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Viability for Stochastic Differential Equation Driven by Fractional Brownian Motion

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1 Introduction

In this lecture we base on the following references:

[1] J.-P.Aubin, and G.Da Prato:*Stochastic viability and Invariance,* Annali Scuola Normale di Pisa, Vol.27 (1990), 595-694.

[2] I.Ciotir and A.Răşcanu:*Viability for Stochastic Differential Equation Driven by Fractional Brownian Motions,* J. Differential Equations 247 (2009) 1505-1528

Our aim is to establish the deterministic necessary and sufficient conditions that guarantee that the solution of a given Stochastic Differential Equation(SDE) driven by the fractional Brownian motion evolves in a prescribed set K.

2 Stochastic viability problems

We consider the following SDE

$$X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dB_r, \quad s \in [t, T], \quad a.s.\, \omega \in \Omega, \qquad (1)$$

where

- $\{B_t, t \geq 0\}$ is a k- dimensional *(fractional)* Brownian motion defined on a complete stochastic base $\left(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \geq 0}\right)$.
- $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$ are continuous functions.

Let

$$\mathcal{K} = \{K(s) : s \in [0,T]\}; \quad K(s) = \overline{K(s)} \subset \mathbb{R}^d.$$

Definition

- $\mathcal{K}$ is viable (weak invariant) for the equation (1) if, $\forall\ t \in [0,T]$ and $\forall\ x \in K(t)$, there exist at least one solution $\{X_s^{t,x} : s \in [t,T]\}$ of the SDE (1) satisfies

$$X_s^{t,x} \in K(s) \qquad \text{for all } s \in [t,T].$$

- $\mathcal{K}$ is invariant (strong invariant) for the equation (1) if, $\forall\ t \in [0,T]$ and $\forall\ x \in K(t)$, all solutions $\{X_s^{t,x} : s \in [t,T]\}$ of the SDE (1) have the property

$$X_s^{t,x} \in K(s) \quad \text{for all } s \in [t,T].$$

Obviously in the case when the equation has a unique solution, viability is equivalent with invariance.

Starting with Nagumo's pioneer work in 1943, the viability property has been extensively studied for deterministic differential equations and inclusions.

To our knowledge the first work that gives a characterization of the viability property in a stochastic framework was written by Aubin and Da Prato(1990)

The key point of their work consists in defining a suitable Bouligand's stochastic tangent cone, which generalizes the cone used in the study of the viability property for deterministic systems.

Another approach has been developed in Buckdahn, Quincampoix, Rainer and Rascanu (2002).

The main point of this work consist in proving that the viability property holds if and only if the square of the distance to the constraint set is a viscosity supersolution of a PDE associated.

I will present the main result of this first, and in our lecture we mainly use the result of [2], which will be presented in the third section.

Theorem(Viability Criterion for SDE driven by BM)

We suppose that in the equation (1), $\{B_t, t \geq 0\}$ is a k- dimensional Brownian motion, and

- $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are continuous functions such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|, \forall x, y \in \mathbb{R}^d$$

- $t \mapsto d^2_{K(t)}(x) : [0,T] \to \mathbb{R}$ is lower semicontinuous(l.s.c.) $\forall x \in \mathbb{R}^d$;

- $\sup_{t\in[0,T]} d^2_{K(t)}(0) \leq M < \infty$

Denote

$$\begin{aligned}\mathcal{A}_t(\varphi)(x) &= \frac{1}{2}\mathbf{Tr}\left[\sigma(t,x)\sigma^*(t,x)\varphi''_{xx}(x)\right] + \langle b(t,x), \varphi'_x(x)\rangle \\ &= \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial \varphi(x)}{\partial x_i}.\end{aligned}$$

Then the next assertions are equivalent:

(I) Equation (1) is $\mathcal{K}$–viable on $[0,T]$.

(II) $u(t,x) = d^2_{K(t)}(x)$ is a l.s.c. viscosity supersolution of the partical differential equation

$$\frac{\partial U(t,x)}{\partial t} + \mathcal{A}_t U(t,x) - C d^2_{K(t)}(x) = 0,\ (t,x) \in [0,T] \in \mathbb{R}^m. \tag{2}$$

i.e. for any $\varphi \in C^{1,2}([0,T[\times\mathbb{R}^d)$, and $(t,x) \in [0,T] \times \mathbb{R}^d$ such that $u - \varphi$ has a local minimum at (t,x), it holds

$$\frac{\partial \varphi(t,x)}{\partial t} + \mathcal{A}_t \varphi(t,x) - C d^2_{K(t)}(x) \leq 0.$$

Example

Let us consider $a(t) \in C^2([0,T];\mathbb{R}^d)$ and $r(t) \in C^2([0,T];\mathbb{R}_+)$ with $r\,(t) \geq \delta > 0$, $t \in [0,T]$.

We put

$$K(t) = \{y \in \mathbb{R}^n : |y - a\,(t)| \leq r(t)\}, \quad t \in [0,T].$$

Then $\mathcal{K} = \{K(t) : t \in [0,T]\}$ is viable if and only if

for all $t \in [0,T]$ and $x \in \partial K(t) = \{y : |y - a\,(t)\,| = r(t)\}$,

$$\begin{cases} \sigma^*(t,x)\,(x - a\,(t)) = 0, \ \ \textit{and} \\ 2 < x - a\,(t)\,, b(t,x) > + \|\sigma(t,x)\|^2 \leq 2r(t)r'(t). \end{cases}$$

Applications:

1. *security tube in the traffic control*

$$\mathcal{K} = \{K(s) : s \in [0,T]\}; \quad \text{where}$$
$$K(s) = \left\{x \in \mathbb{R}^d : |x - a(s)| \le r(s)\right\}$$

2. *comparison of the solutions*

Consider the two dimensional decoupled system

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \ s \ge t \\ \\ Y_s^{t,y} = y + \int_t^s \tilde{b}(r, Y_r^{t,y})dr + \int_t^s \tilde{\sigma}(r, Y_r^{t,y})dB_r, \ s \ge t \end{cases}$$

and

$$\mathcal{K} = \{K(t) : t \in [0,T]\}; \quad \text{where } K(t) = \left\{(x,y) \in \mathbb{R}^2 : x \le y\right\}$$

The viability of $\mathcal{K}$ means:

$$\text{if } x \le y \quad \text{then } X_s^{t,x} \le Y_s^{t,y}, \quad \text{for all } s \ge t.$$

3. *controllability:*

If we choose the constraint sets of the form

$$\mathcal{K} = \{K(t) : t \in [0,T]\}; \quad \text{where}$$

$$K(t) = \begin{cases} \{x_0\}, & \text{if } t = 0, \\ \mathbb{R}^d, & \text{if } 0 \le t < T \\ \{x_T\}, & \text{if } t = T. \end{cases}$$

Then we can study the controlled problem: Given $x_0 \in \mathbb{R}^d$ and $x_T \in \mathbb{R}^d$, to find a control u and $X = X^{0,x_0;u}$ solution of the SDE

$$X_s = x_0 + \int_0^s [b(r, X_r) + u_r]\, dr + \int_0^s \sigma(r, X_r) dB_r, \quad s \in [0,T] \quad a.s.\ \omega \in \Omega,$$

such that

$$X_T = x_T \ ,$$

More generally, we can consider the problem:

$$X_s = x_0 + \int_0^s b(r, X_r, u_r) dr + \int_0^s \sigma(r, X_r, U_r) dB_r, \quad s \in [0,T] \quad a.s.\ \omega \in \Omega,$$

4. *stability:*
Consider the two dimensional decoupled system

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \\ \\ Y_s^{t,y} = y - \lambda \int_t^s Y_r^{t,y} dr, \end{cases}$$

and

$$\mathcal{K} = \{K(t) : t \in [0, T]\}; \quad \text{where } K(t) = \left\{(x, y) \in \mathbb{R}^2 : x \leq y\right\}$$

The viability of $\mathcal{K}$ means:

$$\text{if } x \leq y \quad \text{then } X_s^{t,x} \leq Y_s^{t,y} = ye^{-\lambda(s-t)}, \quad \text{for all } s \geq t.$$

3 Viability for SDE driven by fBm

3.1 Preliminaries

In this part we present a result concerning the viability for SDE driven by fBm. Consider the SDE on $\mathbb{R}^d$

$$X_s = X_0 + \int_0^s b(r, X_r)dr + \int_0^s \sigma\left(r, X_r\right) dB_r^H, \;\; s \in [0, T] .$$

where

- $B = \{B_t, t \geq 0\}$ is a k- dimensional fractional Brownian motion with Hurst parameter $\frac{1}{2} < H < 1$, and the integral with respect to B is a pathwise Riemann-Stieltjes integral;
- X_0 is a d - dimensional random variable defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ are continuous functions.

Assume

$(\mathbf{H}_1)$ there exist some constants $\beta, \delta, 0 < \beta, \delta \leq 1$, and for every $R \geq 0$ there exists $M_R > 0$ such that for all $t \in [0, T]$:

$$i) \quad |\sigma(t,x) - \sigma(s,y)| \leq M_0 \left(|t-s|^\beta + |x-y|\right), \quad \forall x, y \in \mathbb{R}^d,$$

$$ii) \quad |\nabla_x \sigma(t,y) - \nabla_x \sigma(s,z)| \leq M_R \left(|t-s|^\beta + |y-z|^\delta\right), \quad \forall |y|, |z| \leq R,$$

Remark that for all $x \in \mathbb{R}^d$, $\sigma(t,x)$ has sublinear growth.

Let

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}.$$

$(\mathbf{H}_2)$ There exist $\mu \in (1-\alpha_0, 1]$ and for every $R \geq 0$ there exists $L_R > 0$ such that $\forall t \in [0, T]$, :

$i)$ Local (Hölder-Lipschitz) continuity:
$|b(r,x) - b(s,y)| \leq L_R \left(|r-s|^\mu + |x-y|\right), \qquad \forall |x|, |y| \leq R,$
$ii)$ Boundedness : $|b(t,x)| \leq L_0(1+|x|), \quad \forall x \in \mathbb{R}^d.$

D.Nualart and A.Răşcanu prove in *Differential equations driven by fractional Brownian motion.* (2002), the following theorem:

Theorem:

Under the assumpations(H_1) and (H_2) , with $\beta > 1 - H$ and $\delta > \frac{1}{H} - 1$ the SDE

$$X_s^{t,\xi} = \xi + \int_t^s b(r, X_r^{t,\xi})dr + \int_t^s \sigma\left(r, X_r^{t,\xi}\right) dB_r, \ \ s \in [t,T]\,,$$

has a unique solution $X^{t,\xi} \in L^0\left(\Omega, \mathcal{F}, \mathbb{P}\,; W^{\alpha,\infty}(t,T;\mathbb{R}^d)\right)$, for all $\alpha \in (1-H, \alpha_0)$. Moreover, for $\mathbb{P}$-almost all $\omega \in \Omega$

$$X\left(\omega, \cdot\right) \in C^{1-\alpha}\left(0, T; \mathbb{R}^d\right).$$

The definition of $W^{\alpha,\infty}(t,T;\mathbb{R}^d)$ and $C^{1-\alpha}$ will be given in the follow.

Let $t \in [0,T]$ be fixed. Denote

- $W^{\alpha,\infty}(t,T;\mathbb{R}^d)$, $0<\alpha<1$, the space of continuous functions $f:[t,T]\to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty;[t,T]} := \sup_{s\in[t,T]} \left(|f(s)| + \int_t^s \frac{|f\,(s) - f\,(r)|}{(s-r)^{\alpha+1}} dr \right) < \infty.$$

  An equivalent norm can be defined by

$$\|f\|_{\alpha,\lambda;[t,T]} := \sup_{s\in[t,T]} e^{-\lambda s} \left(|f(s)| + \int_t^s \frac{|f\,(s) - f\,(r)|}{(s-r)^{\alpha+1}} dr \right) \quad \forall \lambda \geq 0.$$

- $\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)$, the space of continuous functions $g:[t,T]\to\mathbb{R}^d$ such that

$$\|g\|_{\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)} := |g\,(t)| + \sup_{t<r<s<T} \left(\frac{|g(s)-g(r)|}{(s-r)^{1-\alpha}} + \int_r^s \frac{|g(y)-g(r)|}{(y-r)^{2-\alpha}} dy \right) < \infty$$

- $C^{\mu}([t,T];\mathbb{R}^d)$, $0<\mu<1$, the space of μ-Holder continuous functions $f:[t,T]\to\mathbb{R}^d$, equipped with the norm

$$\|f\|_{\mu;[t,T]} := \|f\|_{\infty;[t,T]} + \sup_{t\le r<s\le T} \frac{|f(s)-f(r)|}{(s-r)^{\mu}} < \infty$$

where $\|f\|_{\infty;[t,T]} := \sup_{s\in[t,T]}|f(s)|$. We have, for all $0<\epsilon<\alpha$

$$C^{\alpha+\epsilon}([t,T];\mathbb{R}^d) \subset W^{\alpha,\infty}(t,T;\mathbb{R}^d)$$

- $W^{\alpha,1}(t,T;\mathbb{R}^d)$ the space of measurable functions f on $[t,T]$ such that

$$\|f\|_{\alpha,1;[t,T]} := \int_t^T \left[\frac{|f(s)|}{(s-t)^{\alpha}} + \int_t^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}}dy\right] ds < \infty.$$

Clearly

$$W^{\alpha,\infty}(t,T;\mathbb{R}^d) \subset W^{\alpha,1}(t,T;\mathbb{R}^d).$$

Definition Let $0 < \alpha < \frac{1}{2}$. If $f \in W^{\alpha,1}(t,T;\mathbb{R}^{d\times k})$ and $g \in \tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^{k})$, then defining

$$\int_t^s f(r)\,dg(r) := (-1)^{\alpha} \int_t^s \left(D_{t+}^{\alpha} f\right)(r) \left(D_{s-}^{1-\alpha} g_{s-}\right)(r)\,dr.$$

the integral $\int_t^s f dg$ exists for all $s \in [t,T]$ and

$$\left|\int_t^T f(r)\,dg(r)\right| \leq \sup_{t\leq r<s\leq T} |(D_{s-}^{1-\alpha} g_{s-})(r)| \int_t^T |(D_{t+}^{\alpha} f)(s)ds| \leq \Lambda_{\alpha}(g;[t,T]) \left\|f\right\|_{\alpha,1;[t,T]}$$

Where

$$\Lambda_{\alpha}(g;[t,T]) = \frac{1}{\Gamma(1-\alpha)} \sup_{t<r<s<T} \left|\left(D_{s-}^{1-\alpha} g_{s-}\right)(r)\right|.$$

$$(D_{t+}^{\alpha} f)(r) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(r)}{(r-t)^{\alpha}} + \alpha \int_t^r \frac{f(r)-f(y)}{(r-y)^{\alpha+1}} dy\right) 1_{(t,T)}(r).$$

$$(D_{s-}^{1-\alpha} g_{s-})(r) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)} \left(\frac{g(s)-g(r)}{(s-r)^{1-\alpha}} + (1-\alpha) \int_r^s \frac{g(r)-g(y)}{(y-r)^{2-\alpha}} dy\right) 1_{(t,s)}(r).$$

We will present now the deterministic approach for the study of viability for SDE(3).

More precisely, let arbitrary fixed $(t,x) \in [0,T] \times \mathbb{R}^d$. and we will consider the deterministic differential equation on $\mathbb{R}^d$:

$$X_s^{tx} = x + \int_t^s b(r, X_r^{tx})dr + \int_t^s \sigma\left(r, X_r^{tx}\right) dg\left(r\right), \quad s \in [t,T], \tag{3}$$

where

$$g \in \tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^k)$$

We also assume (H_1) and (H_2) be satisfied.

We recall here the main definitions and results from I.Ciotir and A.Răşcanu: *Viability for SDE driven by fBM,* (2009) .

Let α be arbitrary fixed such that $0 < \alpha < \alpha_0$.

Let $\mathcal{K} = \{K(t) : t \in [0, T]\}$ be a family of nonempty closed subsets of $\mathbb{R}^d$.

Definition. Let $t \in [0, T]$ and $x \in K(t)$. Let $\frac{1}{2} < 1 - \alpha < H$.

The pair $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$-fractional g-contingent to $K(t)$ in (t, x) if there exist $\bar{h} = \bar{h}^{t,x} > 0$, and a function $Q = Q^{t,x} : \left[t, t + \bar{h}\right] \to \mathbb{R}^d$ such that for all $s, \tau \in \left[t, t + \bar{h}\right]$ and $|x| \leq R$:

$$|Q(s) - Q(\tau)| \leq G_R |s - \tau|^{1-\alpha}, \qquad |Q(s)| \leq \tilde{G}_R |s - t|^{1+\gamma}$$

and

$$x + (s - t) b(t, x) + \sigma(t, x) [g(s) - g(t)] + Q(s) \in K(s),$$

where the constants G_R, $\tilde{G}_R$, γ_R depend only on R, L_R, M_0, L_0, T, α, β, $\Lambda_\alpha(g)$.

Definition. Let $t \in [0,T]$ and $x \in K(t)$. Let $\frac{1}{2} < 1-\alpha < H$.

The pair $(b(t,x), \sigma(t,x))$ is $(1-\alpha)$-fractional g -tangent to $K(t)$ in (t,x) if there exist $\bar{h} = \bar{h}^{t,x} > 0$, and two functions

$$U = U^{t,x} : \left[t, t+\bar{h}\right] \to \mathbb{R}^d, \; U(t) = 0$$
$$V = V^{t,x} : \left[t, t+\bar{h}\right] \to \mathbb{R}^{d\times k}, \; V(t) = 0$$

such that for all $s, \tau \in \left[t, t+\bar{h}\right]$ and $|x| \leq R$:

$$|U(s) - U(\tau)| \leq D_R \left|s-\tau\right|^{1-\alpha}, \qquad |V(s) - V(\tau)| \leq \tilde{D}_R \left|s-t\right|^{min\{\beta, 1-\alpha\}}$$

and

$$x + \int_t^s (b(t,x) + U(r))dr + \int_t^s (\sigma(t,x) + V(r))dg(r) \in K(s),$$

where the constants D_R, $\tilde{D}_R$, depend only on R, L_R, M_0, L_0, T, α, β, $\Lambda_\alpha(g)$.

Theorem.

Assume (H_1) and (H_2) are satisfied and $1-\mu<\alpha<\alpha_0$.

The following assertions are equivalent:

(I) K is viable for the fractional differential equation , i.e. $\forall t\in[0,T]$ and $\forall x\in K(t)$, there exists a solution $X^{t,x}(\cdot)\in C^{1-\alpha}\left([t,T];\ \mathbb{R}^d\right)$ for

$$X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s \sigma(r,X_r^{t,x})dg(r)\,,\ \ s\in[t,T],$$

such that

$$X_s^{t,x}\in K(s)\,,\quad \text{for all } s\in[t,T].$$

(II) $\forall t\in[0,T]$ and all $x\in K(t)$ the pair$(b(t,x),\sigma(t,x))$ is $(1-\alpha)$-fractional g -contingent to $K(t)$ in (t,x)

(III) $\forall t\in[0,T]$ and all $x\in K(t)$ the pair$(b(t,x),\sigma(t,x))$ is $(1-\alpha)$-fractional g-tangent to $K(t)$ in (t,x)

Therefore, choosing $g = B^H$, we have similar definition for B^H-contingent and B^H-tangent. We denote B^H-tangent by $S_{K(t)}(t,x)$.

Then we have the theorem providing the characterization of the viability:

Theorem. Let $1-H < \alpha < \alpha_0$ and $\mathcal{K} = \{K(t) : t \in [0,T]\}$, $K(t) = \overline{K(t)} \subset \mathbb{R}^d$.

Then the following assertions are equivalent:

(I) $\mathcal{K}$ is viable for the fractional SDE, i.e. for all $t \in [0,T]$ and for all $x \in K(t)$ there exists a solution $X^{t,x}(\omega,\cdot) \in C^{1-\alpha}\left([t,T]\,;\ \mathbb{R}^d\right)$ of the equation

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r^H, \quad s \in [t,T], \ a.s.\ \omega \in \Omega, \tag{4}$$

and

$$X_s^{t,x} \in K(s), \qquad \forall\ s \in [t,T].$$

(II) For all $t \in [0,T]$ and all $x \in K(t)$,

$(b(t,x), \sigma(t,x))$ *is* $(1-\alpha)$-fractional B^H -contingent to $K(t)$ in (t,x) .

3.2 Deterministic Sufficient and Necessary Conditions. Main Results

From the result of [2], we obtain

Corollary.

Let $1 - H < \alpha < \alpha_0$ and $\mathcal{K} = \{K(t) : t \in [0, T]\}$, $K(t) = \overline{K(t)} \subset \mathbb{R}^d$. Then the following assertions are equivalent:

(I) $\mathcal{K}$ is viable for the fractional SDE(4).

(II) For all $t \in [0, T]$ and all $x \in K(t)$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$-fractional B^H-contingent to $K(t)$ in (t, x) .

(III) For all $t \in [0, T]$ and all $x \in K(t)$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$-fractional B^H-tangent to $K(t)$ *in* (t, x) .

We give two Lemmas for the basic estimates which we will use later.

Lemma 1. The solution of SDE(4), $X^{t,x}$ is $(1-\alpha)$-Holder continuous and

$$\left\|X_{\cdot}^{t,x}\right\|_{1-\alpha;[t,T]} \leq C_0\left(1+|x|\right)$$

where C_0 is a constant depending only on M_0, L_0 T, α, β, $\Lambda_\alpha(g)$.

Lemma 2. Let (H_1) and (H_2) be satisfied and $1-\mu \leq \alpha < \beta \wedge \frac{1}{2}$. If Y is a Holder continuous function with

$$\|Y\|_{1-\alpha;[t,T]} \leq R$$

then $\exists\ C_R^{(i)}$ depending on $R, M_0, T, \alpha, \beta, \Lambda_\alpha(g)$ s.t. for all $0 \leq t \leq s \leq T$,

$$(a) \quad \left|\int_t^s \left[b(r,Y_r) - b\left(t,Y_t\right)\right] dr\right| \leq C_R^{(1)}\left(s-t\right)^{2-\alpha} \quad \text{and}$$

$$(b) \quad \left|\int_t^s \left[\sigma(r,Y_r) - \sigma\left(t,Y_t\right)\right] dg\left(r\right)\right| \leq C_R^{(2)}\left(s-t\right)^{1+\min\{\beta-\alpha,1-2\alpha\}}.$$

Following the paper of J.-P.Aubin, and G.Da Prato**,** :*Stochastic viability and Invariance,* (1990), we obtain the theorem concerning the *Stochastic Tangent Sets to Direct Images* in the fBM framework.

Theorem 3.2.1 Let φ be a $C^{(2)}$ map from $\mathbb{R}^d$ to $\mathbb{R}^m$. If

$$(b(t,x),\sigma(t,x)) \in S_{K(t)}(t,x)$$

then

$$(\varphi'(x)b(t,x),\varphi'(x)\sigma(t,x)) \in S_{\varphi(K(t))}(t,\varphi(x)).$$

Main idea of Proof.

Let $(b(t,x),\sigma(t,x)) \in S_{K(t)}(t,x)$, Using the Itô formula (see Yuliya S.Mishura–Stochastic Calculus for Fractional Brownian Motion and Related Processes(2007)), and Lemma 1 and $(\mathbf{H}_2)$, we can prove

$$(\varphi'(x)b(t,x),\varphi'(x)\sigma(t,x)) \in S_{\varphi(K(t))}(t,\varphi(x)).$$

$\square$

Also we can prove *the Stochastic Tangent Sets to Inverse Images* in the fBM form.

Theorem 3.2.2 Let φ *be a* $C^{(2)}$ map from $\mathbb{R}^d$ to $\mathbb{R}^m$. Assume that there exist a random variable $\bar{h} = \bar{h}^{t,x} > 0$, such that $\forall s \in [t, t+\bar{h}], \varphi'(X_s)^+$ is bounded and Lipschitz. where $\varphi'(X_s)^+$ denote the right inverse of $\varphi'(X_s)$. *then*

$$(b(t,x), \sigma(t,x)) \in S_{K(t)}(t,x)$$

if and only if

$$(\varphi'(x)b(t,x), \varphi'(x)\sigma(t,x)) \in S_{\varphi(K(t))}(t, \varphi(x)).$$

Proof. It remains to assume that $(\varphi'(x)b(t,x), \varphi'(x)\sigma(t,x)) \in S_{\varphi(K(t))}(t, \varphi(x))$. and to infer that $(b(t,x), \sigma(t,x)) \in S_{K(t)}(t,x)$.

It's similar to the proof of Theorem 3.2.1, but here we use the right inverse of $\varphi'(X_s)$. And we need $\varphi'(X_s)^+$ is bounded and Lipschitz. It's very useful for our estimates.

$\square$

In this part, we mainly use Theorem 3.2.2 to get the deterministic sufficient and necessary conditions for viability when K takes some particular form.

Lemma 3. Let K be the unit sphere, then for all $x \in K$, $(b(t,x), \sigma(t,x)) \in S_K(t,x)$ if and only if

$$\langle x, b(t,x)\rangle = 0, \qquad \langle x, \sigma(t,x)\rangle = 0$$

Proof: Firstly, we take $\varphi(x) = |x|^2$, then

$$\varphi(x)^+ = \frac{x}{2|x|^2}$$

And it is easy to verify that $\varphi(x)$ satisfies our conditions. We can use the method of Lemma 2(b) to get some estimates. Then use Theorem 3.2.2 we can get

$$\langle x, b(t,x)\rangle = 0, \qquad \langle x, \sigma(t,x)\rangle = 0$$

$\square$

Using the same method, we can also get this following Lemma.

Lemma 4. Let $K = \{x \in \mathbb{R}^d; r \leq |x| \leq R\}$ then $\forall x$, such that $|x| = R$, $(b(t,x), \sigma(t,x)) \in S_K(t,x)$ if and only if

$$\langle x, b(t,x)\rangle \leq 0, \qquad \langle x, \sigma(t,x)\rangle = 0$$

and $\forall x$, such that $|x| = r$, $(b(t,x), \sigma(t,x)) \in S_K(t,x)$ if and only if

$$\langle x, b(t,x)\rangle \geq 0, \qquad \langle x, \sigma(t,x)\rangle = 0$$

From Lemma 4. It's also easy to get

Lemma 5. Let K be the unit ball, then $\forall x$, such that $|x| = 1$, $(b(t,x), \sigma(t,x)) \in S_K(t,x)$ if and only if

$$\langle x, b(t,x)\rangle \leq 0, \qquad \langle x, \sigma(t,x)\rangle = 0$$

Remark. Considering that if we want get the conditions for the viability of K ,we only need to think about the starting point $x \in \partial K$.

From this Remark, we have

Corollary 1. Let $1 - H < \alpha < \alpha_0$ and K is the unit sphere. Then the following assertions are equivalent:

(I) K is viable for the fractional SDE.

(II) $\forall t \in [0, T]$ and all $x \in K$,

$$\langle x, b(t,x) \rangle = 0, \qquad \langle x, \sigma(t,x) \rangle = 0.$$

Corollary 2. Let $1 - H < \alpha < \alpha_0$ and K is the unit ball. Then the following assertions are equivalent:

(I) K is viable for the fractional SDE.

(II) $\forall t \in [0, T]$ and all $|x| = 1$,

$$\langle x, b(t,x) \rangle \leq 0, \qquad \langle x, \sigma(t,x) \rangle = 0.$$

Example 1.

Consider the SDE on $\mathbb{R}$

$$X_s = x + \int_0^s b(r, X_r) dr + \int_0^s \sigma\left(r, X_r\right) dB_r^H, \ \ s \in [0, T] \, .$$

where

- $B = \{B_t, t \geq 0\}$ is a k- dimensional fractional Brownian motion.
- b, σ satisfy the assumptions $\mathbf{H}_1, \mathbf{H}_2$.
- $x \geq 0$.

Then it has a positive solution if and only if

$$b(t, 0) \geq 0, \sigma(t, 0) = 0, \forall t \in [0, T]$$

In fact we take $K = [0, +\infty)$, the problem is just that K is viable for the fractional SDE. We can use $x = \tan \frac{\pi}{4}(y+1)$ and we can get $y = \frac{4}{\pi} \arctan x - 1$, it just maps $[0, +\infty)$ to $[-1, 1]$, and we can use Corollary 2.

Example 2.

comparison of the solutions

Consider the two dimensional decoupled system

$$\begin{cases} X_s^{t,x} = x + \int_t^s (f(r)X_r^{t,x} + f_1(r))dr + \int_t^s (g(r)X_r^{t,x} + g_1(r))dB_r,\ s \in [t,T] \\ Y_s^{t,y} = y + \int_t^s (f(r)Y_r^{t,x} + f_2(r))dr + \int_t^s (g(r)Y_r^{t,x} + g_2(r))dB_r,\ s \in [t,T] \end{cases}$$

Then using the viability theorem, we can get if $x \leq y$, then

$$f_1(t) \leq f_2(t), \quad g_1(t) = g_2(t) \qquad \forall t \in [0,T].$$

$$\Longleftrightarrow X_s^{t,x} \leq Y_s^{t,x}.$$

Thank you for your attention !