On the bounded variation of the flow of stochastic differential equation

Youssef Ouknine

Faculté des Sciences Semlalia, Marrakech, Morocco

`(joint work with Mohamed Erraoui)`

Stochastic Control and Finance

March 18-23, 2009 Brest

Motivation

Let us consider the 1-dimensional sde

$$X(t) = X_0 + \int_0^t \sigma(s, X(s))dB(s) + \int_0^t b(s, X(s))ds, \tag{1}$$

We assume that

A.1 σ and b are continuous on $[0, +\infty) \times \mathbb{R}$ s.t

$$|\sigma(t, x)| + |b(t, x)| \leq L(1 + |x|),$$

A.2 The equation (1) has pathwise uniqueness

- assumption (A.1) $\Rightarrow$ weak solution $\{X_x(t),\ t \geq 0\}$, $\forall x \in \mathbb{R}$,
- assumption (A.2) $\Rightarrow$ strong.

For $x \leq y$ fixed, we define stopping time,

$$S = \inf\{t > 0 : X_x(t) > X_y(t)\}.$$

On the set $[S < +\infty]$

- $\{\tilde{B}(t) = B(S + t) - B(S),\ t \geq 0\}$ is a Brownian motion.
- $X_.(S + t) = X_.(S) + \int_0^t \sigma(s, X_.(S + s))d\tilde{B}(s) + \int_0^t b(s, X_.(S + s))ds$

Since $X_x(S) = X_y(S)$ on $[S < +\infty]$ then

- PU $\Rightarrow$

  $$P[X_x(S+t)1_{[S<+\infty]} = X_y(S+t)1_{[S<+\infty]}, \forall\, t \geq 0] = 1.$$

- $P[X_x(t) \leq X_y(t), \forall\, t \geq 0] = 1.$
- P-almost all w, for any $t \geq 0$, $x \mapsto X_x(t)(w)$ is increasing and consequently is differentiable a.e. with respect to Lebesgue measure.

Functional framework

Let $d \geq 1$,

- $\Omega = C_0([0, +\infty), \mathbb{R}^d)$
- the Borel σ-field $\mathcal{F}$.
- $B = \{B_t : t \geq 0\}$ is defined by $B_t(\omega) = \omega(t)$,
- $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is the Wiener measure
- $\mathcal{F}_t$ is the σ-field $\sigma\{B_s, 0 \leq s \leq t\}$.

Let $\mathbf{h}$ be a fixed continuous positive function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \mathbf{h}(\mathbf{x}) d\mathbf{x} = 1 \text{ and } \int_{\mathbb{R}^n} |\mathbf{x}|^2 \mathbf{h}(\mathbf{x}) d\mathbf{x} < +\infty.$$

- $\hat{\Omega} = \mathbb{R}^n \times \Omega$,
- $\hat{\mathcal{F}}$ denotes the Borel σ-field on $\hat{\Omega}$
- $\hat{P} = \mathbf{h}(\mathbf{x})\, d\mathbf{x} \ \times \ P$.

For simplicity we choose **h** the form of

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^n x_i^2/2} = \prod_{i=1}^n h(x_i)$$

where $h(x_i) = \dfrac{1}{\sqrt{2\pi}} e^{-x_i^2/2}$.

Now we give the definition of class **V** as follows

Definition

We define $\mathbf{V}(\mathbb{R}^n \times \Omega)$ the total set of Borel functions F on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that there exists a Borel function $\hat{F}$ on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ satisfying that

(i) $F = \hat{F}$ $\hat{P}$-a.s.

(ii) $\forall\, w \in \hat{\Omega}$, $\mathbf{x} \longrightarrow \hat{F}(\mathbf{x}, w)$ is a function of bounded variation on each compact in $\mathbb{R}^n$.

Theorem

Let $p > 1$ and assume that there exists a sequence $\{F_j : j \in \mathbb{N}\}$ in $L^p(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that

- B.1 *F_j converges to F almost surely,*
- B.2 *$\{F_j : j \in \mathbb{N}\}$ are uniformly bounded in $L^p(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$,*
- B.3 *for all $(\mathbf{x}, w) \in \hat{\Omega}$, $i \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$, $t \longrightarrow F_j(\mathbf{x} + te_i, w)$ is absolutely continuous*
- B.4 *$\{\nabla_i F_j\}$ are uniformly bounded in $L^1(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$.*

Then $F \in \mathbf{V}(\mathbb{R}^n \times \Omega)$.

Proposition

If $F \in \mathbf{V}(\mathbb{R}^n \times \Omega)$ *then for P-almost all* $w \in \Omega$ *and for all* $i \in \{1, \cdots ..., n\}$ *we have*

(j) $t \longrightarrow F(\mathbf{x} + te_i, w)$ *is a function of bounded variation on any finite interval.*

(jj) $$\frac{\partial}{\partial x_i} F(\mathbf{x}, w) = \nabla_i \hat{F}(\mathbf{x}, w) \ d\mathbf{x}\text{-}a.e.$$

Applications to stochastic differential equations

Let us consider

$$\begin{cases} dX^k_{\mathbf{x}}(t) &= \sum_{j=1}^{d} \sigma^k_j(t, X^k_{\mathbf{x}}(t))dB^j(t) + b^k(t, X_{\mathbf{x}}(t))dt \\ X_{\mathbf{x}}(0) &= \mathbf{x}, \end{cases} \quad (2)$$

$$\sigma = \left(\sigma^k_j\right)_{k=1,\ldots,n, j=1,\ldots,d} \in C_b([0,+\infty)\times\mathbb{R} \to \mathbb{R}^n \otimes \mathbb{R}^d),$$

$$b = \left(b^k\right)_{k=1,\ldots,n} \in C_b([0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n),$$

We assume the following

H.1 $\max_k \left| b^k(t,\mathbf{x}) - b^k(t,\mathbf{x}') \right| \leq K \left|\mathbf{x} - \mathbf{x}'\right|_{\mathbb{R}^n}$,

H.2 Equation (2) has **PU.**

Theorem

*Assume that (**H.1**)-(**H.2**) hold. Then, the solution* $\hat{X}^k(t)$ *is in* $\mathbf{V}(\mathbb{R}^n \times \Omega)$ *for all* $t \in [0, +\infty)$ *and each* $k = 1, 2, ..., n$.

bibliography

1. **S. Kusuoka**, Existence of densities of solutions of stochastic differential equations by Malliavin calculus, **Journal of Functional Analysis 2010**.

2. **N. Bouleau and F. Hirsch**,
   a- On the derivability with respect to the initial data of the solution of a stochastic differential equation with Lipschitz coefficients. **Séminaire de Théorie du Potentiel Paris 1989**.

   b- Propriétés d'absolue continuité dans les espaces de Dirichlet et application aux équations différentielles stochastiques", **Séminaire de Probabilités, 1986**.