Introduction. The Trotter scheme

Consider the following evolution equation

$$\text{(E)} \qquad \begin{cases} \dfrac{du}{dt} + Au + Bu = 0, \quad t > 0 \\ u(0) = u_0 \end{cases}$$

where $A : \mathcal{D}(A) \subseteq X \to X$ and $B : \mathcal{D}(B) \subseteq X \to X$ generate the semigroups $\{e^{-tA}, t \geq 0\}$ and $\{e^{-tB}, t \geq 0\}$, respectively.

Let $\varepsilon = \dfrac{t}{n}$, $n \geq 1$. So

$$0 < \varepsilon < 2\varepsilon < ... < n\varepsilon = t.$$

Consider now

$$\text{(*)} \qquad \begin{cases} \dfrac{dv}{dt} + Bv = 0, \quad t \in (0; \varepsilon] \\ v(0) = u_0 \end{cases}$$

and $v(\varepsilon) = e^{-\varepsilon B} u_0$ is the solution of (*) in ε.

Then consider

$$(**) \qquad \begin{cases} \dfrac{dw}{dt} + Aw = 0, & t \in (0; \varepsilon] \\ w(0) = v(\varepsilon) = e^{-\varepsilon B} u_0 \end{cases}$$

so $w(\varepsilon) = e^{-\varepsilon A} e^{-\varepsilon B} u_0$.

We define the approximate solution of (E) in $t_1 = \varepsilon$ as $u^{\varepsilon}(t_1) = e^{-\varepsilon A} e^{-\varepsilon B} u_0$. Then consider the systems (*) and (**) with initial data $u^{\varepsilon}(t_1) = e^{-\varepsilon A} e^{-\varepsilon B} u_0$ instead of u_0 and we obtain

$$u^{\varepsilon}(2\varepsilon) = u^{\varepsilon}(t_2) = (e^{-\varepsilon A} e^{-\varepsilon B})^2 u_0$$

and so on...

Finally, we obtain

$u^\varepsilon(t_n) = u^\varepsilon(n\varepsilon) = u^\varepsilon(t) = (e^{-\varepsilon A}e^{-\varepsilon B})^n u_0 = (e^{-\frac{t}{n}A}e^{-\frac{t}{n}B})^n u_0 \xrightarrow[\substack{!! \\ n\to\infty}]{?} u$ (the solution of (E), formally written as $e^{-t(A+B)}u_0$).

Sometimes we may use the resolvent $\left(I + \frac{t}{n}A\right)^{-1}$ instead of $e^{-\frac{t}{n}A}$ because, formally,

$$\lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} = e^{-tA}.$$

A Trotter approximation scheme for the Navier–Stokes equations

HAVÂRNEANU T., POPA C.
havi@uaic.ro cpopa@uaic.ro

Faculty of Mathematics,
University "Al.I. Cuza" of Iași
Institute of Mathematics,
Romanian Academy
Iași Branch, România

We consider the Navier–Stokes equations for incompressible fluid flow:

$$(1)\quad \frac{\partial v}{\partial t}+(v\cdot\nabla)v-\Delta v+\nabla p = 0 \text{ in } Q=\Omega\times(0,T),$$

$$(2)\qquad \text{div } v = 0 \quad \text{in } Q,$$

$$(3)\qquad v = 0 \quad \text{on } \Sigma = \partial\Omega\times(0,T),$$

$$(4)\qquad v(\cdot,0) = v_0(\cdot) \quad \text{in } \Omega,$$

where $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded domain, v is the velocity $(v : Q\to\mathbb{R}^d)$ and p is the scalar pressure $(p : Q\to\mathbb{R})$.

Also we consider the Euler equations for incompressible fluid flow

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{in } Q, \tag{5}$$

$$\text{div } u = 0 \quad \text{in } Q, \tag{6}$$

$$u \cdot N = 0 \quad \text{on } \Sigma, \tag{7}$$

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega, \tag{8}$$

where N is outward normal to $\partial\Omega$, u is the velocity and π is the scalar pressure.

Here $(w \cdot \nabla)w = \left(\sum_{i=1}^{d} w_i \frac{\partial w_j}{\partial x_i} \right)_{j=\overline{1,d}} \in \mathbb{R}^d$ for $w : \Omega \to \mathbb{R}^d$, $w = (w_j)_{j=\overline{1,d}}$.

Let us denote by $(E(t)u_0)(\cdot)$ the solution $u(t,\cdot)$ of the system (5)–(8) and by $A_p = -P_p\Delta$ the Stokes operator for $p > 1$ where $P_p : (L^p(\Omega))^d \rightarrow H_p$ is the Leray projector and $H_p = \{u \in (L^p(\Omega))^d;$ div $u = 0$ in $\Omega, u \cdot N = 0$ on $\partial\Omega\}$.

We can write the system (1)–(4) as an evolution equation

$$\text{(9)} \qquad \begin{cases} \dfrac{dv}{dt} + A_p v = 0 \text{ in } V_p, \\ v(0) = v_0, \end{cases}$$

where $V_p = \{f \in (W^{1,p}(\Omega))^d;$ div $f = 0$ in Ω, $f \cdot N = 0$ on $\partial\Omega\}$ and $\mathcal{D}(A_p) = (W^{2,p}(\Omega))^d \cap (W_0^{1,p}(\Omega))^d \cap V_p$.

Let's present the splitting scheme!

Let the interval $[0, T]$ be fixed and let an initial free-divergence velocity v_0 be given. The time interval is divided into m subintervals each of size $\varepsilon = \dfrac{T}{m}$

$$0 < \varepsilon < 2\varepsilon < ... < (m-1)\varepsilon < T.$$

The splitting scheme defines recursively an approximate solution of the Navier–Stokes equations. Let $u_0 = v_0$. Having defined v_n (an approximate solution at time $t_n = n\varepsilon$, $0 \leq n \leq m-1$), let v^* be the solution of Euler equations (5)–(8) at the end of an interval of size ε with initial data v_n. Then v_{n+1} is the solution of the stationary Stokes equation

$$u + \varepsilon A_p u = v^* \text{ i.e. } v_{n+1} = (I + \varepsilon A_p)^{-1} v^*$$

and with our notations

$$v_{n+1} = (I + \varepsilon A_p)^{-1} E(\varepsilon) v_n.$$

We propose the following splitting approximation scheme:

Let $m \in \mathbb{N}^*$ and $\varepsilon = \dfrac{T}{m}$. Consider

$$(10)\quad \begin{cases} v_0^\varepsilon = v_0 \\ v_{n+1}^\varepsilon = (I + \varepsilon A_p)^{-1} E(\varepsilon) v_n^\varepsilon, 0 \le n \le m-1, \end{cases}$$

and define the approximation solution of (1)–(4) as

$$(11)\quad \begin{cases} v_E^\varepsilon(t_n + s) = E(s) v_n^\varepsilon, \quad 0 < s \le \varepsilon \\ v^\varepsilon(t_n + s) = (I + \varepsilon A_p)^{-1} v_E^\varepsilon(t_n + s), \\ \qquad\qquad 0 < s \le \varepsilon, 0 \le n \le m-1, \end{cases}$$

i.e.

$$(12)\quad \begin{cases} v^\varepsilon(t_n + s) = (I + \varepsilon A_p)^{-1} E(s) v_n^\varepsilon, \\ \qquad\qquad 0 < s \le \varepsilon, 0 \le n \le m-1 \\ v^\varepsilon(0) = v_0, \end{cases}$$

where $\{v_n^\varepsilon\}_{n \ge 0}$ is given by (10).

The main result

Theorem 1. *If the initial velocity* $v_0 \in V_p \cap (W^{2,p}(\Omega))^d$ *with* $p > d$*, then for a sufficiently small* $T > 0$ *(depending on* v_0*), the approximate solutions* v^ε *(given by* (2)*) is well-defined and satisfies*

$$\sup_{0 \le t \le T} |v^\varepsilon(\cdot, t) - v(\cdot, t)|_{(L^p(\Omega))^d} \le C\varepsilon,$$

where v *is the strong solution of* $(1) - (4)$ *and* $C > 0$ *is a constant independent of* ε*.*

The idea of the proof

We consider the linearization of (5)–(8) around the solution v of (1)–(4).

$$(S) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (v \cdot \nabla) u + \nabla \pi = 0 & \text{in } Q, \\ \text{div } u = 0 & \text{in } Q, \\ u \cdot N = 0 & \text{on } \Sigma, \end{cases}$$

and then we apply the same scheme with (S) instead of (5)–(7).

We prove that, in this case, the approximation scheme is convergent to v (the solution of (1)–(4)), i.e. the new approximation solution $\widetilde{v}^{\varepsilon}$ tends to v.

In the last step we prove that

$$\sup_{0 \le t \le T} |v^{\varepsilon} - \widetilde{v}^{\varepsilon}|_{(L^p(\Omega))^d} \xrightarrow[\varepsilon \to 0]{} 0.$$

So we obtain the conclusion of our theorem.

References

[1] J.T. BEALE and C. GREENGARD, *Convergence of Euler–Stokes splitting of the Navier–Stokes equations*, Comm. Pure Appl. Math., 47(1994), 1083–1115.

[2] C. POPA, *On the convergence of Euler–Stokes splitting of the Navier–Stokes equations*, Differential and Integral Equations, 15(2002), 657–670.

[3] C. POPA and S.S. SRITHARAN, *Fluid-magnetic splitting of the magnetohydrodynamic equations*, Mathematical Models & Methods in Applied Sciences, 13(2003), 893–917.