G-expectations in infinite dimensional spaces and related PDE

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Abstract

Despite of increasing popularity of theory of G-expectation the most part of results are dedicated to the finite dimensionale case. In this talk I would like to present some results obtaining in infinite dimensions.

Let H is a separable Hilbert space. Consider monotone, sublinear, L(H)-continuous functional defined on the set of compact, non-negative, symmetric operators. We will call it G-function. Every G-function can be represented in the form $G(A) = \frac{1}{2} \sup_{B \in \Sigma} \operatorname{Tr}[A \cdot B]$. G also defines G-normal distributed random variable with covarience set Σ : $X \sim N_G(0, \Sigma)$; the sublinear expectation $\mathbb{E}[\cdot]$ which we will call G-expectation, such that $G(A) := \frac{1}{2} \mathbb{E}[\langle AX, X \rangle]$; and G-Brownian motion $B_t \sim N_G(0, t \cdot \Sigma)$.

G -Brownian motion is related to a fully nonlinear partial differential equation in the following way:

If (B_t) is a G -Brownian motion and $u(t,x) := \mathbb{E}[\phi(x + B_t)]$, then u is the viscosity solution of the following G -heat equation:

$$\frac{\partial u}{\partial t}(t,x) + G\left(\frac{\partial^2 u}{\partial x^2}\right)(t,x) = 0$$
$$u(T,x) = \phi(x)$$

Together with it we can introduce the stochastic integral over G-Brownian motion. One of the essential properties of it is obtained Ito isometry inequality:

$$\mathbb{E}\Big[\| \int_0^T \Phi(t)\, dB_t \|_{\mathrm{H}}^2\Big] \leq \int_0^T \mathbb{E}\Big[\sup_{Q \in \Sigma} \operatorname{Tr}\big[\Phi Q \Phi^*\big]\Big] dt.$$

This helps us to introduce correctly Ornstein-Uhlenbeck process $I_t = \int_0^t e^{(t-s)A} dB_s$, where A is generator of C_0 -semigroup (e^{tA}) .

It turns out that if process $X_t := e^{(T-t)A}x + \int_t^T e^{(t-\sigma)A} dB_\sigma$ and $u(t,x) := \mathbb{E}[\phi(X_t)]$, then u is the viscosity solution of the following (Kolmogorov) G -PDE:

$$\frac{\partial u}{\partial t}(t,x) + G\left(\frac{\partial^2 u}{\partial x^2}\right)(t,x) + \langle Ax, \nabla_x u(t,x) \rangle = 0$$
$$u(T,x) = \phi(x)$$

So, in this talk we try to give an explicit notion of G-expectation theory in infinite dimensions and their connection to the viscosity solutions of some types of PDEs.