A Stochastic Optimal Control Problem for the Heat Equation on the Halfline with Dirichlet Boundary-noise and Boundary-control

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PLAN

1. Heat equations with boundary noise;

2. Basic facts on stochastic optimal control;

3. Stochastic optimal control: boundary case;

4. Regularity of the FBSDE;

5. Solution of the related HJB;

6. Synthesis of the optimal control;

7. FBSDE in the infinite horizon case;

8. Stationary HJB and optimal control.

Neumann boundary conditions

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), \qquad s \in [t,T],\ \xi \in (0,\pi), \\ y(t,\xi) = x(\xi), \\ \dfrac{\partial y}{\partial \xi}(s,0) = \dot{W}_s^1 + u_s^1, \quad \dfrac{\partial y}{\partial \xi}(s,\pi) = \dot{W}_s^2 + u_s^2. \end{cases}$$

- $\{W_t^i\}_{t\geq 0}, \quad i = 1,2$ independent real Wiener processes;
- $\{u_t^i\}_{t\geq 0}, \quad i = 1,2$ predictable real valued processes modelling the control;
- $y(t,\xi,\omega)$ state of the system;
- $x \in L^2(0,\pi)$.

no noise as a forcing term!

Reformulation in $H = L^2(0,\pi)$

$$\begin{cases} dX_s^u = AX_s^u ds + F(s, X_s^u)ds + (\lambda - A)b\, u_s ds + (\lambda - A)b\, dW_s & s \in [t,T], \\ X_t^u = x, \end{cases}$$

where

$$F(t,x) = f(t, x(\cdot)),\ W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix},\ u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \text{ and } b(\cdot) = \begin{pmatrix} b^1(\cdot) \\ b^2(\cdot) \end{pmatrix}.$$

$$b^i(\cdot) \in \mathrm{dom}(\lambda - A)^\alpha,\ 0 < \alpha < 3/4,\ b^i(\cdot) \notin \mathrm{dom}(\lambda - A)^\alpha,\ 3/4 < \alpha < 1.$$

We can give sense to the **mild formulation**

$$X_s^u =$$
$$e^{(s-t)A}x + \int_t^s e^{(s-r)A} F(r, X_r^u)\, dr + \int_t^s e^{(s-r)A}(\lambda - A)b\, u_r dr + \int_t^s e^{(s-r)A}(\lambda - A)b\, dW_r$$
$$= e^{(s-t)A}x + \int_t^s e^{(s-r)A} F(r, X_r^u)\, dr$$
$$+ \int_t^s (\lambda - A)^{1-\beta} e^{(s-r)A}(\lambda - A)^{\beta} b\, u_r dr + \int_t^s (\lambda - A)^{1-\beta} e^{(s-r)A}(\lambda - A)^{\beta} b\, dW_r.$$

Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s, \end{cases} \qquad (1)$$

- $y(t,\xi,\omega)$ state of the system; • $\{W_t\}_{t\geq 0}$ real Wiener process;
- $\{u_t\}_{t\geq 0}$ predictable real valued process modelling the control.

References

- Da Prato-Zabczyk (1995): $y(s,\cdot)$ well defined in H^α, for $\alpha < -\dfrac{1}{4}$.
- Alos-Bonaccorsi (2002) and Bonaccorsi-Guatteri (2002): $y(s,\cdot)$ takes values in the weighted space $L^2((0,+\infty);\xi^{1+\theta}d\xi)$.

In Fabbri-Goldys (2009) equation (1) (with $f = 0$) is reformulated as an evolution equation in $\mathcal{H} = L^2((0,+\infty);\rho(\xi)d\xi)$ with $\rho(\xi) = \xi^{1+\theta}$ or $\rho(\xi) = \min(\xi^{1+\theta},1)$ using results in Krylov (1999) and (2001).

- The heat semigroup in $L^2((0,+\infty))$ extends to a bounded C_0 semigroup $(e^{tA})_{t\geq 0}$ in $\mathcal{H}$ with generator still denoted by $A : \mathrm{dom}(A) \subset \mathcal{H} \to \mathcal{H}$. The semigroup $(e^{tA})_{t\geq 0}$ is analytic: for every $\beta > 0$,

$$\|(\lambda - A)^{\beta} e^{tA}\| \leq C_\beta t^{-\beta} \qquad \text{for all} \qquad t \geq 0.$$

- Dirichlet map: $\lambda > 0$, $\psi_\lambda(\xi) = e^{-\sqrt{\lambda}\xi}$, $\mathcal{D}_\lambda : \mathbb{R} \to \mathcal{H}$, $\mathcal{D}_\lambda(a) = a\psi_\lambda$.

- $B = (\lambda - A)\mathcal{D}_\lambda$

$$\Downarrow$$

$$\begin{cases} dX_s^u = AX_s^u ds + Bu_s ds + BdW_s & s \in [t,T], \\ X_t^u = x, \end{cases}$$

- $B : \mathbb{R} \to \mathcal{H}^{\alpha-1}$ bounded
- $\psi_\lambda \in$ dom$(\lambda - A)^\alpha$ and $(\lambda - A)e^{tA}\mathcal{D}_\lambda = (\lambda - A)^{1-\alpha}e^{tA}(\lambda - A)^\alpha\mathcal{D}_\lambda : \mathbb{R} \to \mathcal{H}$ bounded.
- $\alpha \in (\frac{1}{2}, \frac{1}{2} + \frac{\theta}{4})$.

We can give sense to

$$X_s^u = e^{(s-t)A}x + \int_t^s e^{(s-r)A}Bu_r dr + \int_t^s e^{(s-r)A}BdW_r$$

Our framework

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s, \end{cases}$$

Hypothesis 1

1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ measurable, $\forall\, t \in [0,T]$ $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ continuously differentiable and $\exists\, C_f > 0$ s.t.

$$| f(t,0) | + | \frac{\partial f}{\partial r}(t,r) | \leq C_f, \qquad t \in [0,T],\ r \in \mathbb{R}.$$

2) $x(\cdot) \in \mathcal{H}$.

3) admissible control u: predictable process with values in a compact $\mathcal{U} \subset \mathbb{R}$.

Set $F(s,X)(\xi) = f(s,X(\xi))$: $F : [0,T] \times \mathcal{H} \to \mathcal{H}$ measurable and

$$|F(t,0)| + |F(t,x_1) - F(t,x_2)| \leq C_f(1 + |x_1 - x_2|), \qquad t \in [0,T],\ x_1, x_2 \in \mathcal{H}.$$

$\forall t \in [0,T]$, $F(t,\cdot)$ has a Gâteaux derivative $\nabla_x F(t,x)$ and $|\nabla_x F(t,x)| \leq C_f$. $(x,h) \to \nabla_x F(t,x)h$ continuous as a map $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

$$\begin{cases} dX_s^u = AX_s^u ds + F(s, X_s^u) ds + Bu_s ds + BdW_s & s \in [t,T], \\ X_t^u = x, \end{cases}$$

By the Picard approximation scheme we find a mild solution

$$X_s^u = e^{(s-t)A}x + \int_t^s e^{(s-r)A} F(r, X_r^u)\, dr + \int_t^s e^{(s-r)A} Bu_r dr + \int_t^s e^{(s-r)A} BdW_r. \quad (2)$$

$X \in L^2(\Omega; C([t,T], \mathcal{H}))$ $\forall\, p \in [1,\infty)$, $\alpha \in [0, \theta/4)$, $t \in [0,T]$ $\exists\, c_{p,\alpha}$ s.t.

$$\mathbb{E} \sup_{s \in (t,T]} (s-t)^{p\alpha} |X_s^{t,x}|^p_{dom(\lambda - A)^\alpha} \leq c_{p,\alpha}(1 + |x|_{\mathcal{H}})^p.$$

Controlled state equation in H

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + F\left(\tau, X_\tau^u\right) + u_\tau\right] d\tau + G\left(\tau, X_\tau^u\right) dW_\tau, & \tau \in [t,T], \\ X_t^u = x. \end{cases}$$

Cost functional and value function

$$J(t,x,u) = \mathbb{E}\int_t^T g\left(s, X_s^u, u_s\right) ds + \mathbb{E}\phi\left(X_T^u\right),$$
$$J^*(t,x) = \inf_u J(t,x,u).$$

Hamiltonian function: for every $\tau \in [t,T]$, $x \in H$, $q \in H^*$

$$\psi(\tau,x,q) = -\inf\left\{g(\tau,x,u) + qu : u \in \mathcal{U}\right\},$$

$$\Gamma(\tau,x,q) = \left\{u \in \mathcal{U} : g(\tau,x,u) + qu = -\psi(\tau,x,q)\right\}$$

"Analytic" approach

Find a unique, sufficiently regular, solution of the Hamilton Jacobi Bellman equation (HJB) associated

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{A}_t\left[v\left(t,\cdot\right)\right](x) + \psi\left(t,x,\nabla v\left(t,x\right)\right) \\ v(T,x) = \phi\left(x\right), \end{cases}$$

where

$$\mathcal{A}_t f\left(x\right) = \frac{1}{2}\,\mathsf{Tr}\left(G\left(t,x\right)G^*\left(t,x\right)\nabla^2 f\left(x\right)\right) + \left\langle Ax, \nabla f\left(x\right)\right\rangle + \left\langle F\left(t,x\right), \nabla f\left(x\right)\right\rangle$$

For every admissible control u, $J\left(t,x,u\right) \geq v\left(t,x\right)$ and equality holds iff $\mathbb{P}$-a.e. and for a.e. $\tau \in [t,T]$

$$u_\tau \in \Gamma\left(\tau, X_\tau^u, \nabla v\left(\tau, X_\tau^u\right)\right).$$

u^* s. t. $J(t,x,u^*) = J^*(t,x)$ is called optimal control;

$X^*(\cdot)$ associated is called optimal trajectory;

(X^*, u^*) is called optimal pair.

Closed loop equation

Assume that Γ is not empty. Closed loop equation:

$$\begin{cases} d\overline{X}_\tau = \left[A\overline{X}_\tau + F\left(\tau, \overline{X}_\tau\right) + \Gamma\left(\tau, \overline{X}_\tau, \nabla v\left(\tau, \overline{X}_\tau\right)\right)\right] d\tau + G\left(\tau, \overline{X}_\tau\right) dW_\tau, \\ \overline{X}_t = x, \qquad \tau \in [t,T], \quad x \in H. \end{cases}$$

If there exists a solution, the pair $\left(\overline{X}, \Gamma\left(\tau, \overline{X}_\tau, \nabla v\left(\tau, \overline{X}_\tau\right)\right)\right)$ is an optimal pair.

References

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BSDE approach

Controlled state equation in H

$$\begin{cases} dX^u_\tau = \left[AX^u_\tau + F\left(\tau, X^u_\tau\right) + G\left(\tau, X^u_\tau\right) u_\tau\right] d\tau + G\left(\tau, X^u_\tau\right) dW_\tau, \\ X^u_t = x, \qquad \tau \in [t,T], \quad x \in H. \end{cases}$$

Forward-Backward system

$$\begin{cases} dX_\tau = AX_\tau d\tau + F\left(\tau, X_\tau\right) d\tau + G\left(\tau, X_\tau\right) dW_\tau, \qquad \tau \in [t,T] \\ dY_\tau = \psi\left(\tau, X_\tau, Z_\tau\right) d\tau + Z_\tau dW_\tau, \qquad \tau \in [t,T] \\ X_t = x, \\ Y_T = \phi\left(X_T\right). \end{cases}$$

BSDE in integral form

$$Y_\tau + \int_\tau^T Z_\sigma dW_\sigma = \phi(X_T) + \int_\tau^T \psi(\sigma, X_\sigma, Z_\sigma)\, d\sigma$$

There exists a unique adapted solution

$$(X_\tau, Y_\tau, Z_\tau) = \left(X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}\right).$$

$v(t,x) = Y(t,t,x)$ is deterministic and $J(t,x,u) \geq v(t,x)$, for every admissible control u, and equality holds iff $\mathbb{P}$-a.e. and for a.e. $\tau \in [t,T]$

$$u_\tau \in \Gamma(\tau, X_\tau, \nabla v(\tau, X_\tau)\, G(\tau, X_\tau)).$$

Identification of $Z_\tau^{t,x}$ with $\nabla v(\tau, X_\tau^{t,x}) G(\tau, X_\tau^{t,x})$.

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$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), & s \in [t,T],\ \xi \in (0,\pi), \\ y(t,\xi) = x(\xi), \\ \dfrac{\partial y}{\partial \xi}(s,0) = \dot W^1_s + u^1_s, \quad \dfrac{\partial y}{\partial \xi}(s,\pi) = \dot W^2_s + u^2_s. \end{cases}$$

in $H = L^2(0,\pi)$

$$\begin{cases} dX^u_s = AX^u_s ds + F(s, X^u_s) ds + (\lambda - A) b\, u_s ds + (\lambda - A) b\, dW_s & s \in [t,T], \\ X^u_t = x, \end{cases}$$

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$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s. \end{cases}$$

Minimize over all admissible controls the cost functional

$$J(t,x,u(\cdot)) = \mathbb{E}\int_t^T \int_0^{+\infty} \ell(s,\xi,y(s,\xi),u(s))\, d\xi\, ds + \mathbb{E}\int_0^{+\infty} \phi(\xi, y(T,\xi))\, d\xi.$$

Set

$$L(s,x,u) = \int_0^{+\infty} \ell(s,\xi,x(\xi),u)\, d\xi, \qquad \Phi(x) = \int_0^{+\infty} \phi(\xi,x(\xi))\, d\xi,$$

Abstract formulation of the control problem

Given X^u solution of

$$\begin{cases} dX^u_s = AX^u_s ds + F(s,X^u_s)ds + Bu_s ds + BdW_s & s \in [t,T], \\ X^u_t = x, \end{cases}$$

in $\mathcal{H}$, minimize over all admissible controls

$$J(t,x,u(\cdot)) = \mathbb{E}\int_t^T L(s,X^u_s,u_s)\, ds + \mathbb{E}\,\Phi(X^u_T).$$

We need to study solvability and regularity with respect to the initial datum x of the forward-backward stochastic differential system

$$\begin{cases} dX_s^{t,x} = AX_s^{t,x}ds + F(s, X_s^{t,x})ds + BdW_s & s \in [t, T], \\ X_t^{t,x} = x, \\ dY_s^{t,x} = -\Psi(s, X_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, & s \in [0, T], \\ Y_T = \Phi(X_T^{t,x}). \end{cases}$$

- continuity and differentiability: $(t,x) \to X_\cdot^{t,x}$ continuous $[0,T] \times \mathcal{H} \mapsto L^p_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$.

$$\nabla_x X_s^{t,x} h = e^{(s-t)A} h + \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X_\sigma^{t,x}) \nabla_x X_\sigma^{t,x}\, d\sigma, \qquad s \in [t,T],$$

  and $(t,x,h) \to \nabla_x X_\cdot^{t,x} h$ continuous $[0,T] \times \mathcal{H} \times \mathcal{H} \to L^p_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$.

- "differentiability" in the direction $(\lambda - A)^\alpha h$:

$$\Theta^\alpha(s,t,x)h = \left(\nabla_x X_s^{t,x} - e^{(s-t)A}\right)(\lambda - A)^\alpha h \text{ if } s \in [t,T].$$

  $(t,x,h) \to \Theta^\alpha(\cdot,t,x)h$ continuous $[0,T] \times \mathcal{H} \times \mathcal{H} \to L^\infty_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$; $\exists C_{\theta,\alpha}$ s.t.

$$|\Theta^\alpha(\cdot,t,x)h|_{L^\infty_{\mathcal{P}}(\Omega, C([0,T];\mathcal{H}))} \le C_{\theta,\alpha}|h| \text{ for all } t \in [0,T],\ x,h \in \mathcal{H}.$$

- X admits the Malliavin derivative.

$$\Downarrow$$

Let $w \in C([0,T) \times \mathcal{H};\mathbb{R})$ Gâteaux differentiable. Assume $\forall\, t \in [0,T)$, $x \in \mathcal{H}$, $\beta \in (0, \frac{1}{2}+\frac{\theta}{4})$, the linear operator $k \to \nabla w(t,x)(\lambda - A)^{1-\beta} k$ extends to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, denoted by $[\nabla w(\lambda - A)^{1-\beta}](t,x)$.

Then the process $\{w(s, X_s^{t,x}),\ s \in [t,T]\}$ admits a joint quadratic variation process with W, on every interval $[t,s] \subset [t,T)$, given by

$$\int_t^s [\nabla w(\lambda - A)^{1-\beta}](r, X_r^{t,x})\, (\lambda - A)^{\beta} \mathcal{D}_\lambda \, dr.$$

$$\begin{cases} dY_s^{t,x} = -\Psi(s, X_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, & s \in [0,T]. \\ Y_T = \Phi(X_T^{t,x}), \end{cases}$$

BSDE in integral form

$$Y_s^{t,x} + \int_s^T Z_r^{t,x}\, dW_r = \Phi(X_T^{t,x}) + \int_s^T \Psi(r, X_r^{t,x}, Z_r^{t,x})\, dr, \qquad s \in [0,T].$$

Hypothesis 2 on Ψ and Φ

1) $|\Phi(x_1) - \Phi(x_2)|_{\mathcal{H}} \le C_\Phi(1 + |x_1| + |x_2|)|x_2 - x_1|$ for all x_1, x_2 in $\mathcal{H}$.

2) $|\Psi(s, x_1, z) - \Psi(s, x_2, z)| \le C_\psi(1 + |x_1| + |x_2|)|x_2 - x_1|$, $|\Psi(s, x, z_1) - \Psi(s, x, z_2)| \le C_\psi |z_1 - z_2|$, $\sup_{s\in[0,T]} |\Psi(s, 0, 0)| \le C_\ell$ $\forall x, x_1, x_2 \in \mathcal{H}$, $z, z_1, z_2 \in \mathbb{R}$ and $s \in [0,T]$.

3) Φ Gâteaux differentiable, $\Psi(s,\cdot,\cdot)$ Gâteaux differentiable and $(x,h,z) \to \nabla_x \Psi(s,x,z)h$ and $(x,z,\zeta) \to \nabla_z \Psi(s,x,z)\zeta$ continuous on $\mathcal{H} \times \mathcal{H} \times \mathbb{R}$ and $\mathcal{H} \times \mathbb{R} \times \mathbb{R}$ respectively.

differentiability: $x \to (Y_\cdot^{t,x}, Z_\cdot^{t,x})$

"differentiability" in the direction $(\lambda - A)^\alpha k$

$\forall \alpha \in [0,1/2)$, $p \in [2,\infty)$ $\exists$ two families of processes

$\{P^\alpha(s,t,x)k : s \in [0,T]\}$ and $\{Q^\alpha(s,t,x)k : s \in [0,T]\}$; $\quad t \in [0,T)$, $x \in \mathcal{H}$, $k \in \mathcal{H}$

with $P^\alpha(\cdot,t,x)k \in L^p_{\mathcal{P}}(\Omega, C([0,T],\mathbb{R}))$ and $Q^\alpha(\cdot,t,x)k) \in L^p_{\mathcal{P}}(\Omega, L^2([0,T],\mathbb{R}))$ s. t. if $k \in dom(\lambda - A)^\alpha$, $t \in [0,T)$, $x \in \mathcal{H}$, then $\mathbb{P}$-a.s.

$$P^\alpha(s,t,x)k = \begin{cases} \nabla_x Y_s^{t,x}(\lambda - A)^\alpha k & \text{for all } s \in [t,T], \\ \nabla_x Y_t^{t,x}(\lambda - A)^\alpha k & \text{for all } s \in [0,t), \end{cases}$$

$$Q^\alpha(s,t,x)k = \begin{cases} \nabla_x Z_s^{t,x}(\lambda - A)^\alpha k & \text{for a.e. } s \in [t,T], \\ 0 & \text{if } s \in [0,t). \end{cases}$$

Corollary $v(t,x) := Y_t^{t,x}$. $v \in C([0,T] \times \mathcal{H}; \mathbb{R})$ and $\exists C$ s. t. $|v(t,x)| \leq C\,(1 + |x|)^2$, $t \in [0,T]$, $x \in \mathcal{H}$. v Gâteaux differentiable and $(t,x,h) \to \nabla v(t,x)h$ is continuous.

$\forall \alpha \in [0, 1/2)$, $t \in [0,T)$ and $x \in \mathcal{H}$ $k \to \nabla v(t,x)(\lambda - A)^{\alpha} k$ extends to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, denoted $[\nabla v(\lambda - A)^{\alpha}](t,x)$.

$(t,x,k) \to [\nabla v(\lambda - A)^{\alpha}](t,x)k$ continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and $\exists C_{\nabla v,\alpha}$

$$|[\nabla v(\lambda - A)^{\alpha}](t,x)k| \leq C_{\nabla v,\alpha}(T-t)^{-\alpha}(1 + |x|_{\mathcal{H}})|k|_{\mathcal{H}}, \qquad t \in [0,T),\ x,k \in \mathcal{H}.$$

Moreover

$$Z_s^{t,x} = [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{t,x})\ (\lambda - A)^{\beta}\mathcal{D}_{\lambda}, \qquad \text{for almost all } s \in [t,T].$$

Hamilton-Jacobi-Bellman equation

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = -\Psi(t,x,\nabla v(t,x)B), & t \in [0,T],\ x \in \mathcal{H}, \\ v(T,x) = \Phi(x). \end{cases}$$

transition semigroup:

$$P_{t,s}[\phi](x) = \mathbb{E}\,\phi(X_s^{t,x}), \qquad x \in \mathcal{H},\ 0 \le t \le s \le T,$$

$\mathcal{L}_t$ generator of $P_{t,s}$, formally:

$$\mathcal{L}_t[\phi](x) = \frac{1}{2}\langle \nabla^2\phi(x)B, B\rangle + \langle Ax + F(t,x), \nabla\phi(x)\rangle,$$

mild formulation

$$v(t,x) = P_{t,T}[\Phi](x) - \int_t^T P_{t,s}[\Psi(s,\cdot,\nabla v(s,\cdot)B](x)\,ds, \qquad t \in [0,T],\ x \in \mathcal{H},$$

Definition Let $\beta \in [0, \frac{1}{2})$. $v : [0,T] \times \mathcal{H} \to \mathbb{R}$ is a mild solution of HJB equation if

(i) $v \in C([0,T] \times \mathcal{H}; \mathbb{R})$ $\exists C, m \geq 0$ s.t. $|v(t,x)| \leq C\,(1 + |x|)^m$, $t \in [0,T]$, $x \in \mathcal{H}$.

(ii) v is Gâteaux differentiable and $(t,x,h) \to \nabla v(t,x)h$ is continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

(iii) $\forall t \in [0,T)$ and $x \in \mathcal{H}$ $k \to \nabla v(t,x)(\lambda - A)^{1-\beta}k$ extends to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, denoted by $[\nabla v(\lambda - A)^{1-\beta}](t,x)$. $(t,x,k) \to [\nabla v(\lambda - A)^{1-\beta}](t,x)k$ continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and $\exists C, m \geq 0$, $\kappa \in [0,1)$ s. t.

$$|[\nabla v(\lambda - A)^{1-\beta}](t,x)|_{\mathcal{H}^*} \leq C(T-t)^{-\kappa}(1 + |x|)^m, \qquad t \in [0,T),\ x \in \mathcal{H}.$$

(iv) $\forall t \in [0,T]$, $x \in \mathcal{H}$:

$$v(t,x) = P_{t,T}[\Phi](x) + \int_t^T P_{t,s}\left[\Psi\Big(s, \cdot, [\nabla v(\lambda - A)^{1-\beta}](s,\cdot)\ (\lambda - A)^{\beta}\mathcal{D}_\lambda\Big)\right](x)\, ds.$$

Theorem Assume Hypotheses 1 and 2 hold true. Then there exists a unique mild solution of the Hamilton-Jacobi-Bellman equation. The solution v is given by the formula

$$v(t,x) = Y_t^{t,x},$$

where (X, Y, Z) is the solution of the forward-backward system

$$\begin{cases} dX_s^{t,x} = AX_s^{t,x} ds + F(s, X_s^{t,x}) ds + B dW_s & s \in [t,T], \\ X_t^{t,x} = x, \\ dY_s^{t,x} = -\Psi(s, X_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, & s \in [0,T], \\ Y_T = \Phi(X_T^{t,x}). \end{cases}$$

"concrete" cost functional:

$$J(t,x,u(\cdot)) = \mathbb{E}\int_t^T \int_0^{+\infty} \ell(s,\xi,y(s,\xi),u(s))\,d\xi\,ds + \mathbb{E}\int_0^{+\infty} \phi(\xi,y(T,\xi))\,d\xi.$$

1) $\exists\, C_1, C_2$ s.t., for some $\epsilon > 0$, $\forall\ \xi \in [0,+\infty)$, $y_1, y_2 \in \mathbb{R}$

$$|\phi(\xi,y_1) - \phi(\xi,y_2)| \le C_1 \frac{\sqrt{\rho(\xi)}}{(1+\xi)^{1/2+\epsilon}}\,|y_1 - y_2| + C_2\,\rho(\xi)(|y_1| + |y_2|)\,|y_1 - y_2|,$$

2) $\forall\ t \in [0,T]$ and $\xi \in [0,+\infty)$, $\ell(t,\xi,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ continuous and $\exists\ C_1, C_2$ s. t. for some $\epsilon > 0$, $\forall\ t \in [0,T]$, $\xi \in [0,+\infty)$, $y_1, y_2 \in \mathbb{R}$, $u \in \mathcal{U}$,

$$|\ell(t,\xi,y_1,u) - \ell(t,\xi,y_2,u)| \le C_1 \frac{\sqrt{\rho(\xi)}}{(1+\xi)^{1/2+\epsilon}}\,|y_1 - y_2| + C_2\,\rho(\xi)(|y_1| + |y_2|)\,|y_1 - y_2|,$$

3) $\displaystyle\int_0^{+\infty} |\phi(\xi,0)|\,d\xi < \infty$ and $\forall\ t \in [0,T]$ $\displaystyle\int_0^{+\infty} \sup_{u\in\mathcal{U}} |\ell(t,\xi,0,u)|\,d\xi \le C_\ell.$

$$L(s,x,u) = \int_0^{+\infty} \ell(s,\xi,x(\xi),u)\, d\xi, \qquad \Phi(x) = \int_0^{+\infty} \phi(\xi,x(\xi))\, d\xi,$$

"Abstract" cost

$$J(t,x,u(\cdot)) = \mathbb{E}\int_t^T L(s,X_s^u,u_s)\, ds + \mathbb{E}\,\Phi(X_T^u).$$

Hamiltonian

$$\Psi(s,x,z) = \inf_{u\in\mathcal{U}}\{zu + L(s,x,u)\}.$$

$$\Gamma(s,x,z) = \{u \in \mathcal{U} : zu + L(s,x,u) = \Psi(s,x,z)\}$$

Optimal control problem (strong formulation): minimize, for arbitrary $t \in [0, T]$ and $x \in \mathcal{H}$, the cost $J(t, x, u)$, over all admissible controls, where $\{X_s^u : s \in [t, T]\}$ solves $\mathbb{P}$-a.s.

$$\begin{aligned} X_s^u &= e^{(s-t)A}x + \int_t^s e^{(s-r)A} F(r, X_r^u)\, dr + \int_t^s (\lambda - A)^{1-\beta} e^{(s-r)A} (\lambda - A)^{\beta} \mathcal{D}_\lambda \, dW_r \\ &+ \int_t^s (\lambda - A)^{1-\beta} e^{(s-r)A} (\lambda - A)^{\beta} \mathcal{D}_\lambda \, u_r \, dr, \qquad s \in [t, T]. \end{aligned}$$

Theorem Under the previous assumptions $\forall\ t \in [0, T]$, $x \in \mathcal{H}$ and $\forall$ admissible control u we have $J(t, x, u(\cdot)) \geq v(t, x)$, and $J(t, x, u(\cdot)) = v(t, x)$ holds if and only if

$$u_s \in \Gamma\left(s, X_s^{u,t,x}, [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{u,t,x})\ (\lambda - A)^{\beta} \mathcal{D}_\lambda\right)$$

Heat equation

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) - My(s,\xi) + f(y(s,\xi)), & s \geq 0 \ \xi \in (0,+\infty), \\ y(0,\xi) = x(\xi), \\ y(s,0) = u(s) + \dot{W}_s, \end{cases}$$

reformulated in $\mathcal{H}$ as

$$\begin{cases} dX_s^u = (A - MI)X_s^u ds + F(X_s^u)ds + Bu_s ds + BdW_s & s \geq 0, \\ X_0^u = x, \end{cases}$$

Uncontrolled version in mild form,

$$X_s = e^{s(A-MI)}x + \int_0^s e^{(s-r)(A-MI)} F(X_r^x)\, dr + \int_0^s e^{(s-r)(A-MI)} B\, dW_r, \qquad s \geq 0.$$

We know:

- $\forall\ T > 0$, $\exists$ a unique mild solution s.t. $\forall\ p \in [1, +\infty)$, $\alpha \in [0, \theta/4)$,

$$\mathbb{E} \sup_{s\in(0,T]} s^{p\alpha} |X_s^x|^p_{\text{dom}(\lambda-A)^\alpha} \le c_{p,\alpha}(1 + |x|_{\mathcal{H}})^p.$$

- X^x is continuous and Gâteaux differentiable with respect to the initial datum x).

- $\exists\ \Theta^\alpha(\cdot, x)h$ ("differentiability" in the direction $(\lambda - A)^\alpha h$).

- X^x admits the Malliavin derivative in every interval $[0, T]$.

If Hypothesis 1 holds true and if M is sufficiently large

$$|\nabla_x X_t^x| + |\Theta^\alpha(t, x)h| \le C|h|$$

$\forall\ t > 0$ and $x, h \in \mathcal{H}$.

Infinite horizon BSDE

$$dY_s^x = -\Psi(X_s^x, Z_s^x)\, ds + \mu Y_s^x\, ds + Z_s^x\, dW_s, \qquad s \geq 0,$$

i.e. $\mathbb{P}$-a.s., for every $T > 0$,

$$Y_s^x + \int_s^T Z_r^x\, dW_r = Y_T^x + \int_s^T (\Psi(X_r^x, Z_r^x) - \mu Y_r^x)\, dr, \qquad s \geq 0.$$

Hypothesis 3

i) $\Psi : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ continuous and $|\Psi(x, z_1) - \Psi(x, z_2)| \leq K|z_1 - z_2|$

ii) $\sup_{x\in\mathcal{H}} |\Psi(x, 0)| := M < +\infty$

iii) $\mu > 0$.

iv) Ψ is Gâteaux differentiable and $\nabla_x \Psi(x, z) \leq c \;\forall\; x \in \mathcal{H},\; z \in \mathbb{R}$, and for some $c > 0$.

Theorem Let Hypotheses 1 and 2 hold true. $x \to Y_0^x$ is Gâteaux differentiable as a map $\mathcal{H}, \mathbb{R})$ and $|Y_0^x| + |\nabla Y_0^x| \leq C$.

$\forall\ \alpha \in [0, 1/2)$, $p \in [2, \infty)$ $\exists$ $P^\alpha(x)k$ and $Q^\alpha(x)k$, $x \in \mathcal{H}$, $k \in \mathcal{H}$ s.t. if $k \in dom(\lambda - A)^\alpha$, $x \in \mathcal{H}$, then

$$P^\alpha(x)k = \nabla_x Y_0^x (\lambda - A)^\alpha k \quad Q^\alpha(x)k = \nabla_x Z_0^x (\lambda - A)^\alpha k.$$

$(x, k) \to P^\alpha(x)k$ continuous $\mathcal{H} \to \mathbb{R}$. Moreover $\exists$ $C_{\nabla Y, \alpha, p}$ s.t.

$$|P^\alpha(x)k| \leq C_{\nabla Y, \alpha} |k|_{\mathcal{H}}.$$

Corollary Let $v(x) = Y^x$: $v \in C(\mathcal{H}; \mathbb{R})$ and $|v(x)| \leq C\,(1 + |x|)^2$, $x \in \mathcal{H}$. Moreover v is Gâteaux differentiable and $(x, h) \to \nabla v(x)h$ is continuous. $\forall$ $\alpha \in [0, 1/2)$ and $x \in \mathcal{H}$ the linear operator $k \to \nabla v(x)(\lambda - A)^\alpha k$ extends to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, denoted by $[\nabla v(\lambda - A)^\alpha](x)$.

$(x, k) \to [\nabla v(\lambda - A)^\alpha](x)k$ is continuous $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and

$$|[\nabla v(\lambda - A)^\alpha](x)k| \leq C|k|_{\mathcal{H}}, \qquad x, k \in \mathcal{H}.$$

Hamilton-Jacobi-Bellman equation

$$\mathcal{L}[v](x) = \mu v(x) - \Psi(x, \nabla v(t,x)B).$$

transition semigroup:

$$P_s[\phi](x) = \mathbb{E}\,\phi(X_s^x), \qquad x \in \mathcal{H},\ s \geq 0,$$

$\mathcal{L}$ the generator of P_s, formally:

$$\mathcal{L}[\phi](x) = \frac{1}{2}\langle \nabla^2\phi(x)B, B\rangle + \langle Ax + F(x), \nabla\phi(x)\rangle,$$

mild formulation

$$v(x) = e^{-\mu T}P_T[u](x) - \int_0^T e^{-\mu s}P_s[\Psi(\cdot, \nabla v(\cdot)B](x)\, ds, x \in \mathcal{H},$$

Theorem Let Hypotheses 1 and 3 hold true, let M sufficiently large. Then there exists a unique mild solution of the stationary HJB given by $v(x) = Y_0^x$.

"concrete" cost functional

$$J(x,u) = \mathbb{E}\int_0^{+\infty} e^{-\mu s}\int_0^{+\infty} \ell(s,\xi,y(s,\xi),u_s)\, d\xi\, ds.$$

Hypothesis $\ell : [0,+\infty)\times\mathbb{R}\times\mathcal{U}\to\mathbb{R}$ continuous and $\exists\ C>0, \epsilon>0$ and $g\in L^1([0,+\infty))$ s.t.

$$|l(\xi,x,u)| \le Cg(\xi), \quad \text{for every} \quad \xi\in[0,+\infty), x\in\mathbb{R}, u\in\mathcal{U}.$$

$$|l(\xi,x_1,u)-l(\xi,x_2,u)| \le C\frac{|x_1-x_2|}{(1+\xi)^{\frac{1+\epsilon}{2}}}\sqrt{\rho(\xi)} \quad \forall \quad \xi\in[0,+\infty), x_1,x_2\in\mathbb{R}, u\in\mathcal{U}.$$

Reformulation of the cost functional

$$L(x,u) = \int_0^{+\infty} \ell(s,\xi,x(\xi),u)\, d\xi, \quad J(x,u(\cdot)) = \mathbb{E}\int_0^{+\infty} e^{-\mu s} L(X_s^u,u_s)\, ds.$$

hamiltonian

$$\Psi(x,z) = \inf_{u\in\mathcal{U}}\{zu + L(x,u)\},$$

$$\Gamma(x,z) = \{u \in \mathcal{U} : zu + L(x,u) = \Psi(x,z)\}$$

Theorem Under the previous assumptions $\forall\ x \in \mathcal{H}$ and $\forall$ admissible control u we have $J(x,u(\cdot)) \geq v(x)$, and $J(x,u(\cdot)) = v(x)$ holds if and only if

$$u_s \in \Gamma\left(X_s^{u,x}, [\nabla v(\lambda - A)^{1-\beta}](X_s^{u,x})\ (\lambda - A)^{\beta}\mathcal{D}_\lambda\right)$$

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