Roscoff, March 19, 2010

NFL2

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March 19, 2010

Outline

Example

Two-asset 1-period model : $S_0^1 = S_0^2 = 1$, $S_1^1 = 1$, S_1^2 takes values $1+\varepsilon$ and $1-\varepsilon > 0$ with probabilities $1/2$.
The filtration is generated by S.
$K_0^* = \mathrm{cone}\,\{(1,2),(2,1)\}$, $K_1^* = \mathbf{R}_+\mathbf{1}$. Then $\widehat{K}_1^* = \mathbf{R}_+ S_1$.
The process Z with $Z_0 = (1,1)$ and $Z_1 = S_1$ is a strictly consistent price system, so the NA^w-property holds.
Let $v \in C$ where $C^* = \mathrm{cone}\,\{(1,1+\varepsilon),(1,1-\varepsilon)\} \subseteq \widehat{K}_1$.
For $\varepsilon \in]0,1/2[$ the cone C is strictly larger than $\widehat{K}_0 = K_0$.

The investor the initial endowment $v \in C \setminus K_0$ will solvent at $T = 1$ though not solvent at the date zero. One can introduce small transaction costs at time $T = 1$ to get the same conclusion for a model with efficient friction.

Arbitrage of the second kind
Setting

Let $G = (G_t)$, $t = 0, 1, ..., T$, be an adapted cone-valued process, $A_s^T := \sum_{t=s}^T L^0(-G_t, \mathcal{F}_t)$.
The models admits *arbitrage opportunities of the 2nd kind* if there exist $s \le T - 1$ and an $\mathcal{F}_s$-measurable d-dimensional random variable ξ such that $\Gamma := \{\xi \notin G_s\}$ is not a null-set and

$$(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T) \neq \emptyset,$$

i.e. $\xi = \xi_s + ... + \xi_T$ for some $\xi_t \in L^0(G_t, \mathcal{F}_t)$, $s \le t \le T$. If such ξ does exist then, in the financial context where $G = \widehat{K}$, an investor having $I_\Gamma \xi$ as the initial endowments at time s, may use the strategy $(I_\Gamma \xi_t)_{t \ge s}$ and get rid of all debts at time T.

NA2 property
Rasonyi theorem (2008)

The model has *no arbitrage opportunities of the 2nd kind* (i.e. has the *NA*2-property) if s and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_s)$ the intersection $(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T)$ is non-empty only if $\xi \in L^0(G_s, \mathcal{F}_s)$. Alternatively, the *NA*2-property can be expressed as :

$$L^0(\mathbf{R}^d, \mathcal{F}_s) \cap (-A_s^T) = L^0(G_s, \mathcal{F}_s) \qquad \forall s \le T.$$

Theorem

Suppose that the efficient friction condition is fulfilled, i.e. $G_t \cap (-G_t) = \{0\}$ *and* $\mathbf{R}^d_+ \subseteq G_t$ *for all* t*. Then the following conditions are equivalent :*
(*a*) *NA*2 *;*
(*b*) $L^0(\mathbf{R}^d, \mathcal{F}_s) \cap L^0(G_{s+1}, \mathcal{F}_s) \subseteq L^0(G_s, \mathcal{F}_s)$ *for all* $s < T$ *;*
(*c*) $\mathrm{cone}\,\mathrm{int}\, E(G^*_{s+1} \cap \bar{\mathcal{O}}_1(0) | \mathcal{F}_s) \supseteq \mathrm{int}\, G^*_s$ *(a.s.) for all* $s < T$ *;*
(*d*) *for any* $s < T$ *and* $\eta \in L^1(\mathrm{int}\, G^*_s, \mathcal{F}_s)$ *there is* $Z \in \mathcal{M}^T_s(\mathrm{int}\, G^*)$ *such that* $Z_s = \eta$ *(***PCV** *- "Prices are consistently extendable".)*

Tools

Conditional expectations

A subset $\Xi \in L^p$ is called *decomposable* if with two its elements ξ_1, ξ_2 it contains also $\xi_1 I_A + \xi_2 I_{A^c}$ whatever is $A \in \mathcal{F}$.

Proposition

Let Ξ be a closed subset of $L^p(\mathbf{R}^d)$, $p \in [0,\infty[$. Then $\Xi = L^p(\Gamma)$ for some Γ which values are closed sets if and only if Ξ is decomposable, .

Proposition

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let Γ be a measurable mapping which values are non-empty closed convex subsets of $\bar{\mathcal{O}}_1(0) \subset \mathbf{R}^d$. Then there is a $\mathcal{G}$-measurable mapping, $E(\Gamma|\mathcal{G})$, which values are non-empty convex compact subsets of $\bar{\mathcal{O}}_1(0)$ and the set of its $\mathcal{G}$-measurable a.s. selectors coincides with the set of $\mathcal{G}$-conditional expectations of a.s. selectors of Γ.

Outline

Model

- We are given set-valued adapted processes $G = (G_t)_{t\in[0,T]}$ and $G^* = (G^*_t)_{t\in[0,T]}$ whose values are closed cone in $\mathbb{R}^d$,

$$G^*_t(\omega) = \{y : \ yx \geq 0 \ \forall x \in G_t(\omega)\}.$$

  "Adapted" means that
  $\big\{(\omega, x) \in \Omega \times \mathbb{R}^d : \ x \in G_t(w)\big\} \in \mathcal{F}_t \otimes \mathcal{B}^d$.
- G_t are proper $(\mathrm{EF}$-condition$)$: $G_t \cap (-G_t) = \{0\}$.
  We assume also that G_t dominate $\mathbb{R}^d_+$, i.e. $G^*\backslash\{0\} \subset \operatorname{int} \mathbb{R}^d_+$.
- In financial context $G_t = \widehat{K}_t$, the solvency cone in physical units.
- For each $s \in]0, T]$ we are given a convex cone $\mathcal{Y}^T_s$ of optional $\mathbb{R}^d$-valued processes $Y = (Y_t)_{t\in[s,T]}$ with $Y_s = 0$.
- Assumption : if sets $A^n \in \mathcal{F}_s$ form a countable partition of Ω and $Y^n \in \mathcal{Y}^T_s$, then $\sum_n Y^n I_{A^n} \in \mathcal{Y}^T_s$.

Notations

- for d-dimensional processes Y and Y' the relation $Y \geq_G Y'$ means $Y_t - Y'_t \in G_t$ a.s. for every t ;
- $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^d_+$;
- $\mathcal{Y}^T_{s,b}$ denotes the subset of $\mathcal{Y}^T_s$ formed by the processes Y dominated from below : $Y_t + \kappa\mathbf{1} \in G_t$ for some constant κ ;
- $\mathcal{Y}^T_{s,b}(T)$ is the set of random variables Y_T where $Y \in \mathcal{Y}^T_{s,b}$;
- $\mathcal{A}^T_{s,b}(T) = (\mathcal{Y}^T_{s,b}(T) - L^0(G_T, \mathcal{F}_T)) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ and $\overline{\mathcal{A}^T_{s,b}(T)}^w$ is its closure in $\sigma\{L^\infty, L^1\}$;
- $\mathcal{M}^T_s(G^*)$ is the set of martingales $Z = (Z_t)_{t\in[s,T]}$ evolving in G^*, i.e. such that $Z_t \in L^1(G^*_t, \mathcal{F}_t)$.

Conditions

Standing Hypotheses

- **S**$_1$ $E\xi Z_T \le 0$ for all $\xi \in \mathcal{Y}^T_{s,b}(T)$, $Z \in \mathcal{M}^T_s(G^*)$, $s \in [0, T[$.
- **S**$_2$ $\cup_{t \ge s} L^\infty(-G_t, \mathcal{F}_t) \subseteq \mathcal{Y}^T_{s,b}(T)$ for each $s \in [0, T]$.

Properties of Interest

- **NFL** $\overline{\mathcal{A}^T_{s,b}(T)}^w \cap L^\infty(\mathbb{R}^d_+, \mathcal{F}_T) = \{0\}$ for each $s \in [0, T[$.
- **NFL2** For each $s \in [0, T[$ and $\xi \in L^\infty(\mathbb{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{\mathcal{A}^T_{s,b}(T)}^w) \cap L^\infty(\mathbb{R}^d_+, \mathcal{F}_T) \ne \emptyset$$

  only if $\xi \in L^\infty(G_s, \mathcal{F}_s)$.
- **MCPS** For any $\eta \in L^1(\mathrm{int}\ G^*_s, \mathcal{F}_s)$, there is $Z \in \mathcal{M}^T_s(G^* \setminus \{0\})$ with $Z_s = \eta$.
- **B** If ξ is an $\mathcal{F}_s$-measurable $\mathbb{R}^d$-valued random variable such that $Z_s\xi \ge 0$ for every $Z \in \mathcal{M}^T_s(G^*)$, then $\xi \in G_s$.

FTAP

Theorem

NFL $\Leftrightarrow \mathcal{M}_0^T(G^*\backslash\{0\}) \neq \emptyset$.

Proof. ($\Leftarrow$) Let $Z \in \mathcal{M}_0^T(G^*\backslash\{0\})$. Then the components of Z_T are strictly positive and $EZ_T\xi > 0$ for all $\xi \in L^\infty(\mathbb{R}_+^d, \mathcal{F}_T)$ except $\xi = 0$. On the other hand, $E\xi Z_T \le 0$ for all $\xi \in \mathcal{Y}_{s,b}^T(T)$ and so for all $\xi \in \overline{\mathcal{A}_{s,b}^T(T)}^w$.

($\Rightarrow$) The Kreps–Yan theorem on separation of closed cones in $L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ implies the existence of $\eta \in L^1(\operatorname{int}\mathbb{R}_+^d, \mathcal{F}_T)$ such that $E\xi\eta \le 0$ for every $\xi \in \overline{\mathcal{A}_{s,b}^T(T)}^w$, hence, by virtue of the hypothesis $\mathbf{S}_2$, for all $\xi \in L^\infty(-G_t, \mathcal{F}_t)$. Let us consider the martingale $Z_t = E(\eta|\mathcal{F}_t)$, $t \ge s$, with strictly positive components. Since $EZ_t\xi = E\xi\eta \ge 0$, $t \ge s$, for every $\xi \in L^\infty(G_t, \mathcal{F}_t)$, it follows that $Z_t \in L^1(G_t, \mathcal{F}_t)$ and, therefore, $Z \in \mathcal{M}_s^T(G^*\backslash\{0\})$.

Main Result

Theorem

The following relations hold :

$$\textbf{MCPS} \Rightarrow \{\textbf{B},\ \mathcal{M}_0^T(G^*\backslash\{0\}) \neq \emptyset\} \Leftrightarrow \{\textbf{B},\ \textbf{NFL}\} \Leftrightarrow \textbf{B} \Leftrightarrow \textbf{NFL2}.$$

If, moreover, the sets $\mathcal{Y}_{s,b}^T(T)$ are Fatou-closed for any $s \in [0, T[$. Then all five conditions are equivalent.

In the above formulation the Fatou-closedness means that the set $\mathcal{Y}_{s,b}^T(T)$ contains the limit on any a.s. convergent sequence of its elements provided that the latter is bounded from below in the sense of partial ordering induced by G_T.

Discrete-time model,1

B^p If $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$ and $Z_s\xi \geq 0$ for any $Z \in \mathcal{M}_s^T(G^*)$ with $Z_T \in L^p$, then $\xi \in G_s$ (a.s.), $s = 0, ..., T$.

NAA^p $\overline{\mathcal{A}_{0,b}^T(T)}^{L^p} \cap L^p(\mathrm{R}_+^d, \mathcal{F}_T) = \{0\}$.

Lemma

The conditions **NAA**^{p} *for* $p \in [1, \infty[$ *are measure-invariant and any of them is equivalent to* **NAA**^{0} *as well as to the condition* **NFL** *(which, in turn, is equivalent, to the existence of a bounded process* Z *in* $\mathcal{M}_s^T(G^* \setminus \{0\})$*.*

NAA2^p For each $s = 0, 1, ..., T-1$ and $\xi \in L^\infty(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{\mathcal{A}_{s,b}^T(T)}^{L^p}) \cap L^0(\mathrm{R}_+^d, \mathcal{F}_T) \neq \emptyset$$

only if $\xi \in L^\infty(G_s, \mathcal{F}_s)$.

Discrete-time model,2

Lemma

The conditions $\mathbf{NAA2}^p$ *for* $p \in [1, \infty[$ *are measure-invariant and any of them is equivalent to* $\mathbf{NAA2}^0$ *as well as to the condition* **NFL2** *(which, in turn, is equivalent to the condition* **B***).*

Thus, for the discrete-time model with efficient friction

$$\mathbf{MCPS} \Leftrightarrow \{\mathbf{B},\ \mathcal{M}_0^T(G^*\backslash\{0\}) \neq \emptyset\} \Leftrightarrow \{\mathbf{B},\ \mathbf{NFL}\} \Leftrightarrow \mathbf{B} \Leftrightarrow \mathbf{NFL2}$$

Formally, all properties above are different from those in the Rásonyi theorem **PCE** $\Leftrightarrow$ **NA2**. Recall that
$A_s^T := \sum_{t=s}^T L^0(-G_t, \mathcal{F}_t))$ and
NA2 For each $s \in [0, T[$ and $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + A_s^T) \cap L^0(\mathrm{R}_+^d, \mathcal{F}_T) \neq \emptyset$$

only if $\xi \in L^0(G_s, \mathcal{F}_s)$.

Discrete-time model, 3

However, this equivalence follows from two simple observations. First, **NFL2** $\Leftrightarrow$ **NA2**. Indeed, due to the coincidence of L^0-closures of A_s^T and $\mathcal{A}_s^T(T)$, **NFL2** is equivalent to :

NA2′ For each $s \in [0, T[$ and $\xi \in L^0(\mathrm{R}^d, \mathcal{F}_s)$

$$(\xi + \overline{A_s^T}^{L^0}) \cap L^0(\mathrm{R}^d_+, \mathcal{F}_T) \neq \emptyset$$

only if $\xi \in L^0(G_s, \mathcal{F}_s)$.

This us aproperty stronger than **NGV**. On the other hand, successive application of **NGV** in combination with the efficient friction condition implies that the identity $\sum_{t=s}^T \xi_t = 0$ with $\xi_t \in L^0(-G_t, \mathcal{F}_t)$ may hold only if all $\xi_t = 0$. But it is well-known that in such a case A_s^T is closed in L^0.

Discrete-time model, 4

Second, **PCE** $\Leftrightarrow$ **MCPS**. The implication $\Rightarrow$ is trivial. The inverse implication can be proven by backward induction. Indeed, for $s = T$ there is nothing to prove. Suppose that for $s = t + 1 \leq T$ the claim holds. In particular, there is $\tilde{Z} \in \mathcal{M}_{t+1}^T(\mathrm{int}\, G)$ with $|\tilde{Z}_{t+1}| = 1$. Put $\tilde{Z}_t := E(\tilde{Z}_{t+1}|\mathcal{F}_t)$. Let $\eta \in L^1(\mathcal{F}_t, G_t)$ with $|\eta| = 1$. Take α be the $\mathcal{F}_t$-measurable random variable equal to the half of the distance of η_t to ∂G_t. Then $\eta - \alpha\tilde{Z}_t \in L^1(\mathrm{int}\, G_t, \mathcal{F}_t)$. By **MCPS** there exists $Z \in \mathcal{M}_t^T(G \setminus \{0\})$ with $Z_t \in \mathcal{M}_t^T(G \setminus \{0\})$ and $Z_t = \eta - \alpha\tilde{Z}_t$. Since $Z + \alpha\tilde{Z} \in \mathcal{M}_t^T(\mathrm{int}\, G)$ and $Z_t + \alpha\tilde{Z}_t = \eta$, we conclude.