# *Linear Stochastic Schrödinger and Master Equations*

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# Aim of the Work

The aim of the work is...

To introduce memory in the Theory of Quantum Measurements in Continuous Time based on SDEs .

▶The studied case is the diffusive one (no jumps!)◀

# Pattern of the presentation

- Formulation of the Theory in the Hilbert space.
- Formulation of the Theory for statistical operators acting on the Hilbert space.
- Short discussion of a physical model for the heterodyne detection.

# THE THEORY IN THE HILBERT SPACE

# Linear Stochastic Schrödinger Equation

## Fundamental tools

- A finite dimensional complex Hilbert space $\leftrightsquigarrow \mathscr{H} := \mathbb{C}^n$ .
- A fixed stochastic basis in u.c. $\leftrightsquigarrow \mathbb{F} := (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{Q})$ .
- A standard $d$-dimensional Wiener process $\leftrightsquigarrow W = (W_1, \ldots, W_d)$ in $\mathbb{F}$.

## Linear Stochasic Schrödinger Equation $\mathbb{F}$

**Assumption.** The wave function of the system satisfies a linear SDE, the Linear Stochasic Schrödinger Equation:

$$\begin{cases} \mathrm{d}\psi(t) = -\mathrm{i}\left[H(t) + \frac{1}{2}\sum_{j=1}^{d} R_j^*(t)R_j(t)\right]\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t) \\ \psi(0) = \psi_0\,, \qquad \psi_0 \in L^2\left(\Omega, \mathscr{F}_0, \mathbb{Q}; \mathscr{H}\right), \end{cases} \tag{1}$$

where $\{R_j\}_{j=1}^d$ and $H$ are progressive-$M_n(\mathbb{C})$-valued processes in $\mathbb{F}$. $H$ is the effective Hamiltonian of the system.

# Existence and Pathwise Uniqueness of the Solution

There are very general results...but we decide to extend those ones for Classical SDEs

## Proposed conditions

$$\sup_{\omega\in\Omega}\ \sup_{t\in[0,T]} \left\| \sum_{j=1}^{d} R_j^*(t,\omega) R_j(t,\omega) \right\| \leq L(T) < \infty\,, \qquad \forall T > 0\,;$$

$$\sup_{\omega\in\Omega}\ \sup_{t\in[0,T]} \| H(t,\omega) \| \leq M(T) < \infty\,, \qquad \forall T > 0\,.$$

Although these are strong conditions they allow to study very interesting physical models. Furthermore, we are able to obtain $L^p$-estimates of the solution.

## Square Norm of the Solution and Physical Probabilities

It is possible to prove that the process $\{\|\psi(t)\|^2\}_{t\geq 0}$ is a probability density process (it is an exponential mean one martingale),
so, it can be used to introduce the physical probabilities of the system:

$$\mathbb{P}^T_{\psi_0}(F) := \mathbb{E}_{\mathbb{Q}}[1_F \|\psi(T)\|^2], \quad F \in \mathscr{F}_T .$$

▶Under the physical pb. $\longrightarrow$ Wiener $\widehat{W}$ obtained by a Girsanov transformation.

## Non Linear Equation

The process $\psi$ is a.s. non zero if the initial state is a.s. non zero: we can define the normalised states $\hat{\psi}(t) := \psi(t)/\|\psi(t)\|$

$$\Downarrow$$

Under the physical probabilities $\hat{\psi}(t)$ satisfies a non linear equation.
The two equations are equivalent because it is possible to obtain one from the other, in both the senses.

# THEORY IN THE SPACE OF LINEAR OPERATORS

# Extension of the theory to statistical operators

The theory formulated in the Hilbert space can be generalised to statistical operators (here hermitian, positive defined and trace one matrices).
This formulation allow us to model either a possible uncertainty on the initial state of the system, due for example to a preparation procedure carried out on the system itself, or to introduce dissipative phenomena, due to the interaction of the system.

Let $\rho_0$ be a statistical operator and assume that it is the initial state of the system.

Define the process $\sigma$ as

$$\sigma(t) := \sum_{\beta} \psi^{\beta}(t)\psi^{\beta}(t)^* \,.$$

Where $\psi^{\beta}(t)$ is the solution of (1) if $\psi_0^{\beta} \in \mathscr{H}$ is the initial condition when $t = 0$.

# Linear Stochastic Master Equation

The $M_n(\mathbb{C})$-valued process $\sigma$ satisfies a linear SDE with unique solution which can be interpreted as a "Stochastic Master Equation":

## Linear Stochastic Master Equation

$$\begin{cases} \mathrm{d}\sigma(t) = \mathcal{L}(t)\big[\sigma(t)\big]\mathrm{d}t + \sum_{j=1}^{d} \mathcal{R}_j(t)[\sigma(t)]\mathrm{d}W_j(t)\,. \\ \\ \sigma(0) = \rho_0\,. \end{cases} \tag{2}$$

$\mathcal{L}$ and $\{\mathcal{R}_j\}_{j=1}^d$ are $\mathbb{F}$-progressive processes whose sate space is the space of linear maps on $M_n(\mathbb{C})$: they depend on $H$ and $\{R_j\}_{j=1}^d$.
$\mathcal{L}$ is called Liouville operator or Liouvillian of the system. It contains the dynamics.

# Non Linear Equation

**The process $\mathrm{Tr}\{\sigma(t)\}$**

This is a probability density process (mean one and exponential martingale): we can introduce the physical probabilities also in this formulation, in a similar way as we did in the Hilbert space.

We normalise $\sigma$ with respect to its trace, which is a.s. non zero, and we obtain the "a posteriri states" of the system

$$\varrho(t) := \frac{\sigma(t)}{\mathrm{Tr}\{\sigma(t)\}}, \quad \forall t \in [0, T].$$

The process $\varrho$ satisfies a non linear SDE under the physical probabilities.

# Mean States

Let $\eta(t)$ be the mean state of the system at times $t$:

$$\eta(t) := \mathbb{E}_{\mathbb{Q}}\left[\sigma(t)\right], \quad \forall t \in [0, T].$$

Because of the randomness of the Liouvillian operator...

$$\eta(t) = \rho_0 + \int_0^t \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}(s)[\sigma(s)]\right] \mathrm{d}s.$$

Thus, if we take the mean of the state of the system, the mean state does not satisfies a closed equation.
...If $\mathcal{L}(t)$ were deterministic...

## Master Equation for Mean States

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \mathcal{L}(t)[\eta(t)], \quad \eta_0 = \rho_0.$$

# Physical interpretation

- The output of the system is $W$ (or its linear functional).
By mean of a Girsanov transformation obtained using $\mathrm{Tr}\{\sigma\}$, we obtain the following decomposition of the output under the physical probabilities

$$W(t) = \widehat{W}(t) + \int_0^t v(s)\mathrm{d}s\,,$$

thus, a noise ($\widehat{W}$) and a signal ($\int_0^t v(s)\mathrm{d}s$). Signal and noise are correlated and not independent.
- The process $\sigma$ has the interpretation of the non normalised state process for the quantum system .
- The normalised states are given by the process $\varrho$: they have the interpretation of a posteriori state, thus $\varrho(t)$ is the state of the system once the measuring experiment has been carried out and the trajectory $\{W(s)\,, \quad 0 \leq s \leq t\}$ is observed.

# Moments

It is possible to obtain expressions for the moments of the output under the physical probabilities:

- $\mathbb{E}^T_{\rho_0}\left[\dot{W}_j(t)\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_j(t)[\sigma(t)]\right\}\right]\,,$

- $$\mathbb{E}^T_{\rho_0}\left[\dot{W}_j(t)\dot{W}_i(s)\right] = \delta_{ij}\delta(t-s) + 1_{(0,+\infty)}(t-s)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\Lambda(t,s)\circ\mathcal{R}_i(s)[\sigma(s)]\right\}\right] + 1_{(0,+\infty)}(s-t)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\Lambda(s,t)\circ\mathcal{R}_j(t)[\sigma(t)]\right\}\right]\,.$$

The two time-maps valued process $\Lambda(t,s)$ is the propagator of the Linear Stochastic Master Equation

**Important**

The process $\Lambda$ stisfies the tipical composition law of an evolution,

$$\Lambda(t,r) = \Lambda(t,s)\circ\Lambda(s,r)\,,\quad 0\le r\le s\le t$$

# The Reduced Observation: "Non Markov" Effects

Assume $m \leq d$ ($d$ dimension of $W$). Let us set to zero the coefficients $\{R_j\}_{j=m+1}^{d}$.

- The first $m$ components of $W$ represent the observed output.
- The components of $W$ from $j = m+1$ to $d$ are used to introduce memory or other kind of randomness in the system.

The filtration generated by the output $\{W_1, \ldots, W_m\}$ is

$$\mathcal{E}_t := \sigma\{W_j(s), \quad j = 1, \ldots, m, \quad s \in [0,t]\} \vee \mathcal{N}.$$

▶ We do not require that $\{R_j\}_{j=1}^{m}$ and $H$ are adapted with respect to this filtration.

# THE PHYSICAL MODEL

# Introduction to the the physical model

We present a physical model as an application of the built theory.

- We have studied both the *homodyne* and the *heterodyne* revelation for a two level atom... but we present here only the heterodyne case.
- The choice of the two level atom fixes the Hilbert space: $\mathscr{H} := \mathbb{C}^2$.
- The atom is stimulated by a laser and it emits fluorescence light
- The fluorescence light is observed by means of photon counters... after an interference with a reference laser light (local oscillator).

## New aspects

Thank to the theoretical set up with stochastic coefficients in the involved equations we can consider not perfectly monochromatic and not perfectly coherent lasers...thus, a more realistic model.

# Heterodyne Revelation

## Phase-Diffusion Models for the Lasers

- The stimulating laser is characterised by a phase diffusion model with carrier frequency $\omega > 0$. Its effects enter in the Hamiltonian part of the Liouville operator $\mathcal{L}(t)$. The model for the stimulating laser is

$$H \longrightarrow f(t) = \lambda \exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon_3}{2} B_3(t)\right\} .$$

- We assume the same model for the local oscillators in the revelators.

$$R_k \longrightarrow h_k(t) = \exp\left\{\mathrm{i}\nu t - \mathrm{i}\frac{\varepsilon_k}{2} B_k(t)\right\} , \ k = 1,2 \quad \nu > 0 \text{ carr. freq.}$$

$B_k$s are components of $W$ and $\varepsilon_k$ are real and arbitrary parameters. We study the case $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 > 0$ (the case $=0$ is the perfect case, already studied in the literature).

If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \equiv \varepsilon$, $B_1 = B_2 = B_3 \equiv B$ and $\nu \equiv \omega$ we speak of homodyne revelation.

# Heterodyne Spectrum

In the heterodyne case we study the spectrum of the mean observed power (under the physical pb.) for long time.

The output of the measurement is a regular functional of the Wiener process $W$ and, so, a stochastic process .

## Spectrum

The definition of the spectrum of the output is the classical notion of spectrum for a stochastic process and not one given *ad hoc*.
WE CAN COMPUTE THE SPECTRUM USING THE MOMENTS FORMULA.

▶ The presence of stochastic phases in the involved laser ($\varepsilon_1 \neq 0 \neq \varepsilon_3$) introduce some asymmetries and "smoothing" in the spectrum. ◀

# A Graphical Example

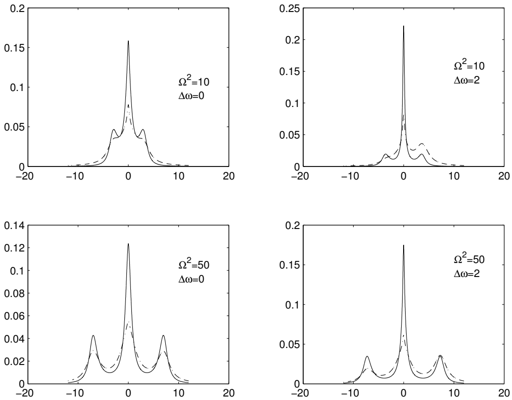


Figura: Spectrum of the observed mean power under the physical probabilities for long time. The continuous line represent the ideal case: $\varepsilon_1^2 = \varepsilon_3^2 = 0$ the dotted line represent the case $\varepsilon_1^2 = \varepsilon_3^2 = 0.2$.

Grazie per l'attenzione...
(...Thank you for your kind attention)

# Essential Bibliography


- P. Baldi. *Equazioni differenziali stocastiche e applicazioni*, Quaderno UMI **28** (Pitagora, Bologna, 2000).
- A. Barchielli, M. Gregoratti. *Quantum Trajectories and Mesurements in Continuous Time*, Lecture Notes in Physics **782** (Springer, Berlin, 2009).
- A. Barchielli, A. S. Holevo. *Constructing quantum measurement processes via classical stochastic calculus*, Stochastic Processes and their Applications **58** (1995) 293-317.
- A. Barchielli, N. Pero. *A quantum stochastic approach to the spectrum of a two-level atom*, J. Opt. B: Quantum Semiclass. Opt. **4** (2002) 272-282.
- G. Lindblad. *On the Generators of Quantum Dynamical Semigroups*, Communications in Mathematical Physics **48** (1976) 119-130.
- M. Mètivier. *Semimartingales: a Course on Stochastic Processes* (de Gruyter Studies in Mathematics 1982).
- P. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Eqautions*, Applied Mathematical Science **44** (Springer-Verlag 1983).