ITN-Marie Curie "Deterministic and Stochastic Controlled Systems and Application"

Some Applications of SDEs (BSDEs) with Oblique Reflection

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Introduction

Forward case (solved):

- Lions and Sznitman, 1984.

- Dupuis and Ishii, 1993.

- Gassous and Rascanu:

$$\begin{cases} dX_t + R\left(X_t\right)\partial\varphi\left(X_t\right)(dt) \ni f\left(t,X_t\right)dt + g\left(t,X_t\right)dB_t, \ t>0, \\ X_0 = \xi \ , \end{cases} \tag{1}$$

where $\varphi : \mathbb{R}^d \to \left]-\infty,+\infty\right]$ is a proper convex lower-semicontinuous function, $\partial\varphi$ is the subdifferential of φ and $R = \left(r_{i,j}\right)_{d\times d} \in C_b^2\left(\mathbb{R}^d;\mathbb{R}^{2d}\right)$ is a *symmetric* matrix such that for all $x \in \mathbb{R}^d$,

$$\frac{1}{c}\left|u\right|^2 \leq \left\langle R\left(x\right)u,u\right\rangle \leq c\left|u\right|^2, \quad \forall \ u \in \mathbb{R}^d \text{ (for some } c \geq 1\text{)}.$$

When $\varphi = I_{\overline{\mathcal{O}}}$, we have

$$\begin{array}{ll} (1) & X_t \in C([0,\infty[,\overline{\mathcal{O}}),\ k_t \in C([0,\infty[,\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}^+,\mathbb{R}^d), \\ (2) & X_t + k_t = x_0 + \int_0^t f(X_s)ds + \int_0^t g\left(X_s\right)dBs,\ \text{for } t \geq 0, \\ (3) & \updownarrow k \updownarrow_t = \int_0^t 1_{bd(\mathcal{O})}\left(x\left(s\right)\right)\ d \updownarrow k \updownarrow_s,\ \ k\left(t\right) = \int_0^t \gamma(x\left(s\right))\ d \updownarrow k \updownarrow_s . \end{array} \tag{2}$$

Backward case (in work):

- Ramasubramanian, 2002 (special domain)

$$\begin{cases} dY_t - R\left(Y_t\right)\partial\varphi\left(Y_t\right)\left(dt\right) \ni -f\left(t, Y_t, Z_t\right)dt + Z_t dB_t,\ \ t \geq 0, \\ Y_T = \xi\ . \end{cases} \tag{3}$$

Application

We consider $(B_t)_{t\geq 0}$ a $k-$dimensional standard BM on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t : t \geq 0\}$ the natural filtration.

1.Reflected Stochastic differential equations in time-dependent domains

Let K be a subset of $\mathbb{R}_+ \times \mathbb{R}^n$ such that the projection of K onto time axis is $[0, T[$, and for each $0 \leq t < T, K(t) = \{x \in \mathbb{R}^n : (t, x) \in K\}$ is a bounded connected open set in $\mathbb{R}^n$.

Let $\mathbf{n}(t, x)$ be the unit inward normal of $K(t)$ and $\overrightarrow{\gamma}$ be the unit inward normal vector field on ∂K.

Theorem

Suppose that $\overrightarrow{\gamma} \cdot \mathbf{n} \geq c_0$ on ∂K for some $c_0 > 0$. Then for each $(s,x) \in \overline{K}$ with $s < T$, there is a unique pair of adapted continuous processes $(X^{s,x}, L^{s,x})$ s.t.

(i) $\quad \left(t, X_t^{s,x}\right) \in \overline{K}$ for $t \in [s,T[$, with $X_s^{s,x} = x$,

(ii) $\quad \left\{L_t^{s,x}, t \in [s,T[\right\}$ is a nondecreasing process with $L_s^{s,x} = 0$ s.t.

$$L_t^{s,x} = \int_s^t 1_{\partial K}(r, X_r)\, dL_r^{s,x},$$

(iii) $\quad X_t^{s,x} = x + \int_s^t b\left(r, X_r^{s,x}\right) ds + \int_s^t \sigma\left(r, X_r^{s,x}\right) dB_r + \int_s^t \mathbf{n}\left(r, X_r^{s,x}\right) dL_r^{s,x}$

Proof We remark that the last equation is equivalent to an equation with an oblique reflection vector field $\mathbf{n}$ verified by the time-space diffusion process (t, $X_t^{s,x}$) in K.

2. Two examples in Economics (BSDE) :

We consider the RBSDE in an $d-$dimensional positive orthant G with oblique reflection $G = \left\{x \in \mathbb{R}^d : x_i > 0,\ 1 \leq i \leq d\right\}$:

$$Y(t) = \xi + \int_t^T b(s, Y(s))\ ds + \int_t^T R(s, Y(s))\ dK(s) - \int_t^T \langle Z(s), dB(s)$$

with $Y(\cdot) \in \overline{G}$ for all $0 \leq t \leq T$;

and $K_i(0) = 0, K_i(\cdot)$ continuous, nondecreasing with

$$K_i(t) = \int_0^t I_{\{0\}}(Y_i(s))\, dK_i(s). \qquad (3)$$

This equation has a unique solution (see [1]).

$\star$ **Backward stochastic analogue of subsidy-surplus model considered in Ramasubramanian [1]**

We consider an economy with d interdependent sectors, with the following interpretation

(a) $Y_i(t) =$ current surplus in Sector i at time t ;

(b) $K_i(t) =$ cumulative subsidy given to Sector i over $[0, t]$;

(c) $\xi_i =$ desired surplus in Sector i at time T;

(d) $\int_s^t b_i(u, Y(u))\, du =$ net production of Sector i over $[s, t]$ due to evolution of the system; this being negative indicates there is net consumption;

(e) $\int_s^t r_{ij}^{-}(u, Y(u))\, dK_j =$ amount of subsidy for Sector j mobilized from Sector i over $[s, t]$;

(f) $\int_s^t r_{ij}^{+}(u, Y(u))\, dK_j =$ amount of subsidy mobilized for Sector j which is actually used in Sector i (but not as subsidy in Sector i) over $[s, t]$.

The condition (3) in RBSDE (ξ, b, R) means that subsidy for Sector i can be mobilized only when Sector i has no surplus.

(The uniform spectral radius condition would mean that the subsidy mobilized from external sources is nonzero; so this would be an ‘open’ system in the jargon of economics).

$\star$ **Backward stochastic (oblique) analogue of projected dynamical system**

Suppose the system represents d traders each specializing in a different commodity. For this model we assume:

$r_{ij}(\cdot,\cdot) \leq 0, i \neq j$;

$Y_i(t) =$ current price of Commodity i at time t ; there is a price floor viz. prices cannot be negative;

$K_i(t) =$ cumulative adjustment involved in the price of Commodity i over $[0, t]$;

$b_i(t, Y(t))\, dt =$ infinitesimal change in price of Commodity i due to evolution of the system;

$\xi_i =$ desired price level of Commodity i at time T.

Condition (3) then means that adjustment $dK_i(t)$ can take place only if the price of Commodity i is zero.

$\int_s^t r_{ij}^{-}(u, Y(u))\, dK_j(u) =$ adjustment from Trader i when price of Commodity j is zero.

Note that $dK_j(\cdot)$ can be viewed upon as a sort of artificial/forced infinitesimal consumption when the price of Commodity j is zero to boost up the price;

hence

$$r^{-}_{ij}\left(t, Y\left(t\right)\right) dK_{j}\left(t\right)$$

is the contribution of Trader i towards this forced consumption. (As before, the uniform spectral radius condition) implies that there is nonzero 'external adjustment', like perhaps governmental intervention/consumption to boost prices when prices crash).

3. Switching Games(Ying-Hu and Shanjian Tang)

Consider two players I and II, who use their respective switching control processes $a(\cdot)$ and $b(\cdot)$ to control the following BSDE :

$$U(t) = \xi + \left(A^{(a)}(T) - A^{(a)}(t)\right) - \left(B^{(b)}(T) - B^{(b)}(t)\right)$$
$$+ \int_t^T f(s, U(s), V(s), a(s), b(s))\ ds - \int_t^T V(s)\, dB(s),$$

where $A^{a(\cdot)}(\cdot)$ and $B^{b(\cdot)}(\cdot)$ are the cost processes associated with the switching control processes $a(\cdot)$ and $b(\cdot)$.

Under suitable conditions, the above BSDE has a unique adapted solution, denoted by $(U^{a(\cdot),b(\cdot)}, V^{a(\cdot),b(\cdot)})$.

Player I chooses the switching control $a(\cdot)$ from a given finite set to minimize the cost

$$\text{min} -- > J(a(\cdot), b(\cdot)) = U^{a(\cdot),b(\cdot)}(0)$$

and each of his instantaneous switching from one scheme $i \in \Lambda$ to another different scheme $i^{'} \in \Lambda$ incurs a positive cost which will be specified by the function $k\left(i, i'\right)$.

While Player II chooses the switching control $b(\cdot)$ from a given finite set Π to maximize the cost

$$\text{max} -- > J(a(\cdot), b(\cdot))$$

and each of his instantaneous switching from one scheme $j \in \Pi$ to another different scheme $j' \in \Pi$ incurs a positive cost which will be specified by the function $l(j, j\prime)$, Let $\left\{\theta_j\right\}_{j=0}^{\infty}$ increasing sequence of stopping time, α_j $\mathcal{F}_{\theta_j}-$measurable r.v with value in Λ, then a admissible switching strategy for player I:

$$a\left(s\right) = \alpha_0 \chi_{\{\theta_0\}}\left(s\right) + \sum_{j=1}^{N} \alpha_{j-1} \chi_{(\theta_{j-1},\theta_j]}\left(s\right),$$

therefore

$$A^{a(\cdot)}\left(s\right) = \sum_{j=1}^{N-1} k\left(\alpha_{j-1}, \alpha_j\right) \chi_{[\theta_j,T]}\left(s\right).$$

We are interested in the existence and the construction of the value process as well as the saddle point.

The solution of the above-stated switching game will appeal the reflected backward stochastic differential equation with oblique reflection:

$$
\begin{cases}
Y_{i,j}(t) = \xi_{i,j} + \int_t^T f\left(s, Y_{ij}(s), Z_{ij}(s), i, j\right) ds \\
\qquad - \int_t^T dK_{ij}(s) + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s)\, dB(s) \\
\qquad Y_{i,j}(t) \leq \min\limits_{i' \neq i} \left\{ Y_{i',j}(t) + k(i,i') \right\}, \\
\qquad Y_{i,j}(t) \geq \max\limits_{i' \neq i} \left\{ Y_{i,j'}(t) - l(j,j') \right\}, \\
\qquad \int\limits_0^T \left(Y_{i,j}(s) - \min\limits_{i' \neq i} \left\{ Y_{i',j}(s) + k\left(i,i'\right) \right\} \right) dK_{ij}(s) = 0, \\
\qquad \int\limits_0^T \left(Y_{i,j}(s) - \max\limits_{i' \neq i} \left\{ Y_{i,j'}(t) - l\left(j,j'\right) \right\} \right) dL_{ij}(s) = 0.
\end{cases}
\tag{4}
$$

We define $(a^*(\cdot), b^*(\cdot))$ as follows:

$$\theta_0^* := 0, \tau_0^* := 0; \;\; \alpha_0^* := i, \beta_0^* := j.$$

We define stopping times $\theta_p^*, \tau_p^*; \;\; \alpha_p^*, \beta_p^*$ in the following inductive manner:

$$\begin{aligned} \theta_p^* := \inf\{s \geq \theta_{p-1}^* \wedge \tau_{p-1}^* : Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(s) = \min_{i' \neq i}\{Y_{i', \beta_{p-1}^*}(s) \\ + k(\alpha_{p-1}^*, i')\}\} \wedge T, \\ \tau_p^* := \inf\{s \geq \theta_{p-1}^* \wedge \tau_{p-1}^* : Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(s) = \max_{j' \neq j}\{Y_{\alpha_{p-1}^*, j'}(s) \\ - l(\beta_{p-1}^*, j')\}\} \wedge T. \end{aligned}$$

Theorem

Under the usual hypothesis. Let (Y, Z, K, L) solution in the space $S^2 \times M^2 \times N^2 \times N^2$ to *RBSDE (4)*. Then we have the representation :

$$Y_{ij}(t) = \operatorname*{ess\,inf}_{a(\cdot) \in \mathcal{A}_t^i} U_j^{a(\cdot)}(t).$$

Theorem

We denote by $\left(Y_{ij}, Z_{ij}, K_{ij}, L_{ij}; i \in \Lambda,\ j \in \Pi\right)$ solution of (4). We assume the usual hypothesis which are standard in the literature of switching games. Then $\left(Y_{ij}; i \in \Lambda,\ j \in \Pi\right)$ is the value process for our switching game, and the switching strategy $a^*(\cdot) := \left(\theta_p^* \wedge \tau_p^*, \alpha_p^*\right)$ for Player I and $b^*(\cdot) := \left(\theta_p^* \wedge \tau_p^*, \beta_p^*\right)$ for Player II is a saddle point of the switching game, it means that

$$Y_{ij}(0) = U^{a^*(\cdot), b^*(\cdot)}(0).$$

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Thank you for your attention !