2-Person Zero-Sum Stochastic Differential Games

based on common work with
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Preliminaries. Framework

$(\Omega, \mathcal{F}, P)$ canonical Wiener space: for a given finite time horizon $T > 0$,

- $\Omega = C_0([0,T]; \mathbb{R}^d)$ (endowed with the supremum norm);
- $B_t(\omega) = \omega(t), t \in [0,T], \omega \in \Omega$ - the coordinate process;

• P - the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$: unique probability measure w.r.t. B is a standard BM;

• $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_P$;

• $\mathbb{F} = (\mathcal{F})_t)_{t\in[0,T]}$ with $\mathcal{F}_t = \mathcal{F}_t^B = \sigma\{B_s, s \le t\} \vee \mathcal{N}_P$.

$(\Omega, \mathcal{F}, \mathbb{F}, P; B)$ - the complete, filtered probability space on which we will work.

Dynamics of the game:

Initial data: $t \in [0,T]$, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; R^d)$;

associated doubly controlled stochastic system:

$$\begin{array}{rcll} dX_s^{t,\zeta;u,v} & = & b(s, X_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)dB_s, & \\ X_t^{t,\zeta;u,v} & = & \zeta, & s \in [t,T], \end{array} \qquad (1)$$

Player I: $u \in \mathcal{U} =: L^0_{\mathbb{F}}(0,T;U)$;

Player II: $v \in \mathcal{V} =: L^0_{\mathbb{F}}(0,T;V)$; U, V - compact metric spaces and where the mappings

$$b : [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n\times d},$$

are continuous over $\mathbb{R}^n \times U \times V$ (for simplicity); Lipschitz in x, uniformly w.r.t (t,u,v), i.e., for some $L \in \mathbb{R}_+$,

$|\sigma(s,x,u,v) - \sigma(s,x',u,v)|, |b(s,x,u,v) - b(s,x',u,v)| \le |x-x'|;$

$|\sigma(s,x,u,v)|, |b(s,x,u,v)| \le (1+|x|).$

Existence and uniqueness of the solution $X^{t,\zeta,u,v} \in \mathcal{S}^2_{\mathbb{F}}(t,T;R^n)$;

from standard estimates: for all $p \ge 2$ there is some $C_p (= C_{p,L}) \in \mathbb{R}_+$ s.t.

$$E\left[\sup_{s\in[t,T]} |X_s^{t,\zeta;u,v} - X_s^{t,\zeta';u,v}|^p \mid \mathcal{F}_t\right] \le C_p|\zeta-\zeta'|^p, \ P\text{-a.s.},$$

$$E\left[\sup_{s\in[t,T]} |X_s^{t,\zeta;u,v}|^p \mid \mathcal{F}_t\right] \le C_p(1+|\zeta|^p), \ P\text{-a.s.}$$

Definition of the cost functionals

The cost functional is defined with the help of a backward SDE (BSDE):

Associated with $(t,\zeta)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n), u\in\mathcal{U}$ and $v\in\mathcal{V}$, we consider the BSDE:

$$\begin{array}{rcl} dY_s^{t,\zeta;u,v} & = & -f(s,X_s^{t,\zeta;u,v},Y_s^{t,\zeta;u,v},Z_s^{t,\zeta;u,v},u_s,v_s)ds+Z_s^{t,x\zeta;u,v}dB_s, \\ Y_T^{t,\zeta;u,v} & = & \Phi(X_T^{t,\zeta;u,v}), \qquad\qquad s\in[t,T], \end{array} \tag{2}$$

where

$\diamond$ Final cost: $\Phi:\mathbb{R}^n\to\mathbb{R}$ Lipschitz

$\diamond$ Running cost: $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\times V\to\mathbb{R}$, continuous; Lipschitz in (x,y,z), uniformly w.r.t (t,u,v).

Under the above assumptions: existence and uniqueness of the solution of BSDE (2):

$$(Y^{t,\zeta;u,v}, Z^{t,\zeta;u,v}) \in \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^d).$$

From standard estimates for BSDEs using the corresponding results for the controlled stochastic system: for all $p \geq 2$ there is some $C_p (= C_{p,L}) \in \mathbb{R}_+$ s.t.

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,\zeta;u,v} - Y_s^{t,\zeta';u,v}|^p + \left(\int_t^T |Z_s^{t,\zeta;u,v} - Z_s^{t,\zeta';u,v}|^2 ds\right)^{p/2} \mid \mathcal{F}_t\right] \leq C_p|\zeta - \zeta'|^p, \ P\text{-a.s.},$$

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,\zeta;u,v}|^p + \left(\int_t^T |Z_s^{t,\zeta;u,v}|^2 ds\right)^{p/2} \mid \mathcal{F}_t\right] \leq C_p(1+|\zeta|^p), \ P\text{-a.s.}$$

In particular, for $C = C_2^{1/2}$,

$|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \leq C|\zeta - \zeta'|$, P-a.s.,

$|Y_t^{t,\zeta;u,v}| \leq C(1+|\zeta|)$, P-a.s.

Let $t \in [0,T]$, $\zeta = x \in R^n$ - deterministic initial data; $u \in \mathcal{U}$, $v \in \mathcal{V}$;
associated Cost functional for the game over the time interval $[t,T]$:
$J(t,x;u,v) := Y_t^{t,x;u,v}\big(\in L^2(\Omega, \mathcal{F}_t, P)\big)$.

Remark 1: (i) If $f \equiv 0 : J(t,x;u,v) = E[\Phi(X_T^{t,x;u,v})|\mathcal{F}_t]$;
(ii) If f doesn't depend on (y, z):

$$J(t,x;u,v) = E[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s, X_s^{t,x;u,v}, u_s, v_s)ds|\mathcal{F}_t].$$

Which kind of game shall we study?

Objective of Player I: maximization of $J(t,x,u,v)$;

Objective of Player II: minimization of $J(t,x,u,v)$;

the both players have the same value function, it's the game for player I, the loss for player II - one speaks of "2-persons zero-sum stochastic differential games";

in non-zero sum games: Player i has cost functional $J_i(t,x,u_1,u_2...)$, $i \geq 1$, the players want to maximize there cost functional; problem of the existence and the characterization of Nash equilibrium points.

Game "Control against Control"?

• In general no value of the game, i.e., the result of the game depends on which player begins, and this even if Isaacs' condition is fulfilled (precision later); example: pursuit games (*Example at the blackboard.*)

• Games "Control against Control" with value if: $n = d$; $\sigma \in \mathbb{R}^{n\times n}(x)$ is independent of (u,v) and invertible (as matrix); $\sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is Lipschitz (S.HAMADENE, J.-P.LEPELTIER, S.PENG).

Game "Strategy against Control":

This concept has been known in the deterministic differential game theory (A.FRIEDMAN, W.H.FLEMING,..)nd has been translated later by W.H.FLEMING, P.E.SOUGANIDIS (1989) to the theory of stochastic differential games.

Here: a generalization of the concept of W.H.FLEMING, P.E.SOUGANIDIS (1989); a comparison of their concept with ours: later.

Admissible controls, admissible strategies

Definition 1: (*admissible controls* for a game over the time interval $[t,T]$)

- For Player I: $\mathcal{U}_{t,T} =: L^0_{\mathbb{F}}(t,T;U)$;
- for Player II: $\mathcal{V}_{t,T} =: L^0_{\mathbb{F}}(t,T;V)$.

Notice: In difference to the concept by FLEMING, SOUGANIDIS, the controls $u \in \mathcal{U}_{t,s}, v \in \mathcal{V}_{t,s}$ are not supposed to be independent of $\mathcal{F}_t$.

Definition 2: (*admissible strategies* for a game over the time interval $[t,T]$)

- For Player II: $\beta : \mathcal{U}_{t,T} \longrightarrow \mathcal{V}_{t,T}$ non anticipating, i.e.,

for any $\mathbb{F}-$ stopping time $S : \Omega \to [t,T]$ and any admissible controls $u_1,\, u_2 \in \mathcal{U}_{t,T}$

$(u_1 = u_2$ dsdP-a.e. on $[\![t,S]\!] \Longrightarrow \beta(u_1) = \beta(u_2)$ dsdP-a.e. on $[\![t,S]\!])$.

$\mathcal{B}_{t,T} := \{\beta : \mathcal{U}_{t,T} \to \mathcal{V}_{t,T} | \beta$ is nonanticipating$\}$.

Analogously we introduce

- for Player I: $\mathcal{A}_{t,T} := \{\alpha : \mathcal{V}_{t,T} \to \mathcal{U}_{t,T} | \alpha$ is nonanticipating$\}$.

Value Functions:

Notice: From $J(t,x,u,v) := Y_t^{t,x,u,v}$ and the standard estimates for $Y_t^{t,x,u,v}$:

$J(t,x,u,v) \in L^\infty(\Omega, \mathcal{F}_t, P)$, $(t,x,u,v) \in [0,T] \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, and:

- $|J(t,x,u,v) - J(t,x',u,v)| \le C|x - x'|$,
- $|J(t,x,u,v)| \le C(1 + |x|)$,

  P-a.s., for all $x, x' \in \mathbb{R}^n$, $(t,u,v) \in [0,T] \times \mathcal{U} \times \mathcal{V}$;

- $Y_t^{t,\zeta,u,v} = J(t,\zeta,u,v) \left(:= J(t,x,u,v)\big|_{x=\zeta} \right)$, P-a.s. (*evt. blackboard*)

The above estimates for $J(t,x,u,v)$ allow to introduce:

• Lower Value Function:

$$W(t,x) := \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u));$$

• Upper Value Function:

$$U(t,x) := \mathrm{esssup}_{\alpha \in \mathcal{A}_{t,T}} \mathrm{essinf}_{v \in \mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

Remarks. • Justification of the names "upper" and "lower" value functions: later we will see $W \leq U$; the proof is far from being obvious and uses the comparison principle for the associated Bellman-Isaacs equations, it will be given later.

• The esssup, essinf should be understood as ones w.r.t. a uniformly bounded, indexed family of $\mathcal{F}_t$-measurable r.v.; see: Dunford/Schwartz (1957). Consequently:

$W(t,x), U(t,x) \in L^\infty(\Omega, \mathcal{F}_t, P)$, and, for some $C \in \mathbb{R}_+$ (independent of (t,x)):

- $|W(t,x) - W(t,x')| + |U(t,x) - U(t,x')| \leq C|x - x'|$, P-a.s.,
- $|W(t,x)| + |U(t,x)| \leq C(1 + |x|)$, P-a.s., for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$.

Although W, U are a priori random variables, we have:

Proposition 1: $W(t,x) = E[W(t,x)]$, $U(t,x) = E[U(t,x)]$, $(t,x) \in [0,T] \times \mathbb{R}^n$, i.e., W and U admit a deterministic version with which we identify the both functions from now on.

Corollary. $W, U : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are such that

$$|W(t,x) - W(t,x')| + |U(t,x) - U(t,x')| \leq C|x - x'|,$$

$$|W(t,x)| + |U(t,x)| \leq C(1 + |x|), \text{ for all } t \in [0,T], x, x' \in \mathbb{R}^n.$$

Some Remarks preceding the proof of the proposition.

1) Concept of W.H.FLEMING, P.E.SOUGANIDIS (1989):

their running cost $f(s,x,y,z)$ don't depend on (y,z), i.e., their cost functional is the classical one;

more essential:

• *admissible controls*: instead of $\mathcal{U}_{t,T}$: $\mathcal{U}^t_{t,T} := L^0_{\mathbb{F}^t}(t,T;U)$,

instead of $\mathcal{V}_{t,T}$: $\mathcal{V}^t_{t,T} := L^0_{\mathbb{F}^t}(t,T;V)$,

$$\mathbb{F}^t = (\mathcal{F}^t_s)_{s\in[t,T]},\ \mathcal{F}^t_s := \sigma\{B_r - B_t,\, r\in[t,s]\}\vee\mathcal{N}_P,\ s\in[t,T];$$

• *admissible strategies*: instead of $\mathcal{B}_{t,T}$: $\mathcal{B}^t_{t,T}$ - the set of all non anticipating mappings $\beta : \mathcal{U}^t_{t,T} \longrightarrow \mathcal{V}^t_{t,T}$,

(non anticipativity is understood in the same sense as that in the definition of $\mathcal{B}_{t,T}$); analogous definition of $\mathcal{A}^t_{t,T}$.

Their cost functional

$$J(t,x;u,v) := E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(s,X_s^{t,x,u,v},u_s,v_s)\,|\,\mathcal{F}_t\right]$$
$$= E\left[\Phi(X_T^{t,x,u,v}) + \textstyle\int_t^T f(s,X_s^{t,x,u,v},u_s,v_s)\right]$$

is automatically deterministic, and so are their upper and lower value functions:

$$\overline{W}(t,x) := \inf_{\beta\in\mathcal{B}_{t,T}} \sup_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u)),\ \overline{U}(t,x) := \inf_{\alpha\in\mathcal{A}_{t,T}} \inf_{v\in\mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

Our approach in comparison with theirs:

(*1st mini-lecture*:)

• Proof that W,U are deterministic is not evident, but after:

• Straight forward approach without approximation by discrete schemes, without further technical notions (like π-controls, r-strategies), without using the Bellman-Isaacs equation for proving the DPP:

- Direct deduction of the DPP from the definition of W, U (with the help of Peng's notion of backward semigroups);

(*2nd mini-lecture*:)

- Direct deduction of the Bellman-Isaacs equations for W, U from the DPP (with the help of a scheme of 3 BSDEs, the so-called Peng's BSDE method developed by him for control problems);

- Adaptation of a uniqueness proof for integro-PDEs (G.BARLES, R.BUCKDAHN, E.PARDOUX) to Bellman-Isaacs equations.

2) Proof that W is deterministic for control problems (1997):

$U \subset R^M$ compact subset; σ, b, f don't depend on v, and are supposed to be Lipschitz in all their variables (x,u) and (x,y,z,u), resp.;

$$W(t,x) := \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x,u).$$

Then:

$$|J(t,x,u)-J(t,x,u')|^2 \le CE\left[\int_t^T |u_s-u'_s|^2 \,|\, \mathcal{F}_t\right], P\text{-a.s., } u,u' \in \mathcal{U}_{t,T}.$$

Let

$$\mathcal{U}_{t,T}^{step} := \Big\{u = \sum_{i,k,\ell=1}^{N} I_{A_i} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} : t = t_0 < t_1 < \cdots < t_N = T,$$

$$\theta_{k,\ell} \in U, A_{k,\ell} \in \mathcal{F}_t, B_{k,\ell} \in \mathcal{F}_{t_{k_1}}^t, N \ge 1\Big\};$$

then

$$W(t,x) = \text{esssup}_{u \in \mathcal{U}_{t,T}^{step}} J(t,x,u).$$

On the other hand, for $u \in \mathcal{U}_{t,T}^{step}$ as above:

$$u = \sum_{i,k,\ell=1}^{N} I_{A_i} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} = \sum_{i=1}^{N} I_{A_i} \Big(\sum_{k,\ell=1}^{N} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} \Big)$$

$$= \sum_{i=1}^{N} I_{A_i} u^i, \qquad \text{where } u^i \in \mathcal{U}_{t,T}^t,\ 1 \le i \le N,$$

and from the uniqueness of the solutions of the controlled forward and backward SDEs:

$J(t,x,u) = \sum_{i=1}^{N} I_{A_i} J(t,x,u^i) \le \sup_{1\le i\le N} J(t,x,u^i) \le \sup_{u'\in\mathcal{U}_{t,T}^t} J(t,x,u')$,

and, consequently, since $\mathcal{U}_{t,T}^t \subset \mathcal{U}_{t,T}$,

$$W(t,x) = \sup_{u'\in\mathcal{U}_{t,T}^t} J(t,x,u');$$

the right-hand side is deterministic and so is $W(t,x)$.

Peng's argument doesn't work for stochastic differential games:

- One cannot restrict to continuous strategies;

- W.r.t. which norm should the spaces of admissible strategies be approximable by which "admissible step strategies"?

<u>Here</u> new approach for the proof that W is deterministic; even continuity of the coefficients in (u,v) is not needed.

Proof the the upper and lower value functions are deterministic: main tool is a Girsanov transformation argument (*at the blackboard*).

Dynamic Programming Principle (DPP)

Some Preparation: *Stochastic Backward Semigroup*, S.Peng,1997: book on his BSDE method for stochastic control problems:

S.Peng, (1997)*BSDE and stochastic optimizations; Topics in stochastic analysis.* J.Yan, S.Peng, S.Fang and L.Wu, Chapter 2, Science Press. Beijing (in Chinese).

Given

$(t,\zeta)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, $\delta>0(t+\delta\leq T)$, $u\in\mathcal{U}_{t,t+\delta}$, $v\in\mathcal{V}_{t,t+\delta}$,

$\eta\in L^2(\Omega,\mathcal{F}_{t+\delta},P;\mathbb{R})$ - terminal condition for time horizon $t+\delta$,

we put

$$G^{t,x;u,v}_{s,t+\delta}[\eta]:=\tilde{Y}_s,\ s\in[t,t+\delta],$$

where $(\tilde{Y},\tilde{Z}) \in \mathcal{S}^2_{\mathbb{F}}(t,t+\delta) \times L^2_{\mathbb{F}}(t,t+\delta;\mathbb{R}^d)$ is the unique solution of the following BSDE with time horizon $t+\delta$:

$$\begin{cases} d\tilde{Y}_s &= -f(s,X_s^{t,x;u,v},\tilde{Y}_s,\tilde{Z}_s,u_s,v_s)ds - \tilde{Z}_s^{t,x;u,v}dB_s, \quad \in [t,t+\delta], \\ \tilde{Y}_{t+\delta} &= \eta; \end{cases}$$

$X^{t,x;u,v}$ is the solution of our doubly controlled stochastic system (the forward SDE).

<u>Remark</u>:

(i) (*The semigroup property*) For $0 \le t \le s \le s' \le t+\delta \le T$,

$$G^{t,x;u,v}_{s,s'}[G^{t,x;u,v}_{s',t+\delta}[\eta]] = G^{t,x;u,v}_{s,t+\delta}[\eta].$$

(ii) $G^{t,x;u,v}_{s,T}[\Phi(X_T^{t,x;u,v})] = Y_s^{t,x;u,v}$, P-a.s., $s \in [t,T]$.

In particular, for $s=t$,

$$G^{t,x;u,v}_{t,T}[\Phi(X_T^{t,x;u,v})] = J(t,x;u,v), \quad \text{P-a.s.}.$$

(iii) $J(t,x;u,v) = Y_t^{t,x;u,v} = G^{t,x;u,v}_{t,T}[\Phi(X_T^{t,x;u,v})]$

$$= G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}] = G_{t,t+\delta}^{t,x;u,v}[J(t+\delta, X_{t+\delta}^{t,x;u,v}; u,v)].$$

The latter relation follows from the uniqueness of the solution of the forward and the backward equations: for $\zeta = X_{t+h}^{t,x;u,v}(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$Y_{t+\delta}^{t,x;u,v} = Y_{t+h}^{t+h, X_{t+h}^{t,x;u,v}; u,v} = Y_{t+h}^{t+h,\zeta;u,v} = J(t+\delta, \zeta; u,v)$$
$$= J(t+\delta, X_{t+h}^{t,x;u,v}; u,v).$$

(iv) If f doesn't depend on (y,z) we have the classical case of conditional expectation:

$$G_{t,t+\delta}^{t,x;u,v}[\eta] = E\left[\eta + \int_t^{t+\delta} f(s, X_s^{t,x;u,v}, u_s, v_s) ds | \mathcal{F}_t\right], \text{ P-a.s.}$$

Taking now $\eta = W(t+\delta, X_{t+\delta}^{t,x;u,v})$ (resp., $U(t+\delta, X_{t+\delta}^{t,x;u,v})$) it becomes clear from the classical DPP from control problems that our DPP shall write as follows:

Theorem 2 (DPP): For any $0 \leq t < t+\delta \leq T,\, x \in \mathbb{R}^n$,

$$W(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})];$$

$$U(t,x) = \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{essinf}_{v \in \mathcal{V}_{t,t+\delta}} G^{t,x;\alpha(v),v}_{t,t+\delta}[U(t+\delta, X^{t,x;\alpha(v),v}_{t+\delta})].$$

Remark: If $f(x,y,z,u,v)$ is independent of (y,z) the above DPP writes:

$$W(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} E[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta}) + \int_t^{t+\delta} f(s, X^{t,x;u,\beta(u)}_s, u_s, v_s)ds|\mathcal{F}_t];$$

analogous for $U(t,x)$.

Sketch of proof: auxiliary function:

$$W_\delta(t,x) := \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})];$$

(i) $W_\delta(t,x)$ is deterministic: same Girsanov transformation argument as for $W(t,x)$.

(ii) For any $\varepsilon > 0$, and for any $\beta \in \mathcal{B}_{t,T}$, there exists some $u^\varepsilon \in \mathcal{U}_{t,T}$ such that

$$W_\delta(t,x) \le J(t,x;u^\varepsilon,\beta(u^\varepsilon)) + \varepsilon,\ P\text{-a.s.},$$

from where: $W_\delta(t,x) \le W(t,x)$. (*Calculus at the blackboard*.)

(iii) For any $\varepsilon > 0$, there exists $\beta^\varepsilon \in \mathcal{B}_{t,T}$ such that $\forall u \in \mathcal{U}_{t,T}$:

$$W_\delta(t,x) \ge J(t,x;u,\beta^\varepsilon(u)) - \varepsilon,\ \text{P-a.s.},$$

from where: $W_\delta(t,x) \ge W(t,x)$. (*Calculus at the blackboard*.)

With the help of the DPP we can prove the following

Theorem 3. $W(.,x)$ and $U(.,x)$ are $\frac{1}{2}-$Hölder continuous, for all $x \in \mathbb{R}^n$: There is some $C \in \mathbb{R}_+$ such that, for every $x \in \mathbf{R}^n,\ t,t' \in [0,T]$,

$$|W(t,x) - W(t',x)| + |U(t,x) - U(t',x)| \le C(1+|x|)|t-t'|^{\frac{1}{2}}.$$

(*Explanation at blackboard.*)

Bellman-Isaacs equations. Existence theorem.

We consider the Hamiltonian

$H(t,x,y,p,S,u,v)$

$:= \frac{1}{2} tr(\sigma\sigma^T(t,x,u,v)S) + b(t,x,u,v).p + f(t,x,y,p.\sigma(t,x,u,v),u,v),$

$(t,x,y,p,S,u,v) \in [0,T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n \times U \times V.$

$$H^-(t,x,y,p,S) := \sup_{u\in U} \inf_{v\in V} H(t,x,y,p,S,u,v);$$
$$H^+(t,x,y,p,S) := \inf_{v\in V} \sup_{u\in U} H(t,x,y,p,S,u,v).$$

We will show that, in viscosity sense, we have the following Bellman-Isaacs equations:

$$\frac{\partial W}{\partial t}(t,x)+H^{-}(t,x,W,DW,D^2W)=0, W(T,x)=\Phi(x), \qquad (3)$$

and

$$\frac{\partial U}{\partial t}(t,x)+H^{+}(t,x,U,DU,D^2U)=0, U(T,x)=\Phi(x). \qquad (4)$$

More precisely,

Theorem 4 (Existence Theorem): $W \in C_\ell([0,T]\times\mathbf{R}^n)$ is a viscosity solution of equation (3), and $U \in C_\ell([0,T]\times\mathbf{R}^n)$ is a viscosity solution of equation (4).

(*Recall of the notion of viscosity solution if necessary*.)

We come after back to the proof of the existence theorem.

Theorem 5 (Comparison Principle): Let $u_1 \in$USC$([0,T]\times\mathbb{R}^n)$ be a viscosity subsolution of (3) (resp., of (4)) and $u_2 \in$LSC$([0,T]\times\mathbb{R}^n)$ be a viscosity supersolution of (3) (resp., of (4)). Moreover, we suppose that both functions belong to the class of measurable functions V with the following growth condition:

$\exists A > 0$ such that, uniformly in $t \in [0,T]$,

$V(t,x)\exp\{-A[\ln|x|]^2\}\left(=\dfrac{V(t,x)}{|x|^{A\ln|x|}}\right) \longrightarrow 0$ as $|x| \to +\infty$.

Then $u_1 \leq u_2$, on $[0,T]\times\mathbb{R}^n$.

Corollary. Let u_1 and u_2 be continuous viscosity solutions of (3) (resp., of (4)). Moreover, we suppose that both functions satisfy the above growth condition. Then $u_1 = u_2$, on $[0,T]\times\mathbb{R}^n$.

Remarks 1: • Barles, Buckdahn, Pardoux (1997) proved that this growth condition is the optimal one for the uniqueness of the (viscosity) solution of the heat equation.

• The proof of the uniqueness theorem adapts the argument of Barles, Buckdahn, Pardoux (1997) to Bellman-Isaacs equations (and, hence, also to Hamilton-Jacobi-Bellman equations).

Remarks 2: • $W \in C_\ell([0,T]\times\mathbb{R}^n)$ (resp., $U \in C_\ell([0,T]\times\mathbb{R}^n)$) is the unique viscosity solution of (3) (resp., (4)) in the class of continuous functions with the above growth condition, and so in particular in $C_p([0,T]\times\mathbb{R}^n)$.

• Notice that $H^- \le H^+$; consequently, W is a viscosity subsolution of (4), and from the comparison principle: $W \le U$. This justifies the names "lower value function" for W and "upper" value function for U.

• If the Isaacs' condition holds: $H^- = H^+$ on $[0,T]\times\mathbb{R}^n\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{S}^n$, then the equations (3) and (4) are the same, and from the uniqueness of the viscosity solution in $C_p([0,T]\times\mathbb{R}^n)$: $W = U$. One says the "game has a value".

• For the case that $f(s,x,y,z,u,v)$ doesn't depend on (y,z), W.H.FLEMING, P.E.SOUGANIDIS have got the same Bellman-Isaacs equations as we have got. From the uniqueness of the viscosity solutions in $C_p([0,T]\times\mathbb{R}^n)$:

$$\overline{W}(t,x)\big(:=\inf_{\beta\in\mathcal{B}^t_{t,T}}\sup_{u\in\mathcal{U}^t_{t,T}} J(t,x;u,\beta(u))\big) = W(t,x);$$

$$\overline{U}(t,x)\big(:= \inf_{\alpha\in\mathcal{A}_{t,T}^t} \inf_{v\in\mathcal{V}_{t,T}^t} J(t,x;\alpha(v),v)\big) = U(t,x).$$

Sketch of the proof of the existence theorem:

We prove that W is a continuous viscosity solution of the PDE

$$\frac{\partial W}{\partial t}(t,x) + H^-(t,x,W,DW,D^2W) = 0, W(T,x) = \Phi(x), \qquad (3)$$

with

$$H^-(t,x,y,p,S) := \sup_{u\in U} \inf_{v\in V} H(x,y,p,S,u,v);$$

and

$$H(x,y,p,S,u,v) := \frac{1}{2}tr(\sigma\sigma^T(x,u,v)S) + f(x,y,p.\sigma(x,u,v),u,v),$$

$(x,y,p,S,u,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U \times V$(for shortness but without restriction of the method: $b=0$; coefficients don't depend on time s).

Let $\varphi \in C^3_{\ell,b}([0,T]\times\mathbb{R}^n)$ be an arbitrary but fixed test function. We define:

$$L_{x,u,v}\varphi(s,x)=\frac{\partial}{\partial s}\varphi(s,x)+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,u,v)D^2\varphi(s,x)),$$

and

$$F(s,x,y,z,u,v):=L_{x,u,v}\varphi(s,x)$$
$$+f(s,x,y+\varphi(s,x)),z+D\varphi(s,x)\sigma(x,u,v),u,v).$$

Notice:

$$\frac{\partial}{\partial t}\varphi(t,x)+H^-(t,x,(\varphi,D\varphi,D^2\varphi)(t,x))=\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v).$$

So we have to prove that if $W-\varphi\leq$ (resp., $\geq$)$=(W-\varphi)(t,x)=0$ then

$\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v)\geq 0$ ($\longrightarrow$ subsolution)

(resp., $\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,u,v)\leq 0$ ($\longrightarrow$ supersolution)).

Peng's BSDE method: *"Approximating BSDEs"*

1st BSDE: For $0<\delta\leq T-t$ small, $u\in\mathcal{U}_{t,t+\delta}$, $v\in\mathcal{V}_{t,t+\delta}$:

$$dY_s^{1,u,v,\delta}=-F(s,X_s^{t,x,u,v},Y_s^{1,u,v,\delta},Z_s^{1,u,v,\delta},u_s,v_s)ds+Z_s^{1,u,v,\delta}dB_s,$$
$$Y_{t+\delta}^{1,u,v,\delta}=0.$$

Notice: • The BSDE admits a unique solution $(Y^{1,u,v,\delta}, Z^{1,u,v,\delta}) \in \mathcal{S}^2_{\mathbb{F}}(t,t+\delta) \times L^2_{\mathbb{F}}(t,t+\delta;\mathbb{R}^d)$.

• $Y_s^{1,u,v,\delta} = G^{t,x,u,v}_{s,t+\delta}[\varphi(t+\delta, X^{t,x,u,v}_{t+\delta})] - \varphi(s, X_s^{t,x,u,v})$, $s \in [t,t+\delta]$, P-a.s.

(*Idea of the proof: evtl. at the blackboard.*)

The 1st BSDE will translate the DPP in BSDE property. Approximation of the 1st BSDE:

2nd BSDE: For $0 < \delta \leq T-t$ small, $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$:

$$dY_s^{2,u,v,\delta} = -F(s,x,Y_s^{2,u,v,\delta}, Z_s^{2,u,v,\delta}, u_s, v_s)ds + Z_s^{2,u,v,\delta} dB_s,\ s \in [t,t+\delta],$$
$$Y^{2,u,v,\delta}_{t+\delta} = 0.$$

$\Big\{$ Recall:

$$dY_s^{1,u,v,\delta} = -F(s,X_s^{t,x,u,v},Y_s^{1,u,v,\delta}, Z_s^{1,u,v,\delta}, u_s, v_s)ds + Z_s^{1,u,v,\delta} dB_s,$$
$$Y^{1,u,v,\delta}_{t+\delta} = 0.$$

Our objective: To approximate the 1st BSDE -the key to use the DPP- by the 2nd BSDE, and the 2nd BSDE by a deterministic ordinary differential equation with terminal condition.

Lemma. There is some $C \in \mathbb{R}_+$ s.t., for all $\delta \in (0, T-t]$ sufficiently small and all $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$:

$$|Y_t^{1,u,v,\delta} - Y_t^{2,u,v,\delta}| \leq C\delta^{3/2}, P\text{-a.s.}$$

(*Idea of proof at blackboard.*)

Let $F_0(s,x,y,z) = \sup_{u \in U} \inf_{v \in V} F(s,x,y,z,u,v)$.

<u>3rd BSDE</u>: For $0 < \delta \leq T-t$ small:

$$dY_s^{0,\delta} = -F_0(s,x,Y_s^{0,\delta},0)ds\big(+0dB_s\big), s \in [t,t+\delta],$$

$$Y_{t+\delta}^{0,\delta} = 0.$$

Lemma. $\text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} \text{essinf}_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v,\delta} = Y_t^{0,\delta}$.

(*Proof at the blackboard.*)

These 3 BSDEs allow to prove:

1)W is a subsolution: (*blackboard*)

2)W is a supersolution: (*blackboard*)

Perspectives (and work which is already done):

• 2-Person zero-sum SDG with reflection at one obstacle, at two obstacles (LI JUAN, R.B., submitted, arXiv)

• 2-Person zero-sum SDG with jumps (in redaction; LI JUAN, R.B.)

• Nonzero-sum SDGs, existence of Nash equilibrium points, Non anticipative Strategies with Delay (NAD-strategies); this concept allows to study games "NAD-strategy against NAD-strategy" (advantage: "symmetry" between both players; disadvantage: Nash equilibria can be studied only by ε-approximations): (P.CARDALIAGUET, C.RAINER, R.B., 2004)

- SDG with asymmetric information (P.CARDALIGUET, C.RAINER, submitted, web page of C.Rainer)
- Measure-valued differential games (P.CARDALIAGUET, M.QUINCAMPOIX)
- A lot of other works.