Constrained portfolio choices in the decumulation phase of a pension plan

M. Di Giacinto[1] **S. Federico**[2] **F. Gozzi**[3] **E. Vigna**[4]

[1]Università degli Studi di Cassino - Italy

[2]LUISS University, Roma - Italy

[3]LUISS University, Roma - Italy

[4]Università degli studi di Torino - Italy

Workshop on Stochastic Control in Finance
Roscoff, March 18–23, 2010

Plan of the talk

- Motivations.
- State equation and optimization problems.
- (P1) Constraints on the strategies.
    - Explicit solution and optimal feedback by verification.
- (P2) Constraints on the strategies and on the wealth.
    - Viscosity approach.
    - Regularity of the value function.
    - Explicit solution and optimal feedback by verification in a special case.
- Future targets.

Motivations

Depending on the laws, in many countries the retiree is allowed for a certain period after retirement:

1. to withdraw a periodic income from the fund;
2. to invest the rest of the fund in the period between retirement and annuitization.

Thus, in this period the pensioner can:

1. decide how much of the fund to withdraw at any time;
2. decide the strategy to adopt to invest the fund at her/his disposal.

→ Investment/consumption Merton problem, which can be solved using, e.g. stochastic optimal control techniques.

We focus on the last problem:

- Fixed withdrawal/consumption rate.
- How to invest optimally?
  ⟶ **Portfolio allocation problem with special features.**

Motivations

Depending on the laws, in many countries the retiree is allowed for a certain period after retirement:

1. to withdraw a periodic income from the fund;
2. to invest the rest of the fund in the period between retirement and annuitization.

Thus, in this period the pensioner can:

1. decide how much of the fund to withdraw at any time;
2. decide the strategy to adopt to invest the fund at her/his disposal.

→ Investment/consumption Merton problem, which can be solved using, e.g. stochastic optimal control techniques.

We focus on the last problem:

- Fixed withdrawal/consumption rate.
- How to invest optimally?

  ⟶ **Portfolio allocation problem with special features.**

Motivations

Depending on the laws, in many countries the retiree is allowed for a certain period after retirement:

1. to withdraw a periodic income from the fund;
2. to invest the rest of the fund in the period between retirement and annuitization.

Thus, in this period the pensioner can:

1. decide how much of the fund to withdraw at any time;
2. decide the strategy to adopt to invest the fund at her/his disposal.

→ Investment/consumption Merton problem, which can be solved using, e.g. stochastic optimal control techniques.

We focus on the last problem:

- Fixed withdrawal/consumption rate.
- How to invest optimally?
  ⟶ **Portfolio allocation problem with special features.**

Motivations

Depending on the laws, in many countries the retiree is allowed for a certain period after retirement:

1. to withdraw a periodic income from the fund;
2. to invest the rest of the fund in the period between retirement and annuitization.

Thus, in this period the pensioner can:

1. decide how much of the fund to withdraw at any time;
2. decide the strategy to adopt to invest the fund at her/his disposal.

$\rightarrow$ Investment/consumption Merton problem, which can be solved using, e.g. stochastic optimal control techniques.
We focus on the last problem:

- Fixed withdrawal/consumption rate.
- How to invest optimally?
  $\longrightarrow$ **Portfolio allocation problem with special features.**

The state equation and the optimization problems

- $t = 0$ retirement time;
- $T > 0$ annuitization time (horizon of the problem);
- x_0 fund wealth at $t = 0$;
- $X(\cdot)$ process representing the fund wealth **(state variable)**;
- $\pi(\cdot)$ process representing the amount of money invested in the risky asset **(control variable)**;
- b_0 consumption rate of the pensioner;
- r, λ, σ usual market parameters in the Black-Scholes model.

$$\begin{cases} dX(s) = [rX(s) + \sigma\lambda\pi(s) - b_0]\, ds + \sigma\pi(s) dB(s), & s \in [0, T] \\ X(0) = x_0. \end{cases}$$

- $F(\cdot)$ is a target we aim to reach.

$$F(s) = \frac{b_0}{r} + \left(F - \frac{b_0}{r}\right) e^{-r(T-s)},$$

where $F \in (0, b_0/r)$ is such that $x_0 < F(0)$.

NOTE: If we reach the target, investing the whole wealth in the riskless asset keeps the wealth on the target.

- $F(\cdot)$ is a target we aim to reach.

$$F(s) = \frac{b_0}{r} + \left(F - \frac{b_0}{r}\right) e^{-r(T-s)},$$

where $F \in (0, b_0/r)$ is such that $x_0 < F(0)$.

NOTE: If we reach the target, investing the whole wealth in the riskless asset keeps the wealth on the target.

- Cost functional:

$$J = E\left[\int_0^T \kappa e^{-\rho s}(F(s) - X(s))^2 ds + e^{-\rho T}(F(T) - X(T))^2\right] \geq 0,$$

where $\kappa \geq 0$.

NOTE: If we reach the target at time t, investing the whole wealth in the riskless asset from t on yields 0 in the remaining part of the functional above. We can say that $F(\cdot)$ is an *optimal absorbing boundary* for the problem.

- Cost functional:

$$J = E\left[\int_0^T \kappa e^{-\rho s}(F(s) - X(s))^2 ds + e^{-\rho T}(F(T) - X(T))^2\right] \geq 0,$$

where $\kappa \geq 0$.

NOTE: If we reach the target at time t, investing the whole wealth in the riskless asset from t on yields 0 in the remaining part of the functional above. We can say that $F(\cdot)$ is an *optimal absorbing boundary* for the problem.

- [Gerrard, Haberman & Vigna, 2004] minimize J without constraints on the strategies and on the wealth.
- We study the minimization of the functional in the cases

(P1) constraint on the strategies (no short selling):

$$\textit{Admissible Strategies} = \left\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \right\};$$

(P2) constraint on the strategies (no short selling) and on the wealth (no ruin):

$$\textit{Admissible Strategies} =$$

$$\left\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \mid X(t;\pi(\cdot)) \geq 0, \ t \in [0,T] \right\}.$$

- [Gerrard, Haberman & Vigna, 2004] minimize J without constraints on the strategies and on the wealth.
- We study the minimization of the functional in the cases

(P1) constraint on the strategies (no short selling):

$$\textit{Admissible Strategies} = \Big\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \Big\};$$

(P2) constraint on the strategies (no short selling) and on the wealth (no ruin):

$$\textit{Admissible Strategies} =$$

$$\Big\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \mid X(t;\pi(\cdot)) \geq 0, \ t \in [0,T] \Big\}.$$

- [Gerrard, Haberman & Vigna, 2004] minimize J without constraints on the strategies and on the wealth.
- We study the minimization of the functional in the cases

(P1) constraint on the strategies (no short selling):

$$\mathit{Admissible\ Strategies} = \left\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \right\};$$

(P2) constraint on the strategies (no short selling) and on the wealth (no ruin):

$$\mathit{Admissible\ Strategies} =$$
$$\left\{ \pi(\cdot) \in L^2(\Omega \times [0,T]; \mathbb{R}^+) \text{ adapted} \mid X(t;\pi(\cdot)) \geq 0,\ \ t \in [0,T] \right\}.$$

(P1) Constraint on the strategies

We follow a classic dynamic programming approach to solve the problem, proceeding along the following steps:

- We define the value function $V(t,x)$ as the optimum for generic initial data $t \in [0,T],\ x \leq F(t)$.
- We associate to the value function the HJB equation.
- We find an explicit solution to the HJB equation.
- We prove a verification theorem which, as a byproduct:
  - says that this solution is indeed the value function;
  - gives a way to define an optimal strategy by this function.

The value function and its properties

Let

$$U = \{(t,x) \mid t \in [0,T),\ x < F(t)\}.$$

- Value function V defined on $\bar{U}$ as

$$V(t,x) := \inf_{\pi(\cdot)\in\Pi(t,x)} E\left[\int_t^T \kappa e^{-\rho s}(F(s)-X(s))^2 ds + e^{-\rho T}(F(T)-X(T))^2\right],$$

  where

$$\Pi(t,x) = \{\pi(\cdot) \in L^2(\Omega \times [t,T];\mathbb{R}^+) \text{ adapted} \mid X(s;t,x,\pi(\cdot)) \le F(s),\ s \in [t,T]\}.$$

- $F(\cdot)$ absorbing boundary for the problem:

$$x = F(t) \ \Rightarrow\ \Pi(t,x) = \{0\} \text{ and } X(s;t,x,0) = F(s),\ s \in [t,T].$$

The HJB equation: explicit solution

- $x \mapsto V(t,x)$ is convex and nonincreasing on $(-\infty, F(t)]$, $\forall t \in [0, T]$.
- $F(\cdot)$ absorbing boundary for the problem:

$$x = F(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = F(s),\; s \in [t, T].$$

- HJB equation:

$$v_t + (rx - b_0)v_x + \kappa e^{-\rho t}(F(t) - x)^2 - \frac{\lambda^2}{2}\frac{v_x^2}{v_{xx}} = 0, \quad \text{on } U,$$

with boundary conditions

$$\begin{cases} v(t, F(t)) = 0, & t \in [0, T], \\ v(T, x) = e^{-\rho T}(F(T) - x)^2, & x \leq F(T). \end{cases}$$

The HJB equation: explicit solution

- $x \mapsto V(t,x)$ is convex and nonincreasing on $(-\infty, F(t)]$, $\forall t \in [0,T]$.
- $F(\cdot)$ absorbing boundary for the problem:

$$x = F(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = F(s),\; s \in [t,T].$$

- HJB equation:

$$v_t + (rx - b_0)v_x + \kappa e^{-\rho t}(F(t)-x)^2 - \frac{\lambda^2}{2}\frac{v_x^2}{v_{xx}} = 0, \quad \text{on } U,$$

with boundary conditions

$$\begin{cases} v(t,F(t)) = 0, & t \in [0,T], \\ v(T,x) = e^{-\rho T}(F(T)-x)^2, & x \le F(T). \end{cases}$$

The HJB equation: explicit solution

- $x \mapsto V(t,x)$ is convex and nonincreasing on $(-\infty, F(t)]$, $\forall t \in [0, T]$.
- $F(\cdot)$ absorbing boundary for the problem:

$$x = F(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = F(s),\; s \in [t, T].$$

- HJB equation:

$$v_t + (rx - b_0)v_x + \kappa e^{-\rho t}(F(t) - x)^2 - \frac{\lambda^2}{2}\frac{v_x^2}{v_{xx}} = 0, \quad \text{on } U,$$

with boundary conditions

$$\begin{cases} v(t, F(t)) = 0, & t \in [0, T], \\ v(T, x) = e^{-\rho T}(F(T) - x)^2, & x \le F(T). \end{cases}$$

Solution:

Let

$$v(t,x) = e^{-\rho t} A(t)(F(t) - x)^2,$$

where $A(\cdot)$ is the unique solution of

$$\begin{cases} A'(t) = \left(\rho + \lambda^2 - 2r\right) A(t) - \kappa, \\ A(T) = 1. \end{cases}$$

Then v solves the HJB equation.

The verification theorem and the optimal feedback strategy

Define the feedback map

$$(s, y) \mapsto G(s, y) := \frac{\lambda}{\sigma}(F(s) - y).$$

Theorem (Verification and Optimal Feedback)

- *There exists a unique process* $X^*(\cdot)$ *solution of the CLE*

$$\begin{cases} dX(s) = [rX(s) + \sigma\lambda G(s, X(s)) - b_0]\, ds + \sigma G(s, X(s)) dB(s), \\ X(t) = x. \end{cases}$$

- $v = V$.
- *The feedback strategy*

$$\pi^*(s) := \Pi(s, X^*(s)), \quad s \in [t, T],$$

is the unique optimal strategy for the problem starting at (t, x).

(P2) Constraints on the wealth and on the strategies

Value function W defined for $t \in [0, T]$, $x \in [0, F(t)]$ as

$$W(t,x) := \inf_{\pi(\cdot)\in\Pi(t,x)} E\left[\int_t^T \kappa e^{-\rho s}(F(s)-X(s))^2 ds + e^{-\rho T}(F(T)-X(T))^2\right],$$

where

$$\Pi(t,x) = \{\pi(\cdot) \in L^2(\Omega \times [t,T]; \mathbb{R}^+) \text{ adapted} \mid 0 \le X(s; t, x, \pi(\cdot)) \le F(s)\ \forall s \in [t,T]\}.$$

The set of admissible strategies

Set

$$S(t) := \frac{b_0}{r} - \frac{b_0}{r} e^{-r(T-t)} < F(t), \quad t \in [0, T],$$

- $\Pi(t,x) \neq \emptyset \iff S(t) \leq x \leq F(t)$.
- The problem is defined over $\bar{\mathcal{C}}$, where

$$\mathcal{C} := \{(t,x) \in [0,T) \times \mathbb{R} \mid x \in (S(t), F(t))\},$$

- $\Pi(t,x)$ can be rewritten as

$$\Pi(t,x) = \{\pi(\cdot) \in L^2(\Omega \times [t,T]; \mathbb{R}^+) \text{ adapted } \mid S(s) \leq X(s; t, x, \pi(\cdot)) \leq F(s) \ \forall s \in [t,T]\}.$$

The set of admissible strategies

Set

$$S(t) := \frac{b_0}{r} - \frac{b_0}{r} e^{-r(T-t)} < F(t), \quad t \in [0, T],$$

- $\Pi(t, x) \neq \emptyset \iff S(t) \leq x \leq F(t)$.
- The problem is defined over $\bar{\mathcal{C}}$, where

$$\mathcal{C} := \{(t, x) \in [0, T) \times \mathbb{R} \mid x \in (S(t), F(t))\},$$

- $\Pi(t, x)$ can be rewritten as

$$\Pi(t, x) = \{\pi(\cdot) \in L^2(\Omega \times [t, T]; \mathbb{R}^+) \text{ adapted } | \\ S(s) \leq X(s; t, x, \pi(\cdot)) \leq F(s) \ \forall s \in [t, T]\}.$$

The set of admissible strategies

Set

$$S(t) := \frac{b_0}{r} - \frac{b_0}{r} e^{-r(T-t)} < F(t), \quad t \in [0, T],$$

- $\Pi(t,x) \neq \emptyset \iff S(t) \leq x \leq F(t)$.
- The problem is defined over $\bar{\mathcal{C}}$, where

$$\mathcal{C} := \{(t,x) \in [0, T) \times \mathbb{R} \mid x \in (S(t), F(t))\},$$

- $\Pi(t,x)$ can be rewritten as

$$\Pi(t,x) = \{\pi(\cdot) \in L^2(\Omega \times [t, T]; \mathbb{R}^+) \text{ adapted } | \\ S(s) \leq X(s; t, x, \pi(\cdot)) \leq F(s) \ \forall s \in [t, T]\}.$$

The set of admissible strategies

Set

$$S(t) := \frac{b_0}{r} - \frac{b_0}{r} e^{-r(T-t)} < F(t), \quad t \in [0, T],$$

- $\Pi(t, x) \neq \emptyset \iff S(t) \leq x \leq F(t)$.
- The problem is defined over $\bar{\mathcal{C}}$, where

$$\mathcal{C} := \{(t, x) \in [0, T) \times \mathbb{R} \mid x \in (S(t), F(t))\},$$

- $\Pi(t, x)$ can be rewritten as

$$\Pi(t, x) = \{\pi(\cdot) \in L^2(\Omega \times [t, T]; \mathbb{R}^+) \text{ adapted } \mid S(s) \leq X(s; t, x, \pi(\cdot)) \leq F(s) \ \forall s \in [t, T]\}.$$

The HJB equation: viscosity solutions

- $S(\cdot)$ absorbing boundary for the problem:

$$x = S(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = S(s),\; s \in [t, T].$$

- $[S(t), F(t)] \to \mathbb{R}^+$, $x \mapsto W(t,x)$ convex and nonincreasing $\forall t \in [0, T]$.
- HJB equation:

$$w_t + (rx - b_0)w_x + \kappa e^{-\rho t}(F(t) - x)^2 - \frac{\lambda^2}{2}\frac{w_x^2}{w_{xx}} = 0, \quad \text{on } \mathcal{C},$$

with boundary conditions

$$\begin{cases} w(T,x) = \kappa e^{-\rho T}(F - x)^2, & x \in [0, F], \\ w_x(t, F(t)) = 0, & t \in [0, T], \\ w(t, S(t)) = g(t) := W(t, S(t)) \;\; (known), & t \in [0, T]. \end{cases}$$

The HJB equation: viscosity solutions

- $S(\cdot)$ absorbing boundary for the problem:

$$x = S(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = S(s),\;\; s \in [t,T].$$

- $[S(t),F(t)] \to \mathbb{R}^+$, $x \mapsto W(t,x)$ convex and nonincreasing $\forall t \in [0,T]$.
- HJB equation:

$$w_t + (rx - b_0)w_x + \kappa e^{-\rho t}(F(t) - x)^2 - \frac{\lambda^2}{2}\frac{w_x^2}{w_{xx}} = 0, \quad \text{on } \mathcal{C},$$

with boundary conditions

$$\begin{cases} w(T,x) = \kappa e^{-\rho T}(F-x)^2, & x \in [0,F], \\ w_x(t,F(t)) = 0, & t \in [0,T], \\ w(t,S(t)) = g(t) := W(t,S(t)) \;\; (known), & t \in [0,T]. \end{cases}$$

The HJB equation: viscosity solutions

- $S(\cdot)$ absorbing boundary for the problem:

$$x = S(t) \;\Rightarrow\; \Pi(t,x) = \{0\} \;\text{ and }\; X(s;t,x,0) = S(s),\; s \in [t, T].$$

- $[S(t), F(t)] \to \mathbb{R}^+$, $x \mapsto W(t,x)$ convex and nonincreasing $\forall t \in [0, T]$.
- HJB equation:

$$w_t + (rx - b_0)w_x + \kappa e^{-\rho t}(F(t) - x)^2 - \frac{\lambda^2}{2}\frac{w_x^2}{w_{xx}} = 0, \quad \text{on } \mathcal{C},$$

with boundary conditions

$$\begin{cases} w(T,x) = \kappa e^{-\rho T}(F - x)^2, & x \in [0, F], \\ w_x(t, F(t)) = 0, & t \in [0, T], \\ w(t, S(t)) = g(t) := W(t, S(t)) \;\; (known), & t \in [0, T]. \end{cases}$$

PROBLEMS:

- Explicit solutions not available anymore in general.
- HJB degenerate ⇒ classical PDEs theory not appliable.

IDEA: pass through the viscosity theory to prove existence and uniqueness of regular solutions for HJB:

- Characterize the value function as unique viscosity solution of the HJB equation.
- Prove $C^{1,2}$ regularity of the value function.

PROBLEMS:

- Explicit solutions not available anymore in general.
- HJB degenerate $\Rightarrow$ classical PDEs theory not appliable.

IDEA: pass through the viscosity theory to prove existence and uniqueness of regular solutions for HJB:

- Characterize the value function as unique viscosity solution of the HJB equation.
- Prove $C^{1,2}$ regularity of the value function.

Consider $\mathcal{L} : [0, T] \times [0, F] \to \bar{\mathcal{C}}$,

$$(t, z) \longmapsto (t, x) = \mathcal{L}(t, z) := \left(t, ze^{-r(T-t)} + \frac{b_0}{r}\left(1 - e^{-r(T-t)}\right)\right).$$

Using $\mathcal{L}$ we can rewrite the HJB as

$$h_t + \kappa b(t)(F - z)^2 - \frac{\lambda^2}{2}\frac{h_z^2}{h_{zz}} = 0, \text{ on } [0, T) \times (0, F). \tag{1}$$

where

$$b(t) = e^{-\rho t - 2r(T-t)},$$

with boundary conditions

$$\begin{cases} h(T, z) = b(T)(F - z)^2, \; z \in [0, F], \\ h_z(t, F) = 0, \; t \in [0, T), \\ h(t, 0) = \psi(t) \; (known), \; t \in [0, T). \end{cases} \tag{2}$$

The HJB equation (1)-(2) is associated to a stochastic control problem with value function H such that

$$H(t,z) = W(\mathcal{L}(t,z)).$$

$\longrightarrow$ We can study H and (1)-(2).

- $[0,F] \rightarrow \mathbb{R}^+$, $z \mapsto H(t,z)$ is convex and nonincreasing $\forall t \in [0,F]$.
- H is continuous on $[0,T] \times [0,F]$.

Theorem

H is the unique viscosity solution of the HJB equation (1)-(2).

The HJB equation (1)-(2) is associated to a stochastic control problem with value function H such that

$$H(t,z) = W(\mathcal{L}(t,z)).$$

$\longrightarrow$ We can study H and (1)-(2).

- $[0,F] \to \mathbb{R}^+$, $z \mapsto H(t,z)$ is convex and nonincreasing $\forall t \in [0,F]$.
- H is continuous on $[0,T] \times [0,F]$.

Theorem

H is the unique viscosity solution of the HJB equation (1)-(2).

Regularity of the value function

- We know that H is the unique viscosity solution of (1)-(2). We want to prove that it is $C^{1,2}$.
- C^2 regularity results for viscosity solution of this kind of equations are proved in the elliptic case. See e.g. [Choulli, Taksar, Zhou; 2003] and [Di Giacinto, F., Gozzi; 2009].
  - The C^1-regularity is proved by an argument of Convex Analysis.
  - The C^2-regularity is proved by a localization argument and classical PDEs theory, once the C^1 regularity is known.
- The same argument does not work in the parabolic case, due to the lack of good information on the dependence of the value function with respect to time.

Regularity of the value function

- We know that H is the unique viscosity solution of (1)-(2). We want to prove that it is $C^{1,2}$.
- C^2 regularity results for viscosity solution of this kind of equations are proved in the elliptic case. See e.g. [Choulli, Taksar, Zhou; 2003] and [Di Giacinto, F., Gozzi; 2009].
  - The C^1-regularity is proved by an argument of Convex Analysis.
  - The C^2-regularity is proved by a localization argument and classical PDEs theory, once the C^1 regularity is known.
- The same argument does not work in the parabolic case, due to the lack of good information on the dependence of the value function with respect to time.

Regularity of the value function

- We know that H is the unique viscosity solution of (1)-(2). We want to prove that it is $C^{1,2}$.
- C^2 regularity results for viscosity solution of this kind of equations are proved in the elliptic case. See e.g. [Choulli, Taksar, Zhou; 2003] and [Di Giacinto, F., Gozzi; 2009].
  - The C^1-regularity is proved by an argument of Convex Analysis.
  - The C^2-regularity is proved by a localization argument and classical PDEs theory, once the C^1 regularity is known.
- The same argument does not work in the parabolic case, due to the lack of good information on the dependence of the value function with respect to time.

- We define a dual problem. The same method has been already used e.g. in [Elie, Touzi; 2008], [Gao; 2008], [Xiao , Zhai, Qin, 2007], [Milevsky, Moore, Young; 2006], [Milevsky, Young; 2007] and [Gerrard, Hojgaard, Vigna; 2010].
- This method allows to remove the fully nonliner term v_x^2/v_{xx}.
- In all these papers the dual equation is linear and explicit solutions are found.
- In our case the dual equation is semilinear and degenerate:
  - we do not have explicit solutions;
  - we study it again passing through the viscosity; then we prove its regularity and othe properties needed to come back to the original problem.

- We define a dual problem. The same method has been already used e.g. in [Elie, Touzi; 2008], [Gao; 2008], [Xiao , Zhai, Qin, 2007], [Milevsky, Moore, Young; 2006], [Milevsky, Young; 2007] and [Gerrard, Hojgaard, Vigna; 2010].
- This method allows to remove the fully nonliner term v_x^2/v_{xx}.
- In all these papers the dual equation is linear and explicit solutions are found.
- In our case the dual equation is semilinear and degenerate:
  - we do not have explicit solutions;
  - we study it again passing through the viscosity; then we prove its regularity and othe properties needed to come back to the original problem.

- We define a dual problem. The same method has been already used e.g. in [Elie, Touzi; 2008], [Gao; 2008], [Xiao , Zhai, Qin, 2007], [Milevsky, Moore, Young; 2006], [Milevsky, Young; 2007] and [Gerrard, Hojgaard, Vigna; 2010].
- This method allows to remove the fully nonliner term v_x^2/v_{xx}.
- In all these papers the dual equation is linear and explicit solutions are found.
- In our case the dual equation is semilinear and degenerate:
  - we do not have explicit solutions;
  - we study it again passing through the viscosity; then we prove its regularity and othe properties needed to come back to the original problem.

- We define a dual problem. The same method has been already used e.g. in [Elie, Touzi; 2008], [Gao; 2008], [Xiao , Zhai, Qin, 2007], [Milevsky, Moore, Young; 2006], [Milevsky, Young; 2007] and [Gerrard, Hojgaard, Vigna; 2010].
- This method allows to remove the fully nonliner term v_x^2/v_{xx}.
- In all these papers the dual equation is linear and explicit solutions are found.
- In our case the dual equation is semilinear and degenerate:
  - we do not have explicit solutions;
  - we study it again passing through the viscosity; then we prove its regularity and othe properties needed to come back to the original problem.

The dual equation

Assume that $H \in C^{1,3}$ and that

$$H_z < 0, \quad H_{zz} > 0, \quad \lim_{z \to 0^+} H_z(t,z) = -\infty.$$

Then, for every $(t,y) \in [0,T) \times (0,+\infty)$ there exists a unique $g(t,y) \in (0,F)$ minimizer of $z \mapsto H(t,z) + zy$, characterized by

$$H_z(t, g(t,y)) = -z. \tag{3}$$

Deriving (3) and using (1)-(2) we can write a semilinear PDE for g:

$$g_t - 2\kappa b(t)(F-g)g_y + \lambda^2 y g_y + \frac{\lambda^2}{2} y^2 g_{yy} = 0, \quad \text{on } [0,T) \times (0,+\infty), \quad (4)$$

with boundary conditions

$$\begin{cases} g(t,0) = F, & t \in [0,T); \\ g(T,y) = \left(F - \frac{y}{2b(T)}\right)^+, & y \in [0,+\infty). \end{cases} \quad (5)$$

Proposition

Suppose that the unique viscosity solution H of (1)-(2) belongs to the class $C^{1,3}$ and satisfies

$$H_z < 0, \quad H_{zz} > 0, \quad \lim_{z \to 0^+} H_z(t,z) = -\infty.$$

Then g defined as above is a classical solution of the dual problem (4)-(5).

Proposition

Conversely, let g *be a classical solution of the dual equation* (4)-(5) *satisfying*

$$\begin{cases} g(t,y) \in (S,F), \ \forall y \in (0,+\infty); \\ g_y(t,y) < 0, \quad \forall t \in [0,T), \ \forall y \in (0,+\infty); \\ \lim_{y \to +\infty} g(t,y) = 0, \quad \forall t \in [0,T); \\ y^2 g_y(t,y) \overset{y \to +\infty}{\longrightarrow} 0, \quad \textit{uniformly in } t \in [0,T); \\ [g(t,\cdot)]^{-1} \textit{ is integrable at } S^+, \quad \forall t \in [0,T). \end{cases} \tag{6}$$

Let

$$\begin{cases} h(t,z) = \psi(t) + b(T)(F-S)^2 - \int_S^z [g(t,\cdot)]^{-1}(\xi) d\xi, \quad (t,z) \in [0,T) \times [S,F], \\ h(T,z) = b(T)(F-z)^2, \quad z \in [S,F], \end{cases}$$

Then h *is a classical solution of* (1)-(2)*. Therefore* $h = H$*.*

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Theorem

There exists a unique g classical solution of (4)-(5) satisfying the assumptions of the previous proposition.

Proof.

- Comparison principle in viscosity sense holds for the equation (standard viscosity theory). Thus uniqueness holds for the equation.
- Existence: by Perron's method exhibiting a suitable subsolution $\underline{g}$ and a suitable supersolution $\bar{g}$. A viscosity solution $\underline{g} \leq g \leq \bar{g}$ is constructed.
- $C^{1,2}$-regularity by a localization argument and by using the standard theory for semilinear uniformly parabolica equations.
- Convexity in y: by convexity preserving adapting the argument of [Korevaar; 1983].
- Other properties follow from the previous ones thanks to the suitable choice of $\underline{g}$, $\bar{g}$.

Corollary

H is the unique classical solution of the HJB (1)-(2).

The case $\kappa = 0$: explicit solution

Take $\kappa = 0$ (no running cost). The dual equation is linear:

$$g_t + \lambda^2 y g_y + \frac{\lambda^2}{2} y^2 g_{yy} = 0 \quad \text{on } [0, T) \times (0, +\infty),$$

with boundary conditions

$$\begin{cases} g(t, 0) = F, & t \in [0, T]; \\ g(T, y) = \left(F - \frac{y}{2b(T)}\right)^+, & y \in (0, +\infty). \end{cases}$$

$\rightarrow$ Black-Scholes equation with boundary conditions of European put option type.

We have the stochastic representation for the solution of this equation:

$$g(t,y) = F\Phi(k(t,y)) - \frac{y}{2b(T)} e^{\lambda^2(T-t)} \Phi(k(t,y) - \lambda\sqrt{T-t}),$$
$$(t,y) \in [0,T] \times [0,+\infty),$$

where

$$k(t,y) = \frac{\log\left(\frac{2F}{y}\right) - \frac{\lambda^2}{2}(T-t)}{\lambda\sqrt{T-t}}$$

and $\Phi(\cdot)$ is the distribution function of $\mathcal{N}(0,1)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi.$$

- We can come back: H is the unique classical solution of the HJB equation and it is explicitly computable in terms of the function Φ.
- The feedback map

$$G : [0, T] \times [0, F] \to \mathbb{R}^+.$$

is explicitely computable.

- Let $y = [g(t, \cdot)]^{-1}(z)$ and let $Y(\cdot; t, y)$ be the solution of

$$\begin{cases} dY(s) = -\beta Y(s) dB(s), \\ Y(t) = y. \end{cases}$$

Consider the process

$$Z^*(s; t, z) = g(s, Y(s; t, y)). \tag{7}$$

- We can come back: H is the unique classical solution of the HJB equation and it is explicitly computable in terms of the function Φ.
- The feedback map

$$G : [0, T] \times [0, F] \to \mathbb{R}^+.$$

is explicitely computable.

- Let $y = [g(t, \cdot)]^{-1}(z)$ and let $Y(\cdot; t, y)$ be the solution of

$$\begin{cases} dY(s) = -\beta Y(s) dB(s), \\ Y(t) = y. \end{cases}$$

Consider the process

$$Z^*(s; t, z) = g(s, Y(s; t, y)). \tag{7}$$

- We can come back: H is the unique classical solution of the HJB equation and it is explicitly computable in terms of the function Φ.
- The feedback map

$$G : [0, T] \times [0, F] \to \mathbb{R}^+.$$

is explicitely computable.

- Let $y = [g(t, \cdot)]^{-1}(z)$ and let $Y(\cdot; t, y)$ be the solution of

$$\begin{cases} dY(s) = -\beta Y(s) dB(s), \\ Y(t) = y. \end{cases}$$

Consider the process

$$Z^*(s; t, z) = g(s, Y(s; t, y)). \tag{7}$$

Theorem (Optimal Feedback)

- *Z^* defined in (7) is the unique solution of the CLE*

$$\begin{cases} dZ(s) = e^{r(T-t))}\left[\sigma\beta G(s, Z(s))ds + \sigma G(s, Z(s))dB(s)\right], \\ Z(t) = z \in (S, F). \end{cases}$$

- *The strategy*

$$\pi^*(s) := G(s, Z^*(s)), \quad s \in [t, T],$$

is the unique optimal strategy for the problem starting at (t, z).

$\longrightarrow$ Numerical simulations are performed by using this explicit solutions.

Theorem (Optimal Feedback)

- *Z^* defined in (7) is the unique solution of the CLE*

$$\begin{cases} dZ(s) = e^{r(T-t))}\left[\sigma\beta G(s,Z(s))ds + \sigma G(s,Z(s))dB(s)\right], \\ Z(t) = z \in (S,F). \end{cases}$$

- *The strategy*

$$\pi^*(s) := G(s, Z^*(s)), \quad s \in [t,T],$$

is the unique optimal strategy for the problem starting at (t,z).

$\longrightarrow$ Numerical simulations are performed by using this explicit solutions.

Future targets

- The problem (P3):
  - "no ruin" for the wealth;
  - "no short selling" and "no borrowing" for the investment strategies.
- Allowing the control in the consumption.

Future targets

- The problem (P3):
  - "no ruin" for the wealth;
  - "no short selling" and "no borrowing" for the investment strategies.
- Allowing the control in the consumption.

References

Albrecht P., and Maurer R., *Self-Annuitization, Consumption Shortfall in Retirement and Asset Allocation: The Annuity Benchmark*, Journal of Pension Economics and Finance, **1**, 269–288, 2002.

Blake D., Cairns A.J.G., and Dowd K., *Pensionmetrics 2: Stochastic pension plan design during the distribution phase*, Insurance: Mathematics and Economics, **33**, 29–47, 2003.

V.H. Fleming, H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 1993, Springer-Verlag, New York.

Gerrard R., Haberman S., and Vigna E., *Optimal investment choices post retirement in a defined contribution pension scheme*, Insurance: Mathematics and Economics, **35**, 321–342, 2004.

Gerrard R., Haberman S., and Vigna E., *The Management of De-cumulation Risks in a Defined Contribution Environment*, North American Actuarial Journal, **10**, 84–110, 2006.

Gerrard R., Højgaard B., and Vigna E. (2008). *Choosing the optimal annuitization time post retirement*, mimeo.

Højgaard B., Vigna E. (2007). *Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes*, mimeo.

Karatzas I., Shreve S.E., *Brownian Motion and Stochastic Calculus*, Second Edition, Springer Verlag, New York, 1991.

Merton R.C., *Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case*, Review of Economics and Statistics, **51**, 247–257, 1969.

Milevsky M.A., *Optimal Annuitization Policies: Analysis of the Options*, North American Actuarial Journal, **5**, 57–69, 2001.

Milevsky M. A., Moore K. S. and Young V. R., *Optimal Asset Allocation and Ruin-Minimization Annuitization Strategies*, Mathematical Finance, **16**, 647–671, 2006.

Milevsky M.A. and Young V. R. *Annuitization and Asset Allocation*, Journal of Economic Dynamics and Control, **31**, 3138–3177, 2007.

H.M. Soner, *Stochastic Optimal Control in Finance*, 2004, Cattedra Galileiana, Scuola Normale Superiore, Pisa.

Yong J., Zhou X.Y., *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer Verlag, New York, 1999.