexistence of *SBR*-measures for some diffeomorphisms that are "Almost Anosov". *Ergod. Th. & Dynam. Sys.*, 15, 1995.

[10] A. Katok and B. Hasselblatt. *Introduction to the modern Theory of Dynamical Systems.* Encyclopedia of mathematics and its applications. Cambridge University Press, 1995.

[11] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms Part I: Characterization of measures satisfying Pesin's entropy formula. *Annals of Mathematics*, 122:509–539, 1985.

[12] Ricardo Mañé. *Ergodic Theory and Differentiable Dynamics.* Springer-Verlag, 1987.

[13] C. Pugh and M. Shub. Ergodic Attractors. *American Mathematical Society*, 32(1), 1989.

[14] V.A. Rohlin. On the fundamental ideas of measure theory. *A.M.S.Translation*, 10:1–52, 1962.