Stochastic variational inequalities with oblique subgradients

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Object

We give an existence and uniqueness result on the SVI with oblique subgradients

$$\begin{cases} X_t + \int_0^t H(X_s)\,dK_s = x_0 + \int_0^t f(s, X_s)\,ds + \int_0^t g(s, X_s)\,dB_s, \quad t \geq 0, \\ dK_t(\omega) \in \partial\varphi(X_t(\omega))(dt), \quad \mathbb{P} - a.s.\ \omega \in \Omega, \end{cases}$$

The approach is via a deterministic generalized Skorohod problem (*a variational inequality with oblique subgradients*)**:**

$$\begin{cases} x(t) + \int_0^t H(x(s))\,dk(s) = x_0 + \int_0^t f(s, x(s))\,ds + m(t), \quad t \geq 0, \\ dk(s) \in \partial\varphi(x(s))(ds), \end{cases}$$

where $m : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function.
Starting papers: Lions & Sznitman (1984), Dupuis & Ishii (1993)

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Outline

- **Preliminaries**
- **Variational inequalities with oblique subgradients**
  - *Existence result* ($m \in C\left([0,T];\mathbb{R}^d\right)$);
  - *Existence and uniqueness* ($m \in m \in C\left([0,T];\mathbb{R}^d\right) \bigcap BV\left([0,T];\mathbb{R}^d\right)$);
  - *Approximation result* $\left(m \in C^1\left([0,T];\mathbb{R}^d\right)\right)$.
- **Stochastic variational inequalities with oblique subgradients**
  - *Existence result (continuous coefficients)*
  - *Existence and uniqueness (Lipschitz coefficients)*
- **Annex**
  - *A priori estimates;*
  - *Yosida's regularization*
  - *Inequalities*
  - *Tightness*
- **References**

1 Preliminaries

Let $H=\left(h_{i,j}\right)_{d\times d}\in C_b^2\left(\mathbb{R}^d;\mathbb{R}^{2d}\right)$ be such that for all $x\in\mathbb{R}^d$,

$$(A_1):\quad\begin{cases}(i) & h_{i,j}(x)=h_{j,i}(x), \quad \text{and } i,j\in\overline{1,d},\\ (ii) & \dfrac{1}{c}|u|^2\leq\langle H(x)u,u\rangle\leq c|u|^2, \quad \forall\, u\in\mathbb{R}^d \text{ (for some } c\geq 1\text{)},\end{cases}$$

Denote $b\overset{def}{=}\sup\left\{|H_x'(x)|+\left|\left(H^{-1}\right)_x'(x)\right|:x\in\mathbb{R}^d\right\}$.
Consider the multivalued differential equation

$$\begin{cases}dx(t)+H(x(t))\,\partial\varphi(x(t))(dt)\ni dm(t), \quad t>0\\ x(0)=x_0\ ,\end{cases}\tag{1}$$

where

$$(A_2):\quad\begin{cases}(i) & x_0\in\mathcal{D}om(\varphi)\\ (ii) & m\in C\left(\mathbb{R}_+;\mathbb{R}^d\right), \quad m(0)=0,\end{cases}$$

and

$$(A_3):\quad \varphi:\mathbb{R}^d\to\left]-\infty,+\infty\right] \text{ is proper l.s.c. convex function.}$$

If $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, we denote

$$E_\varepsilon = \{x \in E : dist\,(x, E^c) \geq \varepsilon\} = \overline{\{x \in E : B\,(x, \varepsilon) \subset E\}}$$

the $\varepsilon-$interior of E.
We formulate the following supplementary assumptions

$$(A_4): \quad \begin{cases} (i) & D = Dom\,(\varphi) \text{ is a closed subset of } \mathbb{R}^d, \\ (ii) & \exists\, r_0 > 0,\ D_{r_0} \neq \emptyset \quad \text{and} \quad h_0 = \sup_{z \in E} d\,(z, D_{r_0}) < \infty, \\ (iii) & \exists\, L \geq 0,\ \text{such that } |\varphi\,(x) - \varphi\,(y)| \leq L + L\,|x - y|\,, \quad \text{for all } x, y \in D \end{cases}$$

By $\partial\varphi$ it is denoted the subdifferential of φ :

$$\partial\varphi\,(x) \overset{def}{=} \left\{\hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x\rangle + \varphi\,(x) \leq \varphi\,(y)\,,\ \text{for all } y \in \mathbb{R}^d\right\}.$$

A vector $\eta \in H\,(x)\,\partial\varphi\,(x)$ will be called *$H-$oblique subgradient*.
If E is a closed convex subset of $\mathbb{R}^d$ then

$$\varphi\,(x) = I_E\,(x) = \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{if } x \notin E, \end{cases}$$

is a convex l.s.c. function and

$$\partial I_E(x) = \{\hat{x} \in \mathbb{R}^d : \quad \langle \hat{x}, y - x \rangle \leq 0, \ \ \forall\, y \in E\} = N_E(x)\,, \text{ if } x \in E.$$

In this case $\nu(x) \in H(x)\,\partial I_E(x)$ is a outward $H-$oblique direction to $Bd(E)$ in the point x.

Definition 1 *Given two functions* $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, *we say that* $dk(t) \in \partial\varphi(x(t))(dt)$ *if*

$$\begin{array}{ll} (a) & x : \mathbb{R}_+ \to \mathbb{R}^d \quad \textit{is continuous,} \\ (b) & \displaystyle\int_0^T \varphi(x(r))\,dr < \infty, \qquad \forall\, T \geq 0, \\ (c) & k \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right), \qquad k(0) = 0, \\ (d) & \displaystyle\int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r))\,dr \leq \int_s^t \varphi(y(r))\,dr, \\ & \qquad\qquad \textit{for all } 0 \leq s \leq t \leq T \quad \textit{and } y \in C\left([0,T]; \mathbb{R}^d\right). \end{array}$$

We state the

Definition 2 *A pair* (x,k) *is a solution of the Skorohod problem (1) with* $H-$*oblique subgradients (and we write* $(x,k)\in\mathcal{SP}(H\partial\varphi;x_0,m)$*) if* $x,k:\mathbb{R}_+\rightarrow\mathbb{R}^d$ *are continuous functions and*

$$\begin{cases} (j) & x(t)+\int_0^t H(x(r))\,dk(r)=x_0+m(t), \quad \forall\, t\geq 0, \\ (jj) & dk(r)\in\partial\varphi(x(r))(dr). \end{cases} \tag{2}$$

In Annex, five technical lemmas are given to arrive of the following a priori estimate of the solutions $(x,k)\in\mathcal{SP}(H\partial\varphi;x_0,m):$

Proposition 1 *If* $(x,k)\in\mathcal{SP}(H\partial\varphi;x_0,m)$, *then under assumptions (*A_1-A_4*) there exists a constant* $C_T(\|m\|_T)=C(T,\|m\|_T,b,c,r_0,h_0)$ *(increasing function with respect to* $\|m\|_T$*), such that for all* $0\leq s\leq t\leq T:$

$$\begin{array}{ll} (a) & \|x\|_T+\updownarrow k\updownarrow_T\leq C_T(\|m\|_T) \\ (b) & |x(t)-x(s)|+\updownarrow k\updownarrow_t-\updownarrow k\updownarrow_s\leq C_T(\|m\|_T)\times\sqrt{(t-s)+\mathbf{m}_m(t-s)} \end{array} \tag{3}$$

2 *Variational inequalities with oblique subgradients*

2.1 Existence result : $m \in C\left([0,T];\mathbb{R}^d\right)$.

Consider the differential equation

$$\begin{cases} dx(t) + H(x(t))\,\partial\varphi(x(t))(dt) \ni f(t,x(t))\,dt + dm(t), & t>0 \\ x(0) = x_0, \end{cases} \tag{4}$$

where

(A_5) : (i) $(t,x) \longmapsto f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function (i.e. measurable w.r. to and t continuous w.r. to x;

(ii) $\int_0^T \left(f^{\#}(t)\right)^2 dt < \infty,$ where $f^{\#}(t) = \sup_{x \in Dom(\varphi)} |f(t,x)|$.

Theorem 1 *Let the assumptions ($A_1 - A_5$) be satisfied. Then the differential equation (4) has at least one solution in the sense of Definition i.e. $x,k : \mathbb{R}_+ \to \mathbb{R}^d$ are continuous functions and*

$$\begin{cases} (j) & x(t) + \int_0^t H(x(r))\,dk(r) = x_0 + \int_0^t f(r,x(r))\,dr + m(t), \quad \forall\, t \geq 0, \\ (jj) & dk(r) \in \partial\varphi(x(r))(dr). \end{cases} \tag{5}$$

Proof. *Step 1. Case* $m \in C^1\left(\mathbb{R}_+; \mathbb{R}^d\right)$.
It is sufficient to prove the existence of a solution on an interval $[0,T]$ arbitrary fixed.
Let $0 < \varepsilon \leq 1$ and the extensions $f(s,x) = 0$ and $m(s) = s \times m'(0+)$ for $s < 0$.
Consider the penalized problem

$$\begin{cases} x_\varepsilon(t) = x_0, \quad \text{if } t < 0, \\ x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s))\, dk_\varepsilon(s) = x_0 + \int_0^t \left[f(s-\varepsilon, \pi_D(x_\varepsilon(s-\varepsilon))) + m'(s-\varepsilon)\right] ds, \\ \hfill t \in [0,T], \end{cases}$$

or equivalent

$$\begin{cases} x_\varepsilon(t) = x_0, \quad \text{if } t < 0, \\ x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s))\, dk_\varepsilon(s) = x_0 + \int_{-\varepsilon}^{t-\varepsilon} \left[f(s, \pi_D(x_\varepsilon(s))) + m'(s)\right] ds, \quad t \in [0,T], \end{cases} \tag{6}$$

where

$$k_\varepsilon(t) = \int_0^t \nabla\varphi_\varepsilon(x_\varepsilon(s))\, ds \quad \text{and}$$

$$\pi_D(x) = \text{the orthogonal projection of } x \text{ on } D = \overline{Dom(\varphi)} = Dom(\varphi).$$

Since $x \longmapsto H(x)\nabla\varphi_\varepsilon(x) : \mathbb{R}^d \to \mathbb{R}^d$ is a sublinear and locally Lipschitz continuous function and $|f(s,\pi_D(x))| \leq f^\#(s)$ for all $(s,x) \in \mathbb{R}\times\mathbb{R}^d$, then recursively on the intervals $[i\varepsilon,(i+1)\varepsilon]$ the approximating equation has a unique solution $x_\varepsilon \in C\left([0,T];\mathbb{R}^d\right)$.
We have

$$\begin{aligned} &|x_\varepsilon(t)-u_0|^2+\varphi_\varepsilon(x_\varepsilon(t))+\int_0^t\langle H(x_\varepsilon(s))\nabla\varphi_\varepsilon(x_\varepsilon(s)),2[x_\varepsilon(s)-u_0]+\nabla\varphi_\varepsilon(x_\varepsilon(s))\rangle\, ds \\ &\quad=|x_0|^2+\varphi_\varepsilon(x_0)+\int_0^t\langle 2[x_\varepsilon(s)-u_0]+\nabla\varphi_\varepsilon(x_\varepsilon(s)),f(s-\varepsilon,\pi_D(x_\varepsilon(s-\varepsilon)))+m'(s-\varepsilon)\,ds\rangle\,. \end{aligned} \tag{7}$$

Let $(u_0,\hat{u}_0)\in\partial\varphi$, $0<\varepsilon\leq 1$. We have

- $|\varphi_\varepsilon(x_\varepsilon)-\varphi_\varepsilon(u_0)|+\varphi(u_0)-2|\hat{u}_0|^2-|x_\varepsilon-u_0|^2\leq\varphi_\varepsilon(x_\varepsilon)\,,$

- $\dfrac{1}{c}|\nabla\varphi_\varepsilon(x_\varepsilon)|^2\leq\langle H(x_\varepsilon)\nabla\varphi_\varepsilon(x_\varepsilon),\nabla\varphi_\varepsilon(x_\varepsilon)\rangle\,,$
- $$\langle H(x_\varepsilon)\nabla\varphi_\varepsilon(x_\varepsilon),2(x_\varepsilon-u_0)\rangle\geq -C\sup_{r\leq s}|x_\varepsilon(r)-u_0|^2-\frac{1}{4c}|\nabla\varphi_\varepsilon(x_\varepsilon)|^2\,,$$

•

$$\begin{aligned}&\left\langle 2\left(x_\varepsilon\left(s\right)-u_0\right)+\nabla\varphi_\varepsilon\left(x_\varepsilon\left(s\right)\right),f\left(s-\varepsilon,\pi_D\left(x_\varepsilon\left(s-\varepsilon\right)\right)\right)+m'\left(s-\varepsilon\right)\right\rangle\\&\le\frac{1}{4c}\left|\nabla\varphi_\varepsilon\left(x_\varepsilon\left(s\right)\right)\right|^2+\frac{1}{c}\left|x_\varepsilon\left(s\right)-u_0\right|^2+2C\left[\left(f^{\#}\left(s-\varepsilon\right)\right)^2+\left|m'\left(s-\varepsilon\right)\right|^2\right].\end{aligned}$$

Using these estimates in (7) and Gronwall's inequality we obtain

$$\sup_{t\in[0,T]}\left|x_\varepsilon\left(t\right)\right|^2+\sup_{t\in[0,T]}\left|\varphi_\varepsilon\left(x_\varepsilon\left(t\right)\right)\right|+\int_0^T\left|\nabla\varphi_\varepsilon\left(x_\varepsilon\left(s\right)\right)\right|^2ds\le C_T\,. \tag{8}$$

Since $\nabla\varphi_\varepsilon\left(x\right)=\dfrac{1}{\varepsilon}\left(x-J_\varepsilon x\right)$, then we also infer

$$\int_0^T\left|x_\varepsilon\left(s\right)-J_\varepsilon\left(x_\varepsilon\left(s\right)\right)\right|^2ds\le\varepsilon C_T. \tag{9}$$

Now from the approximating equation, for all $0\le s\le t\le T$, we have

$$\begin{aligned}\left|x_\varepsilon\left(t\right)-x_\varepsilon\left(s\right)\right|&\le\updownarrow x_\varepsilon\updownarrow_{[s,t]}\\&\le\int_s^t\left|H\left(x_\varepsilon\left(r\right)\right)\nabla\varphi_\varepsilon\left(x_\varepsilon\left(r\right)\right)\right|dr+\int_{s-\varepsilon}^{t-\varepsilon}\left|f\left(r,\pi_D\left(x_\varepsilon\left(r\right)\right)\right)\right|dr+\int_{s-\varepsilon}^{t-\varepsilon}\left|m'\left(r\right)\right|dr\\&\le C_T\sqrt{t-s}.\end{aligned}$$

Hence $\{x_\varepsilon : \varepsilon \in (0,1]\}$ is bounded and uniformly equicontinuous subset of $C\left([0,T];\mathbb{R}^d\right)$. From Acoli-Arzela's theorem it follows there exists $\varepsilon_n \to 0$, and $x \in C\left([0,T];\mathbb{R}^d\right)$ such that

$$\lim_{n\to\infty}\left[\sup_{t\in[0,T]}\left|x_{\varepsilon_n}(t)-x(t)\right|\right]=0.$$

By (9), there exists $h \in L^2\left(0,T;\mathbb{R}^d\right)$ such that on a subsequence denoted also ε_n we have

$$J_{\varepsilon_n}\left(x_{\varepsilon_n}\right)\to x \quad \text{in } L^2\left(0,T;\mathbb{R}^d\right) \text{ and } a.e. \text{ in } [0,T], \quad \text{as } \varepsilon_n \to 0,$$

and

$$\nabla\varphi\left(x_{\varepsilon_n}\right)\rightharpoonup h \quad \text{weakly in } L^2\left(0,T;\mathbb{R}^d\right).$$

Passing to $\liminf_{\varepsilon_n\to 0}$ in the subdifferential inequality

$$\int_s^t \left\langle \nabla\varphi\left(x_{\varepsilon_n}(r)\right), y(r)-x_{\varepsilon_n}(r)\right\rangle dr+\int_s^t \varphi\left(J_{\varepsilon_n}\left(x_{\varepsilon_n}(r)\right)\right)dr \le \int_s^t \varphi\left(y(r)\right)dr$$

we infer

$$\int_s^t \left\langle h(r), y(r)-x(r)\right\rangle dr+\int_s^t \varphi\left(x(r)\right)dr \le \int_s^t \varphi\left(y(r)\right)dr$$

for all $0\le s\le t\le T$ and $y\in C\left([0,T];\mathbb{R}^d\right)$, that is $h(r)\in\partial\varphi\left(x(r)\right)$ a.e. $t\in[0,T]$.

Finally passing to limit for $\varepsilon = \varepsilon_n \to 0$ in the approximating equation (6) we conclude that

$$x(t) + \int_0^t H(x(s))\, dk(s) = x_0 + \int_0^t f(s, x(s))\, ds + m(t),$$

where

$$k(t) = \int_0^t h(s)\, ds.$$

Step 2. $m \in C\left([0,T]; \mathbb{R}^d\right)$.
Extend $m(s) = 0$ for $s \leq 0$ and define for $0 < \varepsilon \leq 1$:

$$m_\varepsilon(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^t m(s)\, ds = \frac{1}{\varepsilon}\int_0^\varepsilon m(t + r - \varepsilon)\, dr$$

Consider the approximating equation

$$\begin{cases} x_\varepsilon(t) + \int_0^t H(x_\varepsilon(r))\, dk_\varepsilon(r) = x_0 + \int_0^t f(r, x_\varepsilon(r))\, dr + m_\varepsilon(t), \ t \geq 0, \\ dk_\varepsilon(r) \in \partial\varphi(x_\varepsilon(r))(dr). \end{cases}$$

By the first step this equation has a unique solution $(x_\varepsilon, k_\varepsilon)$, $dk_\varepsilon(s) = h_\varepsilon(s)\, ds \in \partial\varphi(x_\varepsilon(s))\, ds$

If we denote

$$M_\varepsilon(t) = \int_0^t f(r, x_\varepsilon(r))\,dr + m_\varepsilon(t)$$

then by Proposition 1

$$\|x_\varepsilon\|_T + \updownarrow k_\varepsilon \updownarrow_T \leq C_T(\|M_\varepsilon\|_T)\ , \quad \text{and}$$
$$|x_\varepsilon(t) - x_\varepsilon(s)| + \updownarrow k_\varepsilon \updownarrow_t - \updownarrow k_\varepsilon \updownarrow_s \leq C_T(\|M_\varepsilon\|_T) \times \sqrt{(t-s) + \mathbf{m}_{M_\varepsilon}(t-s)}.$$

Since

$$\|M_\varepsilon\|_T \leq \int_0^T f^{\#}(r)\,dr + \|m\|_T$$

and

$$\mathbf{m}_{M_\varepsilon}(t-s) \leq \sqrt{t-s}\int_0^T \left(f^{\#}(r)\right)^2 dr + \mathbf{m}_m(t-s)$$

then by Acoli-Arzela's theorem there exists $\varepsilon_n \to 0$ and $x, k \in C\left([0,T];\mathbb{R}^d\right)$ such that

$$x_{\varepsilon_n} \to x \quad \text{and} \quad k_{\varepsilon_n} \to k \quad \text{in } C\left([0,T];\mathbb{R}^d\right).$$

Using Helly-Bray theorem we infer $dk(r) \in \partial\varphi(x(r))(dr)$ and (x,k) is a solution of the equation (5).

■

2.2 Existence and uniqueness ($m \in BV\left([0,T];\mathbb{R}^d\right)$)

We introduce a new assumption

$$(A_6): \quad \begin{cases} \exists\ \mu \in L^1_{loc}\left(\mathbb{R}_+;\mathbb{R}_+\right) \text{ s.t. } \forall x, y \in \mathbb{R}^d \\ \qquad |f(t,x) - f(t,y)| \leq \mu(t)\,|x-y|\,, \quad a.e.\ t \geq 0. \end{cases}$$

that will yield the uniqueness.

Proposition 2 Let the assumptions ($A_1 - A_6$) be satisfied. If moreover $m \in BV_{loc}\left(\mathbb{R}_+;\mathbb{R}^d\right)$ then the generalized convex Skorohod problem with oblique subgradients (1) has a unique solution and moreover if (x,k) and $(\hat{x},\hat{k})$ are two solutions, corresponding to m and, respectively, $\hat{m}$, then

$$|x(t) - \hat{x}(t)| \leq Ce^{CV(t)}\left[|x_0 - \hat{x}_0| + \updownarrow m - \hat{m}\updownarrow_t\right]. \tag{10}$$

where $V(t) = \updownarrow x \updownarrow_t + \updownarrow \hat{x} \updownarrow_t + \updownarrow k \updownarrow_t + \updownarrow \hat{k} \updownarrow_t + \int_0^t \mu(r)\,dr$ and C is a constant depending only (b,c).

Proof. The existence was proved in Theorem 1. Let us prove the inequality (10) which yields the uniqueness, too.

Consider the symmetric and strict positive matrix

$$u(r) = \left([H(x(r))]^{-1} + [H(\hat{x}(r))]^{-1}\right)^{1/2}(x(r) - \hat{x}(r)).$$

Then with technical calculus we can show that there exists C a constant depending only c and b such that

$$\langle u(r), du(r)\rangle \leq C\,|u(r)|\, d\updownarrow m-\hat{m}\updownarrow_r + C\,|u(r)|^2\, dV(r)$$

with $V(t) = \updownarrow x\updownarrow_t + \updownarrow \hat{x} \updownarrow_t + \updownarrow k\updownarrow_t + \updownarrow \hat{k} \updownarrow_t + \int_0^t \mu(r)\, dr$. Now by Proposition 4 (Annex) we infer for all $t \geq 0$

$$e^{-CV(t)}\,|u(t)| \leq |x_0 - \hat{x}_0| + C\int_0^t e^{-CV(r)} d\updownarrow m - \hat{m}\updownarrow_r \ .$$

■

2.3 Approximation result $\left(m \in C^1\left([0,T];\mathbb{R}^d\right)\right)$

Proposition 3 Under the assumptions ($A_1 - A_6$) and $m \in C^1\left(\mathbb{R}_+;\mathbb{R}^d\right)$, the solution $(x_\varepsilon)_{0<\varepsilon\leq 1}$ of the approximating equation

$$\begin{cases} x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s))\, dk_\varepsilon(s) = x_0 + \int_0^t f(s, \pi_D(x_\varepsilon(s)))\, ds + m(t), \quad t \geq 0, \\ dk_\varepsilon(s) = \nabla\varphi_\varepsilon(x_\varepsilon(s))\, ds, \end{cases} \tag{11}$$

has the properties:

for all $T>0$*, there exists a constant independent of* $\varepsilon,\delta\in]0,1]$ *such that*

$$\begin{cases} (j) & \sup\limits_{t\in[0,T]}|x_\varepsilon(t)|^2+\sup_{t\in[0,T]}|\varphi_\varepsilon(x_\varepsilon(t))|+\displaystyle\int_0^T|\nabla\varphi_\varepsilon(x_\varepsilon(s))|^2ds\leq C_T\,, \\ (jj) & \updownarrow x_\varepsilon\updownarrow_{[s,t]}\leq C_T\sqrt{t-s},\quad \textit{for all } 0\leq s\leq t\leq T\,, \\ (jjj) & \|x_\varepsilon-x_\delta\|_T\leq C_T\sqrt{\varepsilon+\delta}\,. \end{cases}$$

Moreover there exist $x,k\in C\left([0,T];\mathbb{R}^d\right)$ *and* $h\in L^2\left(0,T;\mathbb{R}^d\right)$ *such that* $dk(t)=h(t)\,dt$,

$$\lim_{\varepsilon\to0}\left[\|x_\varepsilon-x\|_t+|k_\varepsilon(t)-k(t)|\right]=0,\quad\forall\,t\in[0,T]\,,$$

and (x,k) *is the unique solution of the variational inequality with oblique subgradients:*

$$\begin{cases} (j) & x(t)+\displaystyle\int_0^t H(x(r))\,dk(r)=x_0+\int_0^t f(r,x(r))\,dr+m(t)\,,\quad\forall\,t\geq0, \\ (jj) & dk(r)\in\partial\varphi(x(r))(dr)\,. \end{cases}$$

Proof. The proof is similar to those of Theorem 1 . The Cauchy property is proved in a similar manner as the uniqueness. If we denote

$$u_{\varepsilon,\delta}(s)=\left([H(x_\varepsilon(s))]^{-1}+[H(x_\delta(s))]^{-1}\right)^{1/2}(x_\varepsilon(s)-x_\delta(s))$$

then, after some technical calculus, we deduce that

$$\langle u_{\varepsilon,\delta}(s), du_{\varepsilon,\delta}(s)\rangle \leq 4(\varepsilon+\delta)\left|\nabla\varphi(x_\delta(s))\right|\left|\nabla\varphi(x_\varepsilon(s))\right| ds + C\left|u_{\varepsilon,\delta}(s)\right|^2 dV_{\varepsilon,\delta}(s)$$

with

$$V(s) = \updownarrow x_\varepsilon \updownarrow_s + \updownarrow x_\delta \updownarrow_s + \updownarrow k_\varepsilon \updownarrow_s + \updownarrow k_\delta \updownarrow_s + \int_0^s \mu(r)\,dr \leq C_T$$

■

Corollary 1 *If he assumptions ($A_1 - A_6$) are satisfied and*

- $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is a stochastic basis,
- $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is a p.m.s.p., $M_\cdot(\omega) \in C^1\left(\mathbb{R}_+;\mathbb{R}^d\right)$, $a.s.\ \omega \in \Omega$,

then the SDE

$$\begin{cases} X_t(\omega) + \int_0^t H(X_t(\omega))\,dK_t(\omega) = x_0 + \int_0^t f(s, X_s(\omega))\,ds + M_t(\omega), & t \geq 0, \\ dK_t(\omega) \in \partial\varphi(X_t(\omega))(dt) \end{cases}$$

has a unique solution $\{(X_\cdot(\omega), K_\cdot(\omega)) : \omega \in \Omega\}$. *Moreover X and K are p.m.s.p.*

3 Stochastic variational inequalities with oblique subgradients

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis and $\{B_t : t \geq 0\}$ a $\mathbb{R}^k-$valued Brownian motion. We consider the SDE

$$\begin{cases} X_t + \int_0^t H(X_t)\, dK_t = x_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s, \quad t \geq 0 \\ dK_t \in \partial\varphi(X_t)(dt) \end{cases} \tag{12}$$

where $x_0 \in \mathbb{R}^d$, $(t,x) \longmapsto f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $(t,x) \longmapsto g(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$

$$(A_7): \begin{cases} (i) & f \text{ and } g \text{ are Carathéodory functions} \\ & \quad \text{(i.e. measurable w.r.t. to } t \text{ and continuous w.r.t. to } x \\ (ii) & \int_0^T \left(f^{\#}(t)\right)^2 + \int_0^T \left(g^{\#}(t)\right)^4 dt < \infty, \end{cases}$$

where

$$f^{\#}(t) = \sup_{x\in Dom(\varphi)} |f(t,x)| \quad \text{and} \quad g^{\#}(t) = \sup_{x\in Dom(\varphi)} |g(t,x)|$$

Definition 3 (I) *Given* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)_{t\geq 0}$ *a* $\mathbb{R}^k-$*valued* $\mathcal{F}_t-$*Brownian motion, a pair* $(X, K) : \Omega \times$

$[0,\infty[\rightarrow\mathbb{R}^d\times\mathbb{R}^d$ *of continuous p.m.s.p. is a strong solution of the SVI (12) if* $\mathbb{P}-a.s.\ \omega\in\Omega:$

$$\left\{\begin{array}{rl} j) & \varphi\left(X_\cdot\right)\in L^1_{loc}\left(\mathbb{R}_+\right)\\ jj) & K_\cdot\in BV_{loc}\left([0,\infty[\,;\mathbb{R}^d\right),\qquad K_0=0,\\ jjj) & X_t+\displaystyle\int_0^t H\left(X_t\right)dK_t=x_0+\int_0^t f\left(s,X_s\right)ds+\int_0^t g\left(s,X_s\right)dB_s,\ \forall\, t\geq 0,\\ jv) & \forall\, 0\leq s\leq t,\ \forall y:\mathbb{R}_+\rightarrow\mathbb{R}^d\ \textit{continuous}:\\ & \qquad\displaystyle\int_s^t\left\langle y\left(r\right)-X_r,dK_r\right\rangle+\int_s^t\varphi\left(X_r\right)dr\leq\int_s^t\varphi\left(y\left(r\right)\right)dr \end{array}\right. \tag{13}$$

that is

$$\left(X_\cdot\left(\omega\right),K_\cdot\left(\omega\right)\right)=\mathcal{SP}\left(H\partial\varphi;x_0,M_\cdot\left(\omega\right)\right),\quad \mathbb{P}-a.s.\ \omega\in\Omega,$$

with

$$M_t=\int_0^t f\left(s,X_s\right)ds+\int_0^t g\left(s,X_s\right)dB_s\ .$$

(II) *If there exists a stochastic basis* $\left(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t\right)_{t\geq 0}$ *, a* $\mathbb{R}^k-$*valued* $\mathcal{F}_t-$*Brownian motion* $\{B_t:t\geq 0\}$ *and a pair* $\left(X_\cdot,K_\cdot\right):\Omega\times\mathbb{R}_+\rightarrow\mathbb{R}^d\times\mathbb{R}^d$ *of continuous p.m.s.p. such that*

$$\left(X_\cdot\left(\omega\right),K_\cdot\left(\omega\right)\right)=\mathcal{SP}\left(H\partial\varphi;x_0,M_\cdot\left(\omega\right)\right),\quad \mathbb{P}-a.s.\ \omega\in\Omega,$$

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$ *is called a weak solution of the SVI (12).*

Theorem 2 Let the assumptions (A_1, A_3, A_4, A_7) *be satisfied. Then the SDE (12) has at least one weak solution* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$.

Proof. The main ideas of the proof comes from Răşcanu [6].
We extend $f(t,x) = 0$ and $g(t,x) = 0$ for $t < 0$.
Step 1. Approximating problem. Let $0 < \varepsilon \leq 1$ and the approximating equation

$$\begin{cases} X_t^n = x_0, \quad \text{if } t < 0, \\ X_t^n + \displaystyle\int_0^t H(X_t^n)\, dK_t^n = x_0 + M_t^n, \quad t \geq 0, \\ dK_t^n \in \partial\varphi(X_t^n)\, dt. \end{cases} \tag{14}$$

where

$$\begin{aligned} M_t^n &= \int_0^t f\left(s, \pi_D\left(X_{s-1/n}^n\right)\right) ds + n\int_{t-1/n}^t \left[\int_0^s g\left(r, \pi_D\left(X_{r-1/n}^n\right)\right) dB_r,\right] ds \\ &= \int_0^t f\left(s, \pi_D\left(X_{s-1/n}^n\right)\right) ds + \int_0^1 \left[\int_0^{t-\frac{1}{n}+\frac{1}{n}u} g\left(r, \pi_D\left(X_{r-1/n}^n\right)\right) dB_r\right] du. \end{aligned}$$

and

$$\pi_D(x) \text{ is the orthogonal projection of } x \text{ on } D = \overline{Dom(\varphi)}.$$

Since M^n is a C^1-continuous progressively measurable stochastic process, then by Corollary 1 the approximating equation (14) has a unique solution (X^n, K^n) of continuous p.m.s.p.
Step 2. Tightness. Let $T \geq 0$ be arbitrary fixed.

- $\{M^n : n \geq 1\}$ is tight on $C\left([0,T];\mathbb{R}^d\right)$ since

$$\sup_{n\geq 1} \mathbb{E}\left[\sup_{0\leq\theta\leq\varepsilon} |M^n_{t+\theta} - M^n_t|^4\right] \leq \varepsilon\,\gamma(\varepsilon),$$

  where $\gamma(\varepsilon) \to 0$,as $\varepsilon \to 0$.

- $U^n = (X^n, K^n, \updownarrow K^n \updownarrow), n \in \mathbb{N}^*$, is tight on $\mathbb{X} = C\left([0,T];\mathbb{R}^{2d+1}\right)$ since by Proposition 1

$$\|U^n\|_T \leq C_T(\|M^n\|_T)$$
$$\mathbf{m}_{U^n}(\varepsilon) \leq C_T(\|M^n\|_T) \times \sqrt{\varepsilon + \mathbf{m}_{M^n}(\varepsilon)}$$

  and, then, from Lemma 6 the tightness follows.

- By Prohorov theorem there exists a subsequence (denoted also by n) such that as $n \to \infty$

$$(X^n, K^n, \updownarrow K^n \updownarrow, B) \xrightarrow{\mathcal{L}} (X, K, V, B) \quad \text{(in law) on } C\left([0,T];\mathbb{R}^{2d+1+k}\right).$$

- By Skorohod theorem there exist

$$(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n), (\bar{X}, \bar{K}, \bar{V}, \bar{B}) : \left([0,1]\,; \mathcal{B}_{[0,1]}, dt\right) \to C\left([0,T]\,; \mathbb{R}^{2d+1+k}\right)$$

random variables such that

$$\begin{aligned} &(a) \qquad (\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) \overset{\mathcal{L}}{=} (X^n, K^n, \updownarrow K^n \updownarrow, B)\,, \\ &(b) \qquad (\bar{X}, \bar{K}, \bar{V}, \bar{B}) \overset{\mathcal{L}}{=} (X, K, V, B), \\ &(c) \qquad (\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) \xrightarrow{\mathbb{P}-a.s.} (\bar{X}, \bar{K}, \bar{V}, \bar{B}). \end{aligned}$$

- By Lemma 12, $\left(\bar{B}^n, \{\mathcal{F}_t^{\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n}\}\right), n \geq 1$, and $\left(\bar{B}, \{\mathcal{F}_t^{\bar{X}, \bar{K}, \bar{V}, \bar{B}}\}\right)$ are $\mathbb{R}^k-$Brownian motion.

Step 3. Passing to the limit.

- Since $(X^n, K^n, \updownarrow K^n \updownarrow, B) \xrightarrow{\mathcal{L}} (\bar{X}, \bar{K}, \bar{V}, \bar{B})$, then by Lemma 9 for all $0 \leq s \leq t$, $\mathbb{P} - a.s.$

$$\bar{X}_0 = x_0\,, \qquad \bar{K}_0 = 0,$$

$$\updownarrow \bar{K} \updownarrow_t - \updownarrow \bar{K} \updownarrow_s \leq \bar{V}_t - \bar{V}_s \quad \text{and} \quad 0 = \bar{V}_0 \leq \bar{V}_s \leq \bar{V}_s$$

and from

$$\int_s^t \varphi\left(X_r^n\right) dr \leq \int_s^t \varphi\left(y\left(r\right)\right) dr - \int_s^t \left\langle y\left(r\right) - X_r^n, dK_r^n \right\rangle \quad a.s.$$

it follows

$$\int_s^t \varphi\left(\bar{X}_r\right) dr \leq \int_s^t \varphi\left(y\left(r\right)\right) dr - \int_s^t \left\langle y\left(r\right) - \bar{X}_r, d\bar{K}_r\right\rangle \tag{15}$$

for all $0 \leq s < t$. Hence $d\bar{K}_r \in \partial\varphi\left(\bar{X}_r\right)\left(dr\right)$

- By Lebesgue theorem and Lemma 12, as $n \to \infty$

$$\bar{M}^n_\cdot = x_0 + \int_0^\cdot f\left(s, \pi_D\left(\bar{X}^n_{s-1/n}\right)\right) ds + n\int_{\cdot-1/n}^\cdot \left[\int_0^s g\left(r, \pi_D\left(\bar{X}^n_{r-1/n}\right)\right) dB_r,\right] ds$$
$$\longrightarrow \bar{M}_\cdot = x_0 + \int_0^\cdot f\left(s, \bar{X}_s\right) ds + \int_0^\cdot g\left(s, \bar{X}_s\right) d\bar{B}_s, \text{ in } S_d^0\left[0,T\right].$$

- By Lemma 10

$$\mathcal{L}\left(\bar{X}^n, \bar{K}^n, \bar{B}^n, \bar{M}^n\right) = \mathcal{L}\left(X^n, K^n, B^n, M^n\right) \quad \text{on } C\left(\mathbb{R}_+; \mathbb{R}^{d+d+k+d}\right)$$

and therefore, by Lemma 9, from

$$X_t^n + \int_0^t H\left(X_s^n\right) dK_s^n - M_t^n = 0, \quad a.s.,$$

we have

$$\bar{X}_t^n + \int_0^t H\left(\bar{X}_s^n\right) d\bar{K}_s^n - \bar{M}_t^n = 0, \quad a.s.$$

and letting $n \to \infty$,

$$\bar{X}_t + \int_0^t H\left(\bar{X}_s\right) d\bar{K}_s - \bar{M}_t = 0, \quad a.s. \; .$$

that is $\mathbb{P} - a.s.$

$$\bar{X}_t + \int_0^t H\left(\bar{X}_s\right) d\bar{K}_s = x_0 + \int_0^t f\left(s, \bar{X}_s\right) ds + \int_0^t g\left(s, \bar{X}_s\right) d\bar{B}_s, \; \forall \, t \in [0, T] \, .$$

Consequently $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \mathcal{F}_t^{\bar{B}, \bar{X}}, \bar{X}_t, \bar{K}_t, \bar{B}_t\right)_{t \geq 0}$ is a weak solution of the SDE (12). The proof is complete.

■

We also add continuity Lipschitz conditions:

$$(A_8): \quad \begin{array}{ll} & \exists \, \mu \in L^1_{loc}\left(\mathbb{R}_+\right), \;\; \exists \, \ell \in L^2_{loc}\left(\mathbb{R}_+\right) \text{ s.t. } \forall \, x, y \in \mathbb{R}^d, \quad a.e. \; t \geq 0, \\ (i) & \left|f\left(t, x\right) - f\left(t, y\right)\right| \leq \mu\left(t\right) \left|x - y\right|, \\ (ii) & \left|g\left(t, x\right) - g\left(t, y\right)\right| \leq \ell\left(t\right) \left|x - y\right| . \end{array} \tag{16a}$$

Theorem 3 *Let the assumptions* $(A_1, A_3, A_4, A_7, A_8)$ *be satisfied. Then then the SDE (*12*) has a unique strong solution* $(X, K) \in S_d^0 \times S_d^0$.

Proof. It is sufficient to prove the *pathwise uniqueness,* since by Theorem 1.1 page 149 in Ikeda &Watanabe [3] *the existence of a weak solution + the pathwise uniqueness* implies the existence of a strong solution.
Let (X, K), $(\hat{X}, \hat{K}) \in S_d^0 \times S_d^0$ two solutions. Let

$$U_r = \left(H^{-1}(X_r) + H^{-1}(\hat{X}_r)\right)^{1/2}\left(X_r - \hat{X}_r\right).$$

Then

$$dU_r = d\mathcal{K}_r + \mathcal{G}_r dB_r,$$

where

$$\begin{aligned} d\mathcal{K}_r &= (dN_r)\, Q_r^{-1/2} U_r + Q_r^{1/2}\left[H\left(\hat{X}_r\right) d\hat{K}_r - H(X_r)\, dK_r\right] \\ &\quad + Q_r^{1/2}\left[f(r, X_r) - f\left(r, \hat{X}_r\right)\right] dr + \sum_{j=1}^{k} \beta_r^{(j)}\left(g(r, X_r) - g\left(r, \hat{X}_r\right)\right) e_j \\ \mathcal{G}_r &= \Gamma_r + Q_r^{1/2}\left[g(r, X_r) - g\left(r, \hat{X}_r\right)\right] \end{aligned}$$

where for each $j \in \overline{1,k}$, $\beta^{(j)}$ is a $\mathbb{R}^{d\times d}-$valued $\mathcal{P}-$m.s.p. such that $\int_0^T \left|\beta_r^{(j)}\right|^2 dr < \infty$, $a.s.$ and Γ_r is a $\mathbb{R}^{d\times k}$ matrix with the columns $\beta_r^{(1)}(X_r - \hat{X}_r), \ldots, \beta_r^{(k)}(X_r - \hat{X}_r)$.
Hence

$$\langle U_r, d\mathcal{K}_r \rangle + \frac{1}{2}\left|\mathcal{G}_r\right|^2 dt \leq |U_r|^2 dV_r$$

with

$$dV_r = C \times \left(\mu\left(r\right) dr + \ell^2\left(r\right) dr + d\updownarrow N \updownarrow_r + d\updownarrow K \updownarrow_r + d\updownarrow \hat{K} \updownarrow_r + \sum_{j=1}^{k} \left|\beta_r^{(j)}\right|^2 dr \right).$$

By Lemma 7 we infer

$$\mathbb{E}\frac{e^{-2V_s}\left|U_s\right|^2}{1+e^{-2V_s}\left|U_s\right|^2} \leq \mathbb{E}\frac{e^{-2V_0}\left|U_0\right|^2}{1+e^{-2V_0}\left|U_0\right|^2} = 0$$

Consequently $Q_s^{1/2}\left(X_s - \hat{X}_s\right) = U_s = 0$, $\mathbb{P}-a.s.$, for all $s \geq 0$ and by the continuity of X and $\hat{X}$ we conclude that $\mathbb{P}-a.s.$,

$$X_s = \hat{X}_s \quad \text{for all } s \geq 0.$$

■

4 Annex

4.1 A priori estimates

Lemma 1 *If* $(x,k) = \mathcal{SP}(H\partial\varphi; x_0, m)$ *and* $(\hat{x},\hat{k}) = \mathcal{SP}(H\partial\varphi; \hat{x}_0, \hat{m})$ *then for all* $0 \le s \le t$:

$$\int_s^t \left\langle x(r) - \hat{x}(r), dk(r) - d\hat{k}(r) \right\rangle \ge 0;$$

Lemma 2 *Let the assumptions* ($A_1 - A_4$) *be satisfied. If* $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, *then for all* $0 \le s \le t \le T$

$$\mathbf{m}_x(t-s) \le \left[(t-s) + \mathbf{m}_m(t-s) + \sqrt{\mathbf{m}_m(t-s)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)}\right]$$
$$\times \exp\left\{C\left[1 + (t-s) + \left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s + 1\right)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)\right]\right\}$$

where $C = C(b,c,L) > 0$ *and*

$$\mathbf{m}_m(\varepsilon) \overset{def}{=} \sup\left\{|m(u) - g(v)| : u,v \in [0,T],\ |u-v| \le \varepsilon\right\}.$$

Lemma 3 *Let the assumptions ($A_1 - A_4$) be satisfied. If* $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, $0 \le s \le t \le T$ *and*

$$\sup_{r\in[s,t]} |x(r) - x(s)| \le 2\delta_0 = \frac{\rho_0}{2bc} \wedge \rho_0 \quad \textit{with } \rho_0 = \frac{r_0}{2(1 + r_0 + h_0)},$$

then

$$\updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \le \frac{1}{\rho_0} |k(t) - k(s)| + \frac{3L}{\rho_0} (t - s);$$

and

$$|x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \le \sqrt{t - s + \mathbf{m}_m (t - s)} \times e^{C_T\left(1+\|m\|_T^2\right)}$$

where $C_T = C(b, c, r_0, h_0, L, T) > 0$.

Lemma 4 *Let the assumptions ($A_1 - A_4$) be satisfied. Let* $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, $0 \le s \le t \le T$ *and* $x(r) \in D_{\delta_0}$ *for all* $r \in [s,t]$*. Then*

$$\updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \le L\left(1 + \frac{2}{\delta_0}\right) (t - s).$$

and

$$\mathbf{m}_x (t - s) \le C_T \times [(t - s) + \mathbf{m}_m (t - s)]$$

where $C = C\left(b, c, r_0, h_0, L, T\right) > 0.$

Lemma 5 *Let the assumptions (*$A_1 - A_4$*) be satisfied and* $(x, k) \in \mathcal{SP}\left(H\partial\varphi; x_0, m\right)$. *Then there exists a positive constant* $C_T\left(\|m\|_T\right) = C\left(x_0, b, c, r_0, h_0, L, T, \|m\|_T\right)$, *increasing function with respect to* $\|m\|_T$ *,such that for all* $0 \leq s \leq t \leq T$:

$$\begin{array}{ll}(a) & \|x\|_T + \updownarrow k \updownarrow_T \leq C_T\left(\|m\|_T\right), \\ (b) & |x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \leq C_T\left(\|m\|_T\right) \times \sqrt{t - s + \mathbf{m}_m(t-s)}.\end{array}$$

4.2 Yosida's regularization of a convex function

By $\nabla\varphi_\varepsilon$ is denoted the gradient of the Yosida's regularization φ_ε of the function φ,

$$\begin{aligned}\varphi_\varepsilon(x) &= \inf\left\{\frac{1}{2\varepsilon}|z - x|^2 + \varphi(z) : \ z \in \mathbb{R}^d\right\} \\ &= \frac{1}{2\varepsilon}|x - J_\varepsilon x|^2 + \varphi(J_\varepsilon x)\end{aligned}$$

where $J_\varepsilon x = x - \varepsilon\nabla\varphi_\varepsilon(x)$. The function $\varphi_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ is a convex and differentiable. Then for all

$x, y \in \mathbb{R}^d$, $\varepsilon > 0$:

$$\begin{array}{ll} a) & \nabla\varphi_\varepsilon(x) = \partial\varphi_\varepsilon(x) \in \partial\varphi(J_\varepsilon x), \text{ and } \varphi(J_\varepsilon x) \leq \varphi_\varepsilon(x) \leq \varphi(x), \\ b) & |\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)| \leq \dfrac{1}{\varepsilon}|x-y|, \\ c) & \langle \nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y), x-y \rangle \geq 0, \\ d) & \langle \nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y), x-y \rangle \geq -(\varepsilon+\delta)\langle \nabla\varphi_\varepsilon(x), \nabla\varphi_\delta(y) \rangle \end{array}$$

In the case $0 = \varphi(0) \leq \varphi(x)$ for all $x \in \mathbb{R}^d$, then we moreover have

$$\begin{array}{ll} (a) & 0 = \varphi_\varepsilon(0) \leq \varphi_\varepsilon(x) \quad \text{and} \quad J_\varepsilon(0) = \nabla\varphi_\varepsilon(0) = 0, \\ (b) & \dfrac{\varepsilon}{2}|\nabla\varphi_\varepsilon(x)|^2 \leq \varphi_\varepsilon(x) \leq \langle \nabla\varphi_\varepsilon(x), x \rangle, \quad \forall x \in \mathbb{R}^d. \end{array}$$

4.3 Inequalities

Lemma 6 *Let* $x \in BV_{loc}\left([0,\infty[;\mathbb{R}^d\right)$ *and* $V \in BV_{loc}([0,\infty[;\mathbb{R})$ *be continuous functions. Let* $R, N : [0,\infty[\to [0,\infty[$ *continuous increasing functions. If*

$$\langle x(t), dx(t) \rangle \leq dR(t) + |x(t)|\, dN(t) + |x(t)|^2\, dV(t)$$

as signed measures on $[0,\infty[$ *, then for all* $0 \le t \le T$:

$$\left\|e^{-V}x\right\|_{[t,T]} \le 2\left[\left|e^{-V(t)}x(t)\right| + \left(\int_t^T e^{-2V(s)}dR(s)\right)^{1/2} + \int_t^T e^{-V(s)}dN(s)\right]$$

If $R = 0$ *then for all* $0 \le t \le s$:

$$|x(s)| \le e^{V(s)-V(t)}|x(t)| + \int_t^s e^{V(s)-V(r)}dN(r).$$

We give from Pardoux&Răşcanu [5] an estimate on the local semimartingale $X \in S_d^0$ of the form

$$X_t = X_0 + K_t + \int_0^t G_s dB_s, \;\; t \ge 0, \quad \mathbb{P} - a.s.$$

where $G \in \Lambda^0_{d\times k}$ and $K \in S_d^0$; $K_\cdot \in BV_{loc}\left([0,\infty[\,;\mathbb{R}^d\right)$, $K_0 = 0$, $\mathbb{P} - a.s.$;

Lemma 7 *Let* $X \in S_d^0$ *be a local semimartingale of the form* (**??**)*. Assume there exist* $p \ge 1$ *and* V *a* $\mathcal{P}-$*m.b-v.c.s.p.*, $V_0 = 0$, *such that as signed measures on* $[0,\infty[$

$$\langle X_t, dK_t\rangle + \tfrac{1\vee(p-1)}{2}|G_t|^2 dt \le |X_t|^2 dV_t, \;\; \mathbb{P} - a.s.,$$

then for all $\delta \geq 0$, $0 \leq t \leq s$:

$$\mathbb{E}^{\mathcal{F}_t} \frac{\left|e^{-V_s} X_s\right|^p}{\left(1+\delta\left|e^{-V_s} X_s\right|^2\right)^{p/2}} \leq \frac{\left|e^{-V_t} X_t\right|^p}{\left(1+\delta\left|e^{-V_t} X_t\right|^2\right)^{p/2}}, \ \mathbb{P}-a.s..$$

4.4 Tightness

Lemma 8 *Let* $\{X_t^n : t \geq 0\}$, $n \in \mathbb{N}^*$, *be a family of* $\mathbb{R}^d-$*valued continuous stochastic processes defined on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Suppose that for every* $T \geq 0$, *there exist* $\alpha = \alpha_T > 0$ *and* $b = b_T \in C(\mathbb{R}_+)$ *with* $b(0) = 0$, *(both independent of* n*) such that*

$$(j) \qquad \lim_{N \to \infty} \left[\sup_{n \in \mathbb{N}^*} \mathbb{P}(\{|X_0^n| \geq N\})\right] = 0,$$

$$(jj) \qquad \mathbb{E}\left[1 \wedge \sup_{0 \leq s \leq \varepsilon} |X_{t+s}^n - X_t^n|^\alpha\right] \leq \varepsilon \cdot b(\varepsilon), \ \forall\, \varepsilon > 0, \ n \geq 1, \ t \in [0, T],$$

Then $\{X^n : n \in \mathbb{N}^*\}$ *is tight in* $C(\mathbb{R}_+; \mathbb{R}^d)$.

Lemma 9 $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ *is a l.s.c. function. Let* (X, K, V), (X^n, K^n, V^n), $n \in \mathbb{N}$, *be* $C\left([0,T]; \mathbb{R}^d\right)^2 \times$

$C([0,T];\mathbb{R})-$*valued random variables, such that*

$$(X^n,K^n,V^n)\xrightarrow[n\to\infty]{law}(X,K,V)$$

and for all $0\le s<t$, *and* $n\in\mathbb{N}^*$,

$$\begin{aligned}\updownarrow K^n\updownarrow_t-\updownarrow K^n\updownarrow_s\le V_t^n-V_s^n\ \ a.s.\\ \int_s^t\varphi(X_r^n)\,dr\le\int_s^t\langle X_r^n\,,dK_r^n\rangle\,,\ \ a.s.,\end{aligned}$$

then $\updownarrow K\updownarrow_t-\updownarrow K\updownarrow_s\le V_t-V_s\ \ a.s.$ *and*

$$\int_s^t\varphi(X_r)\,dr\le\int_s^t\langle X_r\,,dK_r\rangle\,,\ \ a.s..$$

Lemma 10 *Let* $X,\hat{X}\in S_d^0[0,T]$ *and* $B,\hat{B}$ *be two* $\mathbb{R}^k-$*Brownian motions and* $g:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^{d\times k}$ *be a function satisfying*

$$\begin{aligned}&g(\cdot,y)\ \text{ is measurable }\forall\ y\in\mathbb{R}^d,\ \ \text{and}\\ &y\mapsto g(t,y)\ \text{ is continuous } dt-a.e..\end{aligned}$$

If

$$\mathcal{L}(X,B)=\mathcal{L}(\hat{X},\hat{B}),\ \ \textit{on } C\left(\mathbb{R}_+,\mathbb{R}^{d+k}\right)$$

then

$$\mathcal{L}\left(X,B,\int_0^{\cdot} g(s,X_s)\,dB_s\right)=\mathcal{L}\left(\hat{X},\hat{B},\int_0^{\cdot} g\left(s,\hat{X}_s\right)d\hat{B}_s\right),\ \ \textit{on } C\left(\mathbb{R}_+,\mathbb{R}^{d+k+d}\right).$$

Lemma 11 *Let $g:\mathbb{R}_+\rightarrow\mathbb{R}_+$ be a continuous function satisfying $g(0)=0$ and $G:C\left(\mathbb{R}_+;\mathbb{R}^d\right)\rightarrow\mathbb{R}_+$ be a mapping which is bounded on compact subsets of $C\left(\mathbb{R}_+;\mathbb{R}^d\right)$. Let X^n,Y^n, $n\in\mathbb{N}^*$, be random variables with values in $C\left(\mathbb{R}_+;\mathbb{R}^d\right)$. If $\{Y^n:n\in\mathbb{N}^*\}$ is tight and for all $n\in\mathbb{N}^*$*

$$\begin{array}{ll}(i) & |X_0^n|\leq G(Y^n),\ a.s.\\ (ii) & \mathbf{m}_{X^n}(\varepsilon;[0,T])\leq G(Y^n)\,g\left(\mathbf{m}_{Y^n}(\varepsilon;[0,T])\right),\ a.s.,\ \ \forall\,\varepsilon,T>0,\end{array}$$

then $\{X^n:n\in\mathbb{N}^*\}$ is tight.

Lemma 12 *Let $B,B^n,\bar{B}^n:\Omega\times[0,\infty[\,\rightarrow\mathbb{R}^k$ and $X,X^n,\bar{X}^n:\Omega\times[0,\infty[\,\rightarrow\mathbb{R}^{d\times k}$, be c.s.p. such that*

- B^n is $\mathcal{F}_t^{B^n,X^n}-$Brownian motion $\forall\,n\geq1$;
- $\mathcal{L}(X^n,B^n)=\mathcal{L}\left(\bar{X}^n,\bar{B}^n\right)$ on $C(\mathbb{R}_+,\mathbb{R}^{d\times k}\times\mathbb{R}^k)$ for all $n\geq1$;

- $\int_0^T \left|\bar{X}_s^n - \bar{X}_s\right|^2 ds + \sup_{t\in[0,T]} \left|\bar{B}_t^n - \bar{B}_t\right|$ in probability, as $n \to \infty$, for all $T > 0$.

Then $\left(\bar{B}^n, \{\mathcal{F}_t^{\bar{B}^n,\bar{X}^n}\}\right), n \geq 1$, *and* $\left(\bar{B}, \{\mathcal{F}_t^{\bar{B},\bar{X}}\}\right)$ *are Brownian motions and as* $n \to \infty$

$$\sup_{t\in[0,T]} \left| \int_0^t \bar{X}_s^n d\bar{B}_s^n \longrightarrow \int_0^t \bar{X}_s d\bar{B}_s \right| \longrightarrow 0 \quad \textit{in probability}.$$

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Thank you for your attention !