Semilinear SPDEs driven by an fBM of Hurst parameter H in (0,1/2)

Shuai Jing (井帅), Brest, France*
Jorge A. León, Mexico D.F., Mexico

ITN Spring School and Workshop, Roscoff,
18th, March, 2010

*S. Jing is supported by ITN project: "Deterministic and Stochastic Controlled Systems and Application", FP7-PEOPLE-2007-1-1-ITN, No. 213841-2.

1 Introduction

Our objective: study of the *semilinear SPDE*

$$du(t,x) = (Lu(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x)))ds + \gamma_t u(t,x)dB_t,$$
$$(t,x) \in [0,T] \times R^d,\ u(0,x) = \Phi(x), x \in R^d,$$

where

1. $L := \frac{1}{2}\text{tr}\left(\sigma\sigma^*(x)D^2\right) + b(x)\nabla$;
2. B is a fractional Brownian motion with Hurst parameter $H \in (0,1/2)$;
3. $\gamma \in L^2(0,T)$ + some additional assumptions;
4. $\int_0^t \gamma_s u(s,x)dB_s := \delta(\gamma u(.,x)I_{[0,t]})$ - Skorohod integral (=extended divergence);
5. study of the solution in viscosity sense.

Main idea of our approach: if B were a Brownian motion, the viscosity solution $u(t,x) = Y_t^{t,x}$ of the SPDE would be given by the *backward doubly SDE*:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_0^s Z_r^{t,x} \downarrow dW_r + \int_0^s \gamma_r Y_r^{t,x} dB_r, \quad s \in [0,t]$$

(W - Brownian motion independent of B), associated with the Forward SDE:

$$dX_s^{t,x} = -\sigma(X_s^{t,x}) \downarrow dW_s - b(X_s^{t,x})ds, \, s \in [0,t], \, X_t^{t,x} = x.$$

The difficulty for us: B is a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$; the integral in the BDSDE is in Skorohod's sense $\Rightarrow$ we can't use the approach of Pardoux, Peng for the study of BDSDEs.

Preparing steps: special case of the above backward doubly SDE: if $\Phi \in R$ and f independent of x it's a semilinear SDE driven by a fractional Brownian motion.

I...
S...
F...
F...
T...

Our approach:

• Study of (anticipating) semilinear SDEs driven by a fractional Brownian motion with $H \in (0, 1/2)$ by associating them with the help of the Girsanov transformation with pathwise ordinary differential equations (Jien and Ma[2009]);

• Application of the same technique to associate our backward doubly SDE with a pathwise BSDE (without integral w.r.t. B);

• Study of the associated SPDE with the same techniques; association:

$$\text{semilinear SPDE} \overset{\text{Gt}}{\Leftrightarrow} \text{pathwise PDE} \Leftrightarrow \text{pathwise BSDE} \overset{\text{Gt}}{\Leftrightarrow} \text{BDSDE}$$

Gt - Girsanov transformation.

2 Some basics on frac. Brownian motion

$(\Omega, \mathcal{F}, P)$ - the canonical Wiener space with time horizon $T > 0$:
+ $\Omega = C_0([0,T];R)$ endowed with the supremum norm;
+ P - the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$;
+ $W^0 = (W_t^0)_{t\in[0,T]}$ - the coordinate process on Ω; Brownian motion under P;
+ $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_P$.

For $H \in (0, 1/2)$, our fBm with Hurst parameter H:

$$B_t = \int_0^t K_H(t,s)dW_s^0, t \in [0,T],$$

with the integral kernel

$$K_H(t,s) = C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

where $C_H = \sqrt{\dfrac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}}$.

As an fBm with Hurst parameter H the process B is Gaussian with

$$E[B_t] = 0;$$

$$R_H(t,s) := E[B_t B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), t,s \in [0,T].$$

$\mathcal{H}_H$ - the Hilbert space defined as completion of the space $L^e(0,T)$ of step functions on $[0,T]$ w.r.t. the norm associated with the scalar product

$$\langle I_{[0,t]}, I_{[0,s]}\rangle_{\mathcal{H}_H} = R_H(t,s) = E[B_t B_s], t,s \in [0,T].$$

- $\mathcal{H}_H = \Lambda_T^{1/2-H} = \{f : [0,T] \to R \,|\, \exists \varphi_f \in L^2(0,T),$

  s.t. $f(u) = u^{1/2-H} I_{T-}^{1/2-H}(s^{H-1/2}\varphi_f(s))(u)\}$, where

$$I_{T-}^{\alpha}(f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} du, x \in [0,T], f \in L^1(0,T),$$

is the right-sided fractional integral of f; $\Lambda_T^{1/2-H}$ is a Hilbert space w.r.t. the scalar product

$$\langle f, g\rangle_{\Lambda_T^{1/2-H}} = C_H \Gamma(H + \frac{1}{2})\langle \varphi_f, \varphi_g\rangle_{L^2(0,T)}.$$

• Extension of $I_{[0,t]} \to B_t$ to an isometry $\Lambda_T^{1/2-H} \ni f \to B(f) \in L^2(\Omega, \mathcal{F}, P)$. By the *transfer principle*:

$$B(f) = \int_0^T (\mathcal{K}f)(s) dW_s^0, \text{ where}$$

$(\mathcal{K}f)(s) = C_H \Gamma(H+\frac{1}{2}) s^{1/2-H} (D_{T-}^{1/2-H} u^{H-1/2} f(u))(s), s \in [0,T]$,

$(D_{T-}^{\alpha} f)(s) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(s)}{(T-s)^{\alpha}} + \alpha \int_s^T \frac{f(s)-f(u)}{(u-s)^{1+\alpha}} du \right)$ (right-sided fractional derivative); in particular: $\mathcal{K} I_{[0,t]} = K_H(t,.) I_{[0,t]}, t \in [0,T]$.

The operator $\mathcal{K} : \Lambda_T^{1/2-H} \to L^2(0,T)$ and its dual operator $\mathcal{K}^*$ are crucial for the definition of the stochastic integral δ (Skorohod integral) w.r.t. B (see: Nualart; Cheridito, Nualart; León, Nualart):

I...
S...
F...
F...
T...

• Smooth random variables: $F \in \mathcal{S}_{\mathcal{K}}$ if
$F = f(B(\varphi_1), \ldots, B(\varphi_n)), \varphi_1, \ldots, \varphi_n \in \text{Dom}(\mathcal{K}^* \mathcal{K})$, $f \in C_p^\infty(R)$;
(Dom($\mathcal{K}^* \mathcal{K}$) is dense in $\Lambda_T^{1/2-H}$, see: León, Nualart;)

$$DF = \sum_{i=1}^{n} \partial f_{x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

• Skorohod integral:
Let $u \in L^2(\Omega \times [0,T])$; $u \in \text{Dom}(\delta)$ if $\exists \delta(u) \in L^2(\Omega)$ s.t.

$$E[\langle \mathcal{K}^* \mathcal{K} DF, u \rangle_{L^2(0,T)}] = E[F\delta(u)], \forall F \in \mathcal{S}_{\mathcal{K}}.$$

In this case: $\delta(u) =:$ Skorohod integral (extended divergence of u);
$\int_0^t u_s dB_s := \delta(uI_{[0,t]})$, for $uI_{[0,t]} \in \text{Dom}(\delta)$.

• Girsanov transformation: $T_t(\omega) = \omega + \int_0^{\cdot} (\mathcal{K}\gamma I_{[0,t]})(r)dr, \omega \in \Omega,$
$A_t(\omega) = \omega - \int_0^{\cdot} (\mathcal{K}\gamma I_{[0,t]})(r)dr$, $T_t^{-1} = A_t$, $t \in [0,T]$.

Assumption (H1): $\gamma \in L^2(0,T)$ s.t. $\gamma I_{[0,t]} \in \Lambda_T^{1/2-H}$, for all $t \in [0,T]$.

We have:

$$+\ B(\varphi)(T_t) = B(\varphi) + \int_0^T (\mathcal{K}\gamma I_{[0,t]})(r)(\mathcal{K}\varphi)(r)dr$$
$$= B(\varphi) + \int_0^t \gamma_r(\mathcal{K}^*\mathcal{K}\varphi)(r)dr, \quad \varphi \in \mathrm{Dom}(\mathcal{K}^*\mathcal{K});$$

$$+\ E[F] = E[F(A_t)\varepsilon_t], \text{ with } \varepsilon_t = \exp\left\{\int_0^t \gamma_r dB_r - \frac{1}{2}\int_0^t ((\mathcal{K}\gamma I_{[0,t]})(r))^2 dr\right\}.$$

Lemma. Suppose that, for some $q > \frac{1}{H}$, $\gamma \in L^q(0,T)$ and

$$\int_0^T \left(\int_x^T \frac{|\gamma(t)-\gamma(x)|}{(t-x)^{3/2-H}}dt\right)^q dx < +\infty.$$

Then, for all $p \geq 1$,

$$E\left[\exp\left\{p \sup_{t\in[0,T]}\left|\int_0^t \gamma_s dB_s\right|\right\}\right] \leq C_{H,p,q,\gamma} < \infty.$$

3 Fract. anticipating semilinear SDEs

Let $b : \Omega \times [0, T] \times R \to R$ be measurable,
+ Lipschitz in $x \in R$, uniformly w.r.t. (ω, t), and
+ $b(.,.,0)$ be bounded.

$$X_t = \xi + \int_0^t b(s, X_s) ds + \int_0^t \gamma_s X_s dB_s, t \in [0, T],$$

where $\xi \in L^p(\Omega)$, for some $p > 2$.

Theorem (Jien and Ma [2009]). Under assumption (H) ($\gamma \in L^2(0, T)$ s.t. $\gamma I_{[0,t]} \in \mathrm{Dom}(\delta)$, for all $t \in [0, T]$):

$$X_t = \varepsilon_t \zeta_t (A_t, \xi(A_t)), t \in [0, T],$$

is the unique solution in $L^2(\Omega \times [0, T])$ with $\gamma X I_{[0,t]} \in \mathrm{Dom}(\delta)$, for all $t \in [0, T]$; $(\zeta_t(x))$ is the unique pathwise solution of the equation

$$\zeta_t(x) = x + \int_0^t \varepsilon_s^{-1}(T_s) b(T_s, s, \varepsilon_s(T_s)\zeta_s(x)) ds, t \in [0, T].$$

I...
S...
F...
F...
T...

4 Fractional backward doubly SDEs

$+\ B$ - fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$ defined on the classical Wiener space $(\Omega', \mathcal{F}', P')$ in the same manner as before; $\Omega' = C_0([0,T]; R)$;
$+\ W = (W^1, \ldots, W^d)$ - coordinate process on the classical Wiener space $(\Omega'', \mathcal{F}'', P'')$, where $\Omega'' = C_0([0,T]; R^d)$; W is a d-dimensional Brownian motion
$+\ (\Omega, \mathcal{F}^0, P) := (\Omega', \mathcal{F}', P') \otimes (\Omega'', \mathcal{F}'', P'')$, $\mathcal{F} = \mathcal{F}^0 \vee \mathcal{N}_P$.

Objective: study of the (fractional) backward doubly SDE (BDSDE)

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds - \int_0^t Z_s \downarrow dW_s + \int_0^t \gamma_s Y_s dB_s,\ t \in [0,T],$$

where:
$+\ \xi \in L^2(\Omega, \mathcal{F}^W_{0,T}, P)$; here: $\mathcal{F}^W_{t,s} = \sigma\{W_r - W_t, r \in [t,s]\} \vee \mathcal{N}_P, t \le s$;
$+\ f : \Omega'' \times [0,T] \times R \times R^d \to R$ is measurable s.t.

- $f(., t, y, z) \in \mathcal{F}^W_{t,T}$, for all (t, y, z),

- $f(.,.,0,0) \in L^2(\Omega \times [0,T])$,
- $f(\omega,t,.,.)$ is Lipschitz, uniformly in (ω,t).

Let $\mathbb{G} = (\mathcal{G}_t = \mathcal{F}^W_{t,T} \vee \mathcal{F}^B_t)_{t\in[0,T]}$, $\mathbb{H} = (\mathcal{H}_t = \mathcal{F}^W_{t,T} \vee \mathcal{F}^B_T)_{t\in[0,T]}$.
Solution $(Y,Z) \in L^2_{\mathbb{G}}(0,T;R\times R^d)$ s.t.

$$Y_t = \xi + \int_0^t f(s,Y_s,Z_s)ds - \underbrace{\int_0^t Z_s \downarrow dW_s}_{} + \underbrace{\int_0^t \gamma_s Y_s dB_s}_{},\ t\in[0,T],$$

Itô backw. integral of $Z \in L^2_{\mathbb{H}}(0,T;R^d)$ Skorohod integral $\delta(\gamma Y I_{[0,t]})$
($=$ a Skorohod integral w.r.t. W)
$\Rightarrow$ use of the Malliavin calculus w.r.t. (B,W).

Some basic elements of Malliavin calculus for our case:

- Smooth functionals over Ω: $F \in \mathcal{S}$ of the form
  $F = f(B(\varphi_1),\dots,B(\varphi_n),W(\psi_1),\dots,W(\psi_m))$,
  $\varphi_i \in \mathrm{Dom}(\mathcal{K}^*\mathcal{K}), \psi_j \in C([0,T];R^d), f \in C_p^\infty([0,T];R^{n+m})$;
- Malliavin derivative w.r.t. B: the $\Lambda_T^{1/2-H}$-valued random variable

$$D^B F = \sum_{i=1}^n \partial_{x_i} f(B(\varphi_1),\dots,B(\varphi_n),W(\psi_1),\dots,W(\psi_m))\varphi_i,$$

• Malliavin derivative w.r.t. W:

$$D^W F = \sum_{i=1}^{m} \partial_{x_{i+n}} f(B(\varphi_1), \ldots, B(\varphi_n), W(\psi_1), \ldots, W(\psi_m))\psi_i.$$

• Skorohod integral w.r.t. B (extension of our definition from Ω' to $\Omega = \Omega' \times \Omega''$) :

$u \in L^2(\Omega \times [0,T])$ belongs to $\mathrm{Dom}(\delta^B)$ if $\exists \delta^B(u) \in L^2(\Omega)$ s.t.

$$E\left[\int_0^T (\mathcal{K}^* \mathcal{K} D^B F)(s) u_s ds\right] = E[F\delta^B(u)], F \in \mathcal{S};$$

$\delta^B(u)$ - the Skorohod integral of u;

• Skorohod integral w.r.t. W: $u \in L^2(\Omega \times [0,T])$ belongs to $\mathrm{Dom}(\delta^W)$ if $\exists \delta^W(u) \in L^2(\Omega)$ s.t.

$$E\left[\int_0^T D_s^W F(s) u_s ds\right] = E[F\delta^W(u)], F \in \mathcal{S};$$

$\delta^W(u)$ - the Skorohod integral of u.

Lemma. Let $u \in L^2_{\mathbb{H}}(0,T;R^d)$. Then $u \in \mathrm{Dom}(\delta^W)$ and the Itô backward integral coincides with the Skorohod integral:

$$\int_0^T u_s \downarrow dW_s = \delta^W(u).$$

• Girsanov transformation on $\Omega = \Omega' \times \Omega''$ w.r.t. to B: canonical extension from Ω' to Ω:

$$T_t(\omega', \omega'') = (T_t\omega', \omega''),\ \omega = (\omega', \omega'') \in \Omega = \Omega' \times \Omega'',$$
$$A_t(\omega', \omega'') = (A_t\omega', \omega'');\ T_t^{-1} = A_t,\ t \in [0,T].$$

• A subspace of $L^2_{\mathbb{G}}(0,T;R \times R^d)$ stemming its importance from the invariance w.r.t. Girsanov transformation:

$$I_T^* = \sup_{t\in[0,T]} \left| \int_0^t \gamma_s dB_s \right|;\ \text{recall: } E[\exp\{pI_T^*\}] < +\infty,\ p \geq 1.$$

$$L^{2,*}_{\mathbb{G}}(0,T;R\times R^d) := \Big\{(Y,Z) \in L^2_{\mathbb{G}}(0,T;R\times R^d):$$
$$E\left[\exp\{pI_T^*\}\int_0^T (|Y_t|^2 + |Z_t|^2)dt\right] < \infty, \text{ for all } p \geq 1\Big\}.$$

For the above space the following invariance property holds:

Lemma. For any $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$:

+ $((Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t))) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$;

+ $((Y_t(A_t)\varepsilon_t, Z_t(A_t)\varepsilon_t)) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$.

Theorem. The backward doubly SDE has a unique solution $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$.

Idea of the proof: Reduction of the BDSDE with the help of the Girsanov transformation to the pathwise BSDE

$$\begin{aligned}\widehat{Y}_t &= \xi + \int_0^t f(s,\widehat{Y}_s\varepsilon_s(T_s),\widehat{Z}_s\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s)ds - \int_0^t \widehat{Z}_s \downarrow dW_s \\ &= \xi + \int_0^t F_s(\widehat{Y}_s,\widehat{Z}_s)ds - \int_0^t \widehat{Z}_s \downarrow dW_s,\ t\in[0,T],\end{aligned}$$

where $F_s(y,z) := f(s,y\varepsilon_s(T_s),z\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s)$.

Proposition: The above BSDE has a solution $(\widehat{Y},\widehat{Z}) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)(\subset L^2_{\mathbb{H}}(0,T;R\times R^d))$ which is unique in $L^2_{\mathbb{H}}(0,T;R\times R^d)$.

Theorem. 1) Let $(\widehat{Y},\widehat{Z}) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$ be a solution of the preceding BSDE. Then

$$(Y_t,Z_t)_{t\in[0,T]} = (\widehat{Y}_t(A_t)\varepsilon_t,\widehat{Z}_t(A_t)\varepsilon_t)_{t\in[0,T]} \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$$

is a solution of the backward doubly SDE

$$Y_t = \xi + \int_0^t f(s,Y_s,Z_s)ds - \int_0^t Z_s \downarrow dW_s + \int_0^t \gamma_s Y_s dB_s,\ t\in[0,T];$$

in particular, $\gamma Y I_{[0,t]} \in \mathrm{Dom}(\delta^B)$, for all $t\in[0,T]$.

2) Conversely, if $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$ is a solution of the backward doubly SDE then

$$(\widehat{Y}_t,\widehat{Z}_t)_{t\in[0,T]} = (Y_t(T_t)\varepsilon_t^{-1}(T_t),Z_t(T_t)\varepsilon_t^{-1}(T_t))_{t\in[0,T]} \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$$

is a solution of the BSDE

$$\widehat{Y}_t = \xi + \int_0^t f(s,\widehat{Y}_s\varepsilon_s(T_s),\widehat{Z}_s\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s)ds - \int_0^t \widehat{Z}_s \downarrow dW_s,\ t\in[0,T].$$

Idea of the proof (for 1)): Let $(\widehat{Y},\widehat{Z}) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$ be a solution of the preceding BSDE; (Y,Z) is defined in 1); from a lemma we know that $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;R\times R^d)$.

Recall: $(Y_t,Z_t) = (\widehat{Y}_t(A_t)\varepsilon_t, \widehat{Z}_t(A_t)\varepsilon_t), t \in [0,T]$.
For $F \in \mathcal{S}$:

$$E[F(Y_t-\xi)] = E[FY_t] - E[FY_0] = E[F(T_t)\widehat{Y}_t] - E[F\widehat{Y}_0]$$
$$= E\left[F(T_t)\left(\widehat{Y}_0 + \int_0^t F_s(\widehat{Y}_s,\widehat{Z}_s)ds - \int_0^t \widehat{Z}_s \downarrow dW_s\right) - F\widehat{Y}_0\right];$$

Recall:

$$+\ \frac{d}{dt}F(T_t) = \gamma_t(\mathcal{K}^*\mathcal{K}D^BF)(t,T_t);$$
$$+\ E\left[F(T_t)\int_0^t \widehat{Z}_s \downarrow dW_s\right] = E\left[\int_0^t D^W_s F(T_t)\widehat{Z}_s ds\right].$$

I... S... F... F... T...

Thus,

$$E\left[F(T_t)\int_0^t \widehat{Z}_s \downarrow dW_s\right] = E\left[\int_0^t D_s^W F(T_t)\widehat{Z}_s ds\right]$$

$$= E\left[\int_0^t D_s^W F(T_s)\widehat{Z}_s ds\right] + E\left[\int_0^t \int_s^t D_s^W \left\{\gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r,T_r)\right\} dr\widehat{Z}_s ds\right];$$

but, from the Fubini theorem,

$$E\left[\int_0^t \int_s^t D_s^W \left\{\gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r,T_r)\right\} dr\widehat{Z}_s ds\right]$$

$$= \int_0^t \gamma_r \underbrace{E\left[\int_0^r D_s^W \left((\mathcal{K}^*\mathcal{K}D^B F)(r,T_r)\right)\widehat{Z}_s ds\right]} dr$$

$$= \int_0^t \gamma_r E\left[(\mathcal{K}^*\mathcal{K}D^B F)(r,T_r)\int_0^r \widehat{Z}_s \downarrow dW_s\right] dr.$$

Consequently, using $\frac{d}{dt}F(T_t) = \gamma_t(\mathcal{K}^*\mathcal{K}D^B F)(t,T_t)$ also for the term $E\left[F(T_t)\left(\widehat{Y}_0 + \int_0^t F_s(\widehat{Y}_s,\widehat{Z}_s)ds\right)\right]$, we get:

$$E[F(Y_t-\xi)] = E\left[\int_0^t F(T_s)F_s(\widehat{Y}_s,\widehat{Z}_s)ds - \int_0^t D_s^W F(T_s)\widehat{Z}_s ds\right]$$

$$+E\left[\int_0^t \gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r,T_r)\underbrace{\left(\widehat{Y}_0 + \int_0^t F_s(\widehat{Y}_s,\widehat{Z}_s)ds - \int_0^r \widehat{Z}_s \downarrow dW_s\right)}_{=\widehat{Y}_r} dr\right]$$

Consequently, from the Girsanov transformation and the definition of $F_s(y,z)$ and of (Y,Z):

$$E[F(Y_t-\xi)] =$$

$$E\left[\int_0^t F f(s,Y_s,Z_s)ds\right] - \underbrace{E\left[\int_0^t D_s^W F Z_s ds\right]} + E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)Y_s ds\right]$$

$$\left(\text{since } Z\in L^2_{\mathbb{H}}(0,T;R^d) := E\left[F\int_0^t Z_s \downarrow dW_s\right]\right)$$

$$= E\left[F\left(\int_0^t f(s,Y_s,Z_s)ds - \int_0^t Z_s \downarrow dW_s\right)\right] + E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)Y_s ds\right],$$

i.e.,

$$E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)Y_s ds\right]$$

$$= E\left[F\underbrace{\left(Y_t - \xi - \int_0^t f(s,Y_s,Z_s)ds + \int_0^t Z_s \downarrow dW_s\right)}_{\in L^2(\Omega),}\right], F \in \mathcal{S},$$

from where: $\gamma Y I_{[0,t]} \in \mathrm{Dom}(\delta^B)$ and

$$\int_0^t \gamma_s Y_s dB_s = Y_t - \xi - \int_0^t f(s,Y_s,Z_s)ds + \int_0^t Z_s \downarrow dW_s, t \in [0,T].$$

5 The associated SPDE

SPDE

$$du(t,x) = (Lu(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x)))ds + \gamma_t u(t,x)dB_t,$$
$$(t,x) \in [0,T] \times R^d,\ u(0,x) = \Phi(x),\ x \in R^d,$$

where $L := \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*(x)D^2\right) + b(x)\nabla$.

Our objective: study of the existence and uniqueness of a solution in the viscosity sense for the above SPDE.

Key in our approach: the before studied backward doubly SDE which is a Feynman-Kac type formula for the solution $u(t,x)$.

For this end, study of the BDSDE in a "Markovian" framework:

$\sigma : R^d \to R^{d\times d}, b : R^d \to R^d$ Lipschitz;

Forward SDE: Given $(t,x) \in [0,T] \times R^d$, we consider

$$dX_s^{t,x} = -\sigma(X_s^{t,x}) \downarrow dW_s - b(X_s^{t,x})ds,\ s \in [0,t],\ X_t^{t,x} = x;$$

which has a unique solution $X^{t,x} \in \mathcal{S}^2_{(\mathcal{F}^W_{s,t})_{s\in[0,t]}}$.

With the forward SDE we associate a BDSDE with driving coefficient $f:[0,T]\times R^d\times R\times R^d\to R$ continuous, and $f(t,.,.,.)$ Lipschitz, uniformly w.r.t. t; $\Phi:R^d\to R$ Lipschitz;

Backward doubly SDE:

$$Y_s^{t,x}=\Phi(X_0^{t,x})+\int_0^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr-\int_0^s Z_r^{t,x}\downarrow dW_r+\int_0^s \gamma_r Y_r^{t,x}dB_r,\quad s\in[0,t];$$

existence of a unique solution $(Y^{t,x},Z^{t,x})$ given by

$$(Y_s^{t,x},Z_s^{t,x})=(\widehat{Y}_s^{t,x}(A_s)\varepsilon_s,\widehat{Z}_s^{t,x}(A_s)\varepsilon_s),\ s\in[0,t],$$

where, for $s\in[0,t]$,

$$\widehat{Y}_s^{t,x}=\Phi(X_0^{t,x})+\int_0^s f\Big(r,X_r^{t,x},\widehat{Y}_r^{t,x}\varepsilon_r(T_r),\widehat{Z}_r^{t,x}\varepsilon_r(T_r)\Big)\,\varepsilon_r^{-1}(T_r)dr-\int_0^s \widehat{Z}_r^{t,x}\downarrow dW_r.$$

Pardoux, Peng (1990, 1992): study of BSDEs in a Markovian context in which the driver $F_r(x,y,z)$ is deterministic; here, in our framework:

$$F_r(x,y,z):=f\left(r,x,y\varepsilon_r(T_r),z\varepsilon_r(T_r)\right)\varepsilon_r^{-1}(T_r),\quad (r,x,y,z)\in[0,T]\times R^d\times R\times R^d.$$

- Standard estimates for the solution

For all $q \geq 1$ there is some $C_q \in R_+$ s.t., for all $(t,x),(t',x') \in [0,T] \times R^d$, P-a.s.,

$$+ E\left[\sup_{s\in[0,t]} |\widehat{Y}_s^{t,x}|^q + \left(\int_0^t |\widehat{Z}_s^{t,x}|^2 ds\right)^{q/2} \Big| \mathcal{F}_T^B\right] \leq C_q(1+|x|^q)\exp\{qI_T^*\};$$

$$+ E\left[\sup_{s\in[0,t\wedge t']} |\widehat{Y}_s^{t,x} - \widehat{Y}_s^{t',x'}|^q + \left(\int_0^{t\wedge t'} |\widehat{Z}_s^{t,x} - \widehat{Z}_s^{t',x'}|^2 ds\right)^{q/2} \Big| \mathcal{F}_T^B\right]$$
$$\leq C_q(1+|x|^q) + |x'|^q)\exp\{qI_T^*\}(|t-t'|^{q/2} + |x-x'|^q).$$

- We introduce the stochastic field:

$$\widehat{u}(t,x) := \widehat{Y}_t^{t,x} \in \mathcal{F}_t^B,\ (t,x) \in [0,T] \times R^d;$$

Lemma: $\widehat{u}$ possesses a continuous version. Moreover: $|\widehat{u}(t,x)| \leq C\exp\{I_T^*\}(1+|x|)$.

Our objective: characterization of $\widehat{u}$ as viscosity solution of the PDE with stochastic coefficient F
(recall $F_r(x,y,z) := f(r,x,y\varepsilon_r(T_r), z\varepsilon_r(T_r))\,\varepsilon_r^{-1}(T_r)$),

$$\begin{cases} \dfrac{\partial}{\partial t}\widehat{u}(t,x) = L\widehat{u}(t,x) + F_t(x,\widehat{u}(t,x),\nabla\widehat{u}(t,x)\sigma(x)), & (t,x)\in[0,T]\times R^d, \\ \widehat{u}(0,x) = \Phi(x), & x\in R^d. \end{cases} \tag{1}$$

Definition: A real valued continuous random field $\widehat{u} : \Omega' \times [0,T] \times R^d \mapsto R$ is called a pathwise viscosity solution of equation (1) if there exists a subset $\overline{\Omega'}$ of Ω' with $P'(\overline{\Omega'}) = 1$, such that for all $\omega' \in \overline{\Omega'}$, $\widehat{u}(\omega', \cdot, \cdot)$ is a viscosity solution for the PDE.

Proposition: $\widehat{u} = \widehat{Y}_t^{t,x}$ is a pathwise viscosity solution of the PDE

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = Lu(t,x) + F_t(x, u(t,x), \nabla u(t,x)\sigma(x)), & (t,x) \in [0,T] \times R^d, \\ u(0,x) = \Phi(x), & x \in R^d; \end{cases}$$

this solution is unique in the class of continuous stochastic fields $\widetilde{u} : \Omega' \times [0,T] \times R^d \to R$ such that, for some r.v. $\eta \in L^0(\mathcal{F}_T^B)$ and some $q \geq 1$, P-a.s.,

$$|\widetilde{u}(t,x)| \leq \eta(1 + |x|^q), \text{ for all } (t,x) \in [0,T] \times R^d.$$

Idea of the proof: + Existence: adaption of the version of the proof given in El Karoui, Peng, Quenez (1997) to our framework, in the set $\widehat{\Omega}' := \{\omega' \in \Omega' | I_T^*(\omega') < +\infty$ and $\widehat{u}(\omega', \cdot, \cdot)$ is continuous$\}$ which satisfies $P'(\widehat{\Omega}') = 1$.

+ Uniqueness: because we work on pathwise viscosity solution, we could adapt the proof given by Pardoux (1998) to our framework pathwisely.

We come now back to our SPDE:

$$du(t,x) = (Lu(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x)))dt + \gamma_t u(t,x)dB_t,$$
$$t \in [0,T]\times R^d,\ u(0,x) = \Phi(x), x \in R^d,$$

where $L := \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*(x)D^2\right) + b(x)\nabla$.

In analogy to the relation between the solutions (Y,Z) and $(\widehat{Y},\widehat{Z})$ of the associated BDSDE and the BSDE, resp., we shall expect that

$$u(t,x) := \widehat{u}(A_t,t,x)\varepsilon_t,\ (t,x)\in[0,T]\times R^d,$$

is a solution of the above SPDE.

This conjecture is confirmed by

Proposition: Suppose that $u,\widehat{u}$ are $C^{0,2}$-stochastic fields over $\Omega'\times[0,T]\times R^d$ s.t. there are some $\delta>0$ and some $C_{\delta,x}$ (only depending on δ,x) with:

$$E\left[|v(t,x)|^{2+\delta} + \int_0^t \left(|\nabla v(s,x)|^{2+\delta} + |D^2 v(s,x)|^{2+\delta}\right)ds\right] \le C_{\delta,x},$$
$$t\in[0,T],\ v = u,\widehat{u}.$$

Then u is a classical solution of the above SPDE iff $\widehat{u}$ is a classical solution of the PDE with stochastic coefficient.

Idea of the proof: the proof is based of an argument similar to that used for the proof of the relation between BDSDE and pathwise BSDE, with the help of Girsanov transformation.

Remark: Difficulty to get the regularity of $\widehat{u}$ under not too restrictive assumptions (like coefficients of class $C^3_{\ell,b}$, linearity of f in z).

However, we have:

Lemma: The random field $u(t,x) := \widehat{u}(A_t,t,x)\varepsilon_t$, $(t,x) \in [0,T] \times R^d$ has a continuous version.

The above proposition motivates the following definition:
Definition: A continuous random field $u : \Omega' \times [0,T] \times R^d \to R$ is a (stochastic) viscosity solution of our SPDE iff

$$\widehat{u}(t,x) = u(T_t,t,x)\varepsilon_t^{-1}(T_t),\ (t,x) \in [0,T] \times R^d$$

is a (pathwise) viscosity solution of our PDE with stochastic coefficient.

From our preceding discussion:

Theorem. The continuous stochastic field

$$u(t,x) := \widehat{u}(t,x)(A_t)\varepsilon_t = \widehat{Y}_t^{t,x}(A_t)\varepsilon_t = Y_t^{t,x},\ (t,x) \in [0,T] \times R^d,$$

+ is a viscosity solution of our semilinear SPDE;
+ the unique viscosity solution inside the class $\mathcal{C}_p^B$ of continuous stochastic fields $\widetilde{u} : \Omega' \times [0,T] \times R^d \to R$ s.t.

$\exists C \in R_+$, s.t., for all $(t,x) \in [0,T] \times R^d$,

$$|\widetilde{u}(t,x)| \le C \exp\{I_T^*\}(1+|x|).$$

Remark: 1) $\widetilde{u} \in \mathcal{C}_p^B \Leftrightarrow (\widetilde{u}(A_t,t,x)\varepsilon_t) \in \mathcal{C}_p^B \Leftrightarrow (\widetilde{u}(T_t,t,x)\varepsilon_t^{-1}(T_t)) \in \mathcal{C}_p^B$.

2) BDSDE with solution $Y^{t,x}$ - Feynman-Kac type formula for $u(t,x)$: $u(t,x) = Y_t^{t,x}$.

Merci de votre attention!

Thank you for your attention!

谢谢大家!