Semigroups and stochastic partial (pseudo) differential equations on measure spaces

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1. Introduction

$[X, \mathfrak{X}, \mu]$ σ-finite measure space (or a certain locally finite metric measure space)
consider the formal Cauchy problem on $[0, T] \times X$

$$\frac{\partial u}{\partial t} = -A^\theta u + F(u) + G(u) \cdot \frac{\partial Z^*}{\partial t}, \quad t \in (0, T], \qquad (1)$$

with initial condition $u(0, x) = f(x)$, where

- $-A$ is the generator of an ultracontractive strongly continuous Markovian symmetric semigroup $(P_t)_{t \geq 0}$ on $L_2(\mu)$
- A^θ, $\theta \leq 1$, is a fractional power of A
- F and G are sufficiently regular functions on $\mathbb{R}$
- Z^* is a random element in $C^{1-\alpha}([0, T], H_{2,\infty}^{\theta\beta}(\mu)^*)$ (resp. in $C^{1-\alpha}([0, T], H_q^{\theta\beta}(\mu)^*)$)
- $f \in H_{2,\infty}^{2\gamma+\theta\beta+\varepsilon}(\mu)$

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$$\begin{aligned} u(t,x) &= P_t f(x) + \int_0^t P_{t-s} F(u(s,\cdot)(x))\, ds \\ &+ \int_0^t P_{t-s} \left(G(u(s,\cdot)) \cdot \frac{\partial Z^*}{\partial s}(s) \right)(x) ds \end{aligned}$$

the last formal integral to be determined,

for $\theta < 1$ use the subordinated semigroup P^θ with generator $-A^\theta$ instead of P_t

Main ideas:

- the smoothness is measured in terms of potential spaces $H_2^\sigma(\mu)$ generated by the semigroup, and the latter lifts certain dual spaces to function spaces,
- the paraproduct is introduced by duality relations
- the time integral is realized by means of *Banach space valued fractional calculus*

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our approach is independent of series expansions

Related literature: Gubinelli, Lejay, Tindel 2006

$$dU(t) = AU(t) + G(U(t))\, dX(t)\,, \quad U(0) = U_0\,,$$

$$U(t) = P_t U_0 + \int_0^t P_{t-s} G(U(s))\, dX(s)\,, \quad t \leq T$$

(semigroups P_t in Banach spaces B, potential spaces $B_\alpha = Dom(A^\alpha)$, the noise process X takes values in B_α^*, G as mapping from $B_\delta \mapsto L(B_\alpha^*, B_\rho)$ satisfying some Lipschitz conditions, the time integral is realized as *Young integral*, solution $U \in C^\kappa([0,T], B_\delta)$ for certain parameters)

abstract approach, application to the above situation yields some partial results

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2. Bessel potential spaces associated with semigroups on (metric) measure spaces

Main assumptions:

- $((X, \mu)$ σ-finite measure space (resp.(X, d, μ) locally compact metric measure space, μ Radon measure, $X = \mathrm{supp}\mu$)) admitting a
- strongly continuous Markov semigroup $(P_t)_{t\geq 0}$ on $L_2(\mu)$ (with transition density $p_t(x, y)$)
- $$P_t = e^{-At}, \quad -A \text{ infinitesimal generator},$$
- P_t is ultracontractive: $||P_t||_{2\to\infty} \leq \mathrm{const}\, t^{-d_S/4}$, d_S spectral dimension of the semigroup (the transition densities possess sub-Gaussian estimates)

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Fractional powers of A are determined by:

$$A^\alpha u = \mathrm{const}(\alpha, l) \int_0^\infty t^{-\alpha-1}(I - P_t)^l u \, dt$$

for $l > \alpha > 0$ and

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for $\alpha > 0$ and all $\varphi \in L_2(\mu)$ if 0 is in the resolvent of A

define for $\sigma \geq 0$ and some $\omega > 0$:
Bessel potential operators: (take $e^{-\omega t}P_t$ instead of P_t)

$$J^\sigma := (\omega I + A)^{-\sigma/2}$$

Bessel potential spaces: $H_2^\sigma(\mu) := J^\sigma(L_2(\mu))$ with norm

$$||u||_{H_2^\sigma(\mu)} := ||(\omega I + A)^{\sigma/2}(u)||_{L_2(\mu)} \sim ||u||_{L_2(\mu)} + ||A^{\sigma/2}(u)||_{L_2(\mu)}$$

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Dual spaces:

$$H^{-\sigma}_{2,\infty}(\mu) := H^{\sigma}_{2,\infty}(\mu)^{*} \quad \text{(resp. } H^{-\sigma}_{p}(\mu) := H^{\sigma}_{p}(\mu)^{*}\text{)}$$

by duality the operators $J^{\sigma}(\mu)$ can be extended to the dual spaces and act isomorphically:

$$J^{\alpha}_{:} H^{\beta}_{2}(\mu) \mapsto H^{\beta+\alpha}_{2}(\mu)\,, \quad \alpha, \beta \in \mathbb{R}$$

and

$$P_t J^{\sigma} u = J^{\sigma} P_t u$$

for $\sigma \geq 0$, and thus the semigroup can be extended to the dual spaces with the above equality for $\sigma \in \mathbb{R}$

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$$J^{\alpha}_{:} H^{\beta}_{2}(\mu) \mapsto H^{\beta+\alpha}_{2}(\mu)\,, \quad \alpha, \beta \in \mathbb{R}$$

and

$$P_t J^{\sigma}\, u = J^{\sigma} P_t\, u$$

for $\sigma \geq 0$, and thus the semigroup can be extended to the dual spaces with the above equality for $\sigma \in \mathbb{R}$

we also consider the spaces

$$H^\sigma_{A,\infty}(\mu) := H^\sigma_A(\mu) \cap L_\infty(\mu)$$

with norm

$$||u||_{H^\sigma_{A,\infty}}(\mu) := ||u||_{H^\sigma_A(\mu)} + ||u||_{L_\infty(\mu)}$$

Dual spaces:

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mapping properties of P_t in the above spaces are implied
(for L_∞ properties the ultracontractivity is needed, fulfilled for many examples, classical and fractal cases)

Application to parabolic SPDE on fractals:

- up to on certain fractals mainly elliptic (and some parabolic) PDE with respect to Laplace operators have been considered (Falconer, Hu, Grigoryan, Koshnevisan, ...), without noise terms
- fractal Laplacians: Lindstrøm, Barlow, Bass, Kusuoka, Strichartz, Kigami and many others)
- these fractals are special metric measure spaces fulfilling the above assumptions,

our pathwise approach to the above parabolic equations with random noise is related to some methods from the Euclidean case (Hinz, Z.: J. Funct. Anal. 2009) and results on generalized Bessel potential spaces (Hu, Z. : Studia Math. 2005, Potential Anal. 2009)

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3. Rigorous definition and solution of the stochastic partial (pseudo) differential equation

general situation as above: (P_t) with generator $-A$ (or P_t^θ with generator $-A^\theta$ instead);
recall that u is a mild solution of the Cauchy problem (1) if

$$\begin{aligned} u(t,x) &= P_t f(x) + \int_0^t P_{t-s} F(u(s,\cdot)(x))\, ds \\ &+ \int_0^t P_{t-s} \left(G(u(s,\cdot)) \cdot \frac{\partial Z^*}{\partial s}(s) \right)(x)\, ds \end{aligned}$$

rewrite the last formal integral as

$$\int_0^t \Phi_t(s) \left(\frac{\partial Z^*}{\partial s}(s) \right)(x) ds$$

where

$\Phi_t(s)(w) := P_{t-s}\left(G(u(s,\cdot)\cdot w)\right)$ is for $u \in H^{\delta}_{2,\infty}(\mu)$ shown to be a mapping

$$\Phi_t : [0,T] \to L\left(H^{\rho}_{2,\infty}(\mu)^*, H^{\delta}_{2,\infty}(\mu)\right)$$

(for some ρ) with fractional order of smoothness α' slightly larger than α, by assumption Z^* has fractional order of smoothness $1-\alpha'$, so that we can define

$$\int_0^t \Phi_t(s)\left(\frac{\partial Z^*}{\partial s}(s)\right) ds := \int_0^t D^{\alpha'}_{0+}\Phi_t(s)\left(D^{1-\alpha'}_{t-}Z^*_t\right) ds$$

for left and right sided fractional derivatives $D^{\alpha'}_{0+}$ and $D^{1-\alpha'}_{t-}$ (and $Z^*_t := Z^* - Z^*(t-)$)

If the noise coefficient function G is **linear**, in the **metric case** the L_∞-norms can be omitted. This leads to solutions for all spectral dimensions: $0 < \theta \leq 1$

Theorem. If $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$, $\beta < \delta$ and $2\gamma + \theta\delta < 2(1-\alpha) - \theta\beta$, then problem (1) has a unique mild solution $u \in W^{\gamma}\left([0,T], H^{\theta\delta}_2(\mu)\right)$.

$$||u||_{W^\gamma([0,T],H)} := \sup_{0\leq t\leq T}\left(||u(t)||_H + \int_0^t \frac{||u(t)-u(s)||_H}{(t-s)^{\gamma+1}} ds\right)$$

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In the general case we prove a contraction principle for the mild solution of (1) in $W^{\gamma}\left([0,T], H^{\theta\delta}_{2,\infty}(\mu)\right)$ under some additional conditions on the parameters $\alpha, \beta, \gamma, \delta$ involving the spectral dimension d_S, which can be satisfied only for $\mathbf{d_S < 2}$.

For the metric case with nonlinear G such a result remains true for $\mathbf{d_S} < 4$ and $H_2^{\theta\delta}(\mu)$ (without the spatial sup-norm).

Standard example for $\mathbf{Z}^*$: let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal system of eigenfunctions of A in $L_2(\mu)$ (if exist) and λ_i be the corresponding eigenvalues, $\{B_i^H(t)\}_{i\in\mathbb{N}}$ are i.i.d fractional Brownian motions with Hurst exponent $0 < H < 1$, and take for Z^* the formal series

$$b_t^H := \sum_{i=1}^{\infty} B_i^H(t)\, q_i\, e_i \quad \text{with} \quad \sum_{i=1}^{\infty} q_i^2\, \lambda_i^{-2\beta'} < \infty\,,$$

for real coefficients q_i, then

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