From persistent random walk to the telegraph noise

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Outline

1. Introduction

Let $(Y_n)_{n\geq 0}$ be a Markov chain taking its values in $\{-1, 1\}$ with transition matrix :

$$\pi = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \qquad 0 < \alpha < 1, \quad 0 < \beta < 1.$$

Associated with (Y_n) consider the process

$$X_n := Y_0 + Y_1 + \cdots + Y_n, \quad n \geq 0.$$

(X_n) is said to be a **persistent random walk.**
Two particular cases are interesting :

- $\beta = 1-\alpha$: (X_n) is a classical random walk whose increment is distributed as $(1-\alpha)\delta_{-1} + \alpha\delta_1$.
- $\beta = \alpha$, (X_n) is a *Kac* random walk : $Y_{n+1} = Y_n$ with probability $1-\alpha$ and $-Y_n$ otherwise.

2. Study at a fixed time and applications

Proposition 1 *Let* $\rho := 1 - \alpha - \beta$ *the asymmetry factor. Then :*

$$E[X_t|Y_0 = -1] = \frac{\alpha - \beta}{1 - \rho}(t + 1) - \frac{2\alpha}{(1 - \rho)^2}(1 - \rho^{t+1}).$$

$$E[X_t|Y_0 = +1] = \frac{\alpha - \beta}{1 - \rho}(t + 1) - \frac{2\beta}{(1 - \rho)^2}(1 - \rho^{t+1}).$$

Remark *1) In the classical random walk case, we have :* $\rho = 0$.
2) it is actually possible to compute explicitly the second moment of X_t*, see C. Tapiero and P.V. : Memory-based persistence in a counting random walk process, Physica A, 2007*

Let us introduce :

$$\Phi(\lambda, t) = E[\lambda^{X_t}], \quad (\lambda > 0).$$

Proposition 2 *The generating function of X_t equals:*

$$\Phi(\lambda, t) = a_+ \theta_+^t + a_- \theta_-^t$$

with

$$a_+ = \frac{1 - \alpha + \lambda(\lambda\alpha - \theta_-)}{\lambda^2 \sqrt{\mathcal{D}}} \quad \textit{and} \quad a_- = \frac{1}{\lambda} - a_+ \quad \textit{when } X_0 = Y_0 = -1$$

$$\theta_\pm := \frac{1}{2}\Big(\frac{1-\alpha}{\lambda} + (1-\beta)\lambda \pm \sqrt{\mathcal{D}}\Big)$$

$$\mathcal{D} = \Big(\frac{1-\alpha}{\lambda} + (1-\beta)\lambda\Big)^2 - 4(1-\alpha-\beta).$$

Sketch of the proof of Proposition 2

We decompose $\Phi(\lambda, t)$ as follows :

$$\Phi(\lambda, t) = \Phi_-(\lambda, t) + \Phi_+(\lambda, t),$$

with

$$\Phi_-(\lambda, t) = E[\lambda^{X_t} 1_{\{Y_t=-1\}}], \qquad \Phi_+(\lambda, t) = E[\lambda^{X_t} 1_{\{Y_t=1\}}].$$

The, we obtain the recursive relations :

$$\begin{aligned} \Phi_-(\lambda, t+1) &= \frac{1-\alpha}{\lambda}\,\Phi_-(\lambda, t) + \frac{\beta}{\lambda}\,\Phi_+(\lambda, t) \\ \Phi_+(\lambda, t+1) &= \alpha\lambda\Phi_-(\lambda, t) + (1-\beta)\lambda\Phi_+(\lambda, t) \end{aligned}$$

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An application to insurance

- A "normal claim" is labelled 0 and its values at time i is Z_i^0;
- an "unusual claim" (for instance "large") is labelled 1 and equals Z_i^1 at time i.
- The claims $\left(Z_i^0, i \geq 1\right)$ are i.i.d, $\left(Z_i^1, i \geq 1\right)$ are i.i.d and the two families of r.v.'s are independent.
- The process which attributes labels is (Y_i'). So, $Y_i' = 0$ if at time i a normal claim occurs.

Note that $Y_i' \in \{0, 1\}$. Set $Y_i = 2Y_i' - 1$. Then

$$Y_i \in \{-1, 1\} \quad \text{and} \quad Y_i' = 1 \ \Leftrightarrow Y_i = 1.$$

- We suppose that :

* (Y_i') is a Markov chain (then (Y_i) is a Markov chain as above);
* all the claims $\left(Z_i^j, j = 0, 1, \ i \geq 1\right)$ and (Y_i') are independent.

The sum of claims at time t is :

$$\xi_t = \sum_{i=0}^{t} Z_t^1 1_{\{Y_i'=1\}} + \sum_{i=0}^{t} Z_t^0 1_{\{Y_i'=0\}}.$$

Proposition 3 *1) The first moment of ξ_t is :*

$$E(\xi_t) = (t+1)E(Z_1^0) + \left[E(Z_1^1) - E(Z_1^0)\right] E(X_t')$$

where $X_t' = \sum_{i=0}^{t} Y_i'$.

2) The Laplace transform of ξ_t equals :

$$E\left(e^{-\lambda \xi_t}\right) = \left[E\left(e^{-\lambda Z_1^0}\right)\right]^{t+1} \widetilde{\Phi}(z,t) \quad \lambda > 0$$

where

$$z := \frac{E\left(e^{-\lambda Z_1^1}\right)}{E\left(e^{-\lambda Z_1^0}\right)}, \quad \widetilde{\Phi}(z,t) := E\left(z^{X_t'}\right).$$

Remark *Reference : C. Tapiero and P.V. A claims persistence process and Insurance, Insurance : Mathematics and Economics (2009).*
2) Recall that :

$$X_t = 2X'_t - (t+1).$$

Then,

$$E(X'_t) = \frac{1}{2}\big(E(X_t) + t + 1\big), \quad E\big(z^{X'_t}\big) = z^{\frac{t+1}{2}} E\big(z^{X_t/2}\big)$$

3. From discrete to continuous time

S. Herrmann and P.V. : From persistent random walks to the Telegraph noise. Accepted in Stochastics and Dynamics (2009).

3.1 Notations

a) Denote α_0, β_0 two real numbers : $0 < \alpha_0 \leq 1, \quad 0 < \beta_0 \leq 1$.
b) Δ_x is a "small" parameter such that :

$$\alpha := \alpha_0 + c_0\Delta_x \in [0,1], \quad \beta := \beta_0 + c_1\Delta_x \in [0,1].$$

c) $(Y_t,\ t \in \mathbb{N})$ is a Markov chain which takes its values in $\{-1, 1\}$ with transition matrix :

$$\pi^{\Delta} = \begin{pmatrix} 1-\alpha_0 - c_0\Delta_x & \alpha_0 + c_0\Delta_x \\ \beta_0 + c_1\Delta_x & 1-\beta_0 - c_1\Delta_x \end{pmatrix}$$

d) The re-normalized random walk associated with (Y_t) is defined as :

$$Z_s^{\Delta} = \Delta_x X_{s/\Delta_t}, \quad s \in \Delta_t \, \mathbb{N} \quad (\Delta_t > 0).$$

e) $(\widetilde{Z}_s^{\Delta}, \, s \geq 0)$ is the continuous time process which is obtained by linear interpolation from (Z_s^{Δ}).

f) Set

$$\rho_0 = 1 - \alpha_0 - \beta_0$$

ρ takes into account the "distance" of the persistent random walk to the classical r. w.

Remark *Note that :*

$$\rho_0 = 1 \Leftrightarrow 1 - \alpha_0 - \beta_0 = 0 \Leftrightarrow \alpha_0 + \beta_0 = 0 \Leftrightarrow \alpha_0 = \beta_0 = 0.$$

3.2 Convergence to the Brownian motion with drift, $\rho_0 \neq 1$

Theorem 4 *We assume that* $\alpha_0, \beta_0 > 0$ *(i.e.* $\rho_0 \neq 0$*) and*

$$r\Delta_t = \Delta_x^2 \quad (r > 0).$$

Then the processes

$$\xi_t^\Delta = \widetilde{Z}_t^\Delta + \frac{\sqrt{r}\eta_0}{1-\rho_0}\frac{t}{\sqrt{\Delta_t}}$$

converge in distribution to the process $(\xi_t^0,\ t \geq 0)$*, as* $\Delta_x \to 0$*, with :*

$$\xi_t^0 = r\Big(\frac{-\overline{c}}{1-\rho_0} + \frac{\eta_0 c}{(1-\rho_0)^2}\Big)t + \sqrt{\frac{r(1+\rho_0)}{1-\rho_0}\Big(1 - \frac{\eta_0^2}{(1-\rho_0)^2}\Big)}\ W_t,$$

where $(W_t, t \geq 0)$ *stands for a standard Brownian motion and :*

$$\eta_0 = \beta_0 - \alpha_0, \quad c = c_0 + c_1, \quad \overline{c} = c_1 - c_0.$$

Remark $\eta_0 = 0$ *corresponds to the Kac random walk.*

Theorem 4 is a consequence of the central limit theorem :

$$
\begin{aligned}
Z_1^{\Delta} &= \sqrt{n}\Big(\frac{Y_1 + \cdots + Y_n}{n}\Big) \quad (\Delta_t = \frac{1}{n},\ \Delta_x = \frac{1}{\sqrt{n}}) \\
&= \sqrt{n}\Big(\frac{\big\{Y_1 - E(Y_1)\big\} + \cdots + \big\{Y_n - E(Y_n)\big\}}{n}\Big) + R_n,
\end{aligned}
$$

with $R_n := \sqrt{n}\Big(\frac{E(Y_1) + \cdots + E(Y_n)}{n}\Big)$.

Since $\nu := \frac{\beta}{\alpha+\beta}\delta_{-1} + \frac{\alpha}{\alpha+\beta}\delta_1$ is the invariant probability measure associated with the Markov chain (Y_n) we have

$$
\lim_{n\to\infty} E(Y_n) = \int x\nu_0(dx) = \frac{\alpha_0 - \beta_0}{\alpha_0 + \beta_0} = \frac{\eta_0}{1-\rho_0}.
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3.3 Convergence when $\rho_0 = 1$

In this case, the transition matrix of (Y_t) equals

$$\pi^{\Delta} = \begin{pmatrix} 1 - c_0\Delta_x & c_0\Delta_x \\ c_1\Delta_x & 1 - c_1\Delta_x \end{pmatrix} \quad (c_0, c_1 > 0).$$

Consider a sequence $(e_n,\ n \geq 1)$ of independent r.r.v.'s such that : $(e_{2n},\ n \geq 1)$ (resp. $(e_{2n-1};\ n \geq 1)$) are iid with common exponential distribution with parameter $\frac{1}{c_1}$ (resp. $\frac{1}{c_0}$) i.e. $E[e_{2n}] = c_1$ (resp. $E[e_{2n-1}] = c_0$). Let

$$N_t^{c_0,c_1} = \sum_{k \geq 1} 1_{\{e_1 + \ldots + e_k \leq t\}}, \quad t \geq 0.$$

be the counting process associated with $(e_n;\ n \geq 1)$.

Theorem 5 *We suppose :*

$$\alpha_0 = \beta_0 = 0, \quad Y_0 = -1, \quad \Delta_x = \Delta_t.$$

Then, the interpolated persistent random walk $(\widetilde{Z}_s^{\Delta},\ s \geq 0)$ *converges in distribution, as* $\Delta_x \to 0$*,to the process* $\big(-Z_s^{c_0,c_1}\big)$ *where :*

$$Z_s^{c_0,c_1} = \int_0^s (-1)^{N_u^{c_0,c_1}}\, du \quad s \geq 0.$$

In the case where $c_0 = c_1$*, then* $\big(N_u^{c_0,c_1}\big)$ *is the Poisson process with parameter* c_0*.*

Remarks

- *In the symmetric case, Theorem 5 is a stochastic version of analytical approaches developed by Kac (1974). See for instance the G. Weiss book (1994).*
- *The process* $\left(Z_s^{c,c}\right)$ *has been already introduced by D. Stroock (1982).*
- *The convergence in terms of continuous processes allows to obtain for instance the convergence in distribution of* $\max_{0\le s\le 1} \widetilde{Z}_s^{\Delta}$ *to the r.v.* $\max_{0\le s\le 1}\left(-Z_s^{c_0,c_1}\right)$*, as* $\Delta_x \to 0$*.*

Sketch of the proof of Theorem 5

We only consider $Y_0 = -1$. Let :

$$T_1 = \inf\{n \geq 1;\ Y_n = 1\}.$$

Then $T_1 \sim \mathcal{G}(c_0 \Delta_x)$.
Consequently :

$$\begin{aligned} T_1' &= \inf\{s;\ \widetilde{Z}_s^{\Delta} > \widetilde{Z}_{s-}^{\Delta}\} \\ &= \inf\{n\Delta_t;\ Y_n = 1\} \\ &= T_1 \Delta_t. \end{aligned}$$

it is easy to deduce the convergence in distribution of T_1' to e_1, as $\Delta_x = \Delta_t \to 0$. Recall that e_1 is exponentially distributed with parameter $1/c_0$.

3.4 A few properties of $(Z_t^{c_0,c_1})$

Recall

$$Z_s^{c_0,c_1} = \int_0^s (-1)^{N_u^{c_0,c_1}} du.$$

a) $\big((Z_t^{c_0,c_1}),\ t \geq 0\big)$ is not Markov, however the process $\Big((N_t^{c_0,c_1}, Z_t^{c_0,c_1}),\ t \geq 0\Big)$ is Markov. Its semigroup and the joint law of $(N_t^{c_0,c_1}, Z_t^{c_0,c_1})$ can be determined explicitly.
Note that

$$\frac{dZ_t^{c_0,c_1}}{dt} = (-1)^{N_t^{c_0,c_1}}, \quad t \geq 0.$$

b) We can calculate :

$$P\big(N_t^{c_0,c_1} = 2k\big), \quad P\big(N_t^{c_0,c_1} = 2k+1\big)$$

c) The distribution of $Z_t^{c_0,c_1}$ is the following :

$$P\left(Z_t^{c_0,c_1} \in dx\right) = e^{c_0 t}\delta_t(dx) + \frac{1}{2}e^{-c_0 t} f(t,x) 1_{[-1,1]}(x)dx$$

with δ_t the Dirac measure at t,

$$f(t,x) = \sqrt{\frac{c_0 c_1 (t+x)}{t-x}} I_1\left(\sqrt{c_0 c_1 (t^2 - x^2)}\right) + c_0 I_0\left(\sqrt{c_0 c_1 (t^2 - x^2)}\right)$$

and

$$I_\nu(x) = \sum_{k\geq 0} \frac{1}{\Gamma(\nu + k + 1) k!}\left(\frac{x}{2}\right)^{\nu + 2k}.$$

d) The Laplace transform of $Z_t^{c_0,c_1}$ can be determined.

3.5 Link with the telegraph equation

For simplicity we suppose that $c_0 = c_1 = c$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ with bounded derivatives.
Introduce :

$$u(x,t) = \frac{1}{2}\Big\{ f(x+\sigma t) + f(x-\sigma t) \Big\}.$$

Then, u is the unique solution of the wave equation in dimension 1 :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \sigma^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0. \end{cases}$$

Proposition 6 *The function*

$$w(x,t) = E\Big[u\Big(x, Z_t^{c,c}\Big)\Big], \quad (x \in \mathbb{R}, t \geq 0)$$

where

$$Z_t^{c,c} = \int_0^t (-1)^{N_s^{c,c}} ds$$

is the solution of the telegraph equation :

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + 2c\dfrac{\partial w}{\partial t} = \sigma^2 \dfrac{\partial^2 w}{\partial x^2}, \\ w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = 0. \end{cases}$$

Remark We have a proof based on stochastic calculus.

4. Extensions

4.1 The case where Y_t takes its values in a finite set

Let us deal with the case where (Y_t) is $\{y_1, \cdots, y_k\}$-valued. Denote $\left(\pi^{\Delta}(i,j), 1 \leq i,j \leq k\right)$ its transition matrix. Suppose :

$$\pi^{\Delta}(i,j) = \begin{cases} c(i,j)\Delta_t & si \ i \neq j \\ 1 - \Big(\sum_{l=1}^{k} c(i,l)\Big)\Delta_t & si \ i = j \end{cases}$$

where $c(i,j) \geq 0$ and $c(i,i) = 0$.

Then, the process $\left(\widetilde{Z}_t^{\Delta}, t \geq 0\right)$ converges en distribution as $\Delta_t \to 0$, to the process

$$\int_0^t R_s ds, \quad t \geq 0$$

where (R_s) is a continuous time Markov chain which takes its values in the set $\{y_1, \cdots, y_k\}$.

4.2 The case where Y_t is Markov chain with order 2

Let (Y_t) be a Markov chain with order 2, i.e. (Y_t, Y_{t+1}) is a classical Markov chain valued in $E := \{(-1,-1), (-1,1), (1,-1), (1,1)\}$. Denote π^Δ its transition probability matrix :

$$\pi = \begin{pmatrix} 1 - c_0\Delta_t & c_0\Delta_t & 0 & 0 \\ 0 & 0 & 1 - p_0 & p_0 \\ p_1 & 1 - p_1 & 0 & 0 \\ 0 & 0 & c_1\Delta_t & 1 - c_1\Delta_t \end{pmatrix}$$

where $\Delta_t, c_0, c_1, p_0, p_1 > 0$ and $c_0\Delta_t, c_1\Delta_t, p_0, p_1 < 1$.
Let us introduce :

$$v_i := \frac{p_i}{1 - (1 - p_0)(1 - p_1)}, \quad c_i' := c_i v_i, \qquad i = 0, 1.$$

Suppose that $Y_0 = 1$ and $Y_1 = -1$.
Then, the interpolated persistent random walk $(\widetilde{Z}_s^{\Delta},\ s \geq 0)$ converges in distribution, as $\Delta_x \to 0$, to the process :

$$\Big(-(1-\epsilon) \int_0^s (-1)^{N_u^{c_0', c_1'}} du + \epsilon \int_0^s (-1)^{N_u^{c_1', c_0'}} du,\ s \geq 0 \Big)$$

where ϵ is independent from $\big(N_u^{c_0', c_1'}\big)$ and $\big(N_u^{c_1', c_0'}\big)$ and with distribution :

$$P(\epsilon = 0) = v_1, \quad P(\epsilon = 1) = 1 - v_1.$$