Solvability of G-SDE with Integral-Lipschitz Coefficients

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1. Statement of the Problem

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng introduced G-Brownian motion. The expectation $\mathbb{E}[\cdot]$ associated with G-Brownian motion is a sublinear expectation which is called G-expectation. In the papers, Peng [2007] & Gao [2009], the stochastic calculus with respect to the G-Brownian motion has been established .

In this work, we study the solvability of the following stochastic differential equation driven by G-Brownian motion:

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t h(s, X(s))d\langle B, B\rangle_s + \int_0^t \sigma(s, X(s))dB_s, \quad (1)$$

where $t \in [0, T]$, the initial condition $x \in \mathbb{R}^n$ is given and $(\langle B, B\rangle_t)_{t\geq 0}$ is the quadratic variation process of G-Brownian motion $(B_t)_{t\geq 0}$.

The solvability of (1) has been studied in Peng [2007] & Gao [2009], where the coefficients b, h and σ are subject to a Lipschitz condition, and the existence and uniqueness of the solution to (1) has been obtained by the contraction mapping principle and Picard iteration respectively.

In this work, we establish the existence and uniqueness of the solution to (1) under the following so-called integral-Lipschitz condition:

$$|b(t,x_1)-b(t,x_2)|^2+|h(t,x_1)-h(t,x_2)|^2+|\sigma(t,x_1)-\sigma(t,x_2)|^2\leq\rho(|x_1-x_2|^2), \quad (2)$$

where $\rho:(0,+\infty)\rightarrow(0,+\infty)$ is a continuous, increasing, concave function satisfying

$$\rho(0+)=0,\ \int_0^1\frac{dr}{\rho(r)}=+\infty.$$

A typical example of (2) is:

$$|b(t,x_1)-b(t,x_2)|+|h(t,x_1)-h(t,x_2)|+|\sigma(t,x_1)-\sigma(t,x_2)|\leq|x_1-x_2|(\ln\tfrac{1}{|x_1-x_2|})^{\frac{1}{2}}.$$

Furthermore, we also establish the existence and uniqueness of the solution to equation (1) under a "weaker" condition on b and h, i.e.,

$$|b(t,x_1)-b(t,x_2)|\leq\rho(|x_1-x_2|);\ |h(t,x_1)-h(t,x_2)|\leq\rho(|x_1-x_2|). \quad (3)$$

A typical example of (3) is:

$$|b(t,x_1)-b(t,x_2)|\leq|x_1-x_2|\ln\frac{1}{|x_1-x_2|};\ |h(t,x_1)-h(t,x_2)|\leq|x_1-x_2|\ln\frac{1}{|x_1-x_2|}.$$

Reference:

• Watanabe-Yamada [1971] & Yamada [1981] studied the solvability of (1) under the condition (2) for classical finite dimensional case and the uniqueness of solutions to (1) under the condition (3) for classical case.

• Hu-Lerner [2002] studied the solvability of (1) under the condition (2) for classical infinite dimensional case and the existence of solutions to (1) under the condition (3) for classical case.

In this work, we prove the existence and uniqueness of the solution to (1) still hold under either condition (2) or condition (3)

At the end of the work, we also consider the following type of G-BSDE:

$$Y_t = \mathbb{E}[\xi + \int_t^T f(s, Y_s)ds + \int_t^T g(s, Y_s)d\langle B, B\rangle_s | \mathscr{F}_t], \tag{4}$$

where $t \in [0, T]$ and $\xi \in L_G^1(\mathscr{F}_T; \mathbb{R}^n)$. While the solvability for (4) under a Lipschitz condition on the coefficients f and g has been given by Peng [2007], we prove that under the integral-Lipschitz condition:

$$|g(s, y_1) - g(s, y_2)| + |f(s, y_1) - f(s, y_2)| \le \rho(|y_1 - y_2|),$$

the existence and uniqueness of the solution to (4) are obtained as well.

Adapting approach in Peng [2007], we have the following basic notions:

Ω: a given nonempty fundamental space.

$\mathscr{H}$: a space of random variables which is a linear space of real functions defined on Ω such that :

• $1 \in \mathscr{H}$;
• $\mathscr{H}$ is stable with respect to local Lipschitz functions, i.e., for all $n \geq 1$, and for all $X_1, \ldots, X_n \in \mathscr{H}$, $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $\varphi(X_1, \ldots, X_n) \in \mathscr{H}$.

$\mathbb{E}$: a sublinear expectation which is a functional $\mathbb{E} : \mathscr{H} \to \mathbb{R}$ with the following properties : for all $X, Y \in \mathscr{H}$, we have

• **Monotonicity:** if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
• **Preservation of constants:** $\mathbb{E}[c] = c$, for all $c \in \mathbb{R}$;
• **Sub-additivity**: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$;
• **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0$.

Definition (Independent)

For arbitrary $n, m \geq 1$, *a random vector* $Y \in \mathscr{H}^n$ *is said to be independent of* $X \in \mathscr{H}^m$ *under* $\mathbb{E}[\cdot]$ *if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}^{n+m})$ *we have*

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}].$$

Definition (Identically Distributed)

Given two sublinear expectation spaces $(\Omega, \mathscr{H}, \mathbb{E})$ *and* $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$, *two random vectors* $X \in \mathscr{H}^n$ *and* $Y \in \widetilde{\mathscr{H}}^n$ *are said to be identically distributed if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}^n)$

$$\mathbb{F}_X[\varphi] = \widetilde{\mathbb{F}}_Y[\varphi],$$

where $\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)]$ *and* $\mathbb{F}_Y[\varphi] := \mathbb{E}[\varphi(Y)]$ *respectively.*

Definition (G-Normal Distributed)

A d-dimensional random vector X in a sublinear expectation space $(\Omega, \mathscr{H}, \mathbb{E})$ is called G-normal distributed if for each $\varphi \in C_{l,Lip}(\mathbb{R}^d)$,

$$u(t,x) := \mathbb{E}[\varphi(x+\sqrt{t}X)], \ t \geq 0, x \in \mathbb{R}^d$$

is the viscosity solution of the following PDE defined on $[0,\infty) \times \mathbb{R}^d$:

$$\frac{\partial u}{\partial t} - G(D^2 u) = 0, \ u|_{t=0} = \varphi,$$

where $G = G_X(A) : \mathbb{S}^d \to \mathbb{R}$ is defined by

$$G_X(A) := \frac{1}{2}\mathbb{E}[\langle AX, X\rangle], A \in \mathbb{S}^d,$$

and $D^2 u = (\partial^2_{x_i x_j} u)^d_{i,j=1}$.

Remark

In particular, $\mathbb{E}[\varphi(X)] = u(1,0)$, and by Peng [2007] it is easy to check that, for a G-normal distributed random vector X, there exists a bounded, convex and closed subset Γ of $\mathbb{R}^d$, which is the space of all $d \times d$ matrices, such that for each $A \in \mathbb{S}^d$, $G(A) = G_X(A)$ can be represented as

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} tr[\gamma\gamma^T A].$$

Consequencely, we can denote the G-normal distribution by $N(0, \Sigma)$, where $\Sigma := \{\gamma\gamma^T, \gamma \in \Gamma\}$.

In order to establish the stochastic calculus in the framework of G-expectation, we firstly give a more explicit definition of the triple $(\Omega, \mathscr{H}, \mathbb{E})$:

Sample space:

$$\Omega := \{\omega : [0,\infty) \to \mathbb{R}^d | (\omega_t)_{t\geq 0} \textit{ is continuous and } \omega_0 = 0\}.$$

Distance on Ω:

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

Ramdom variable space:

$$L_{ip}^0(\mathscr{F}_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0,T], \varphi \in \mathscr{C}_{l,Lip}(\mathbb{R}^{d\times n})\},$$

and

$$L_{ip}^0(\mathscr{F}) := \bigcup_{n=1}^{\infty} L_{ip}^0(\mathscr{F}_n),$$

where $B_t(\omega) = \omega_t, t \in [0, +\infty)$ is the canonical process.

Definition (G-Expectation)

Sublinear expectation $\mathbb{E} : L^0_{ip}(\mathscr{F}) \to \mathbb{R}$ *on* $(\Omega, L^0_{ip}(\mathscr{F}))$ *is a G-expectation if and only if* $(B_t)_{t\geq 0}$ *is a G-Brownian motion under* $\mathbb{E}$*, that is,*

• $B_0(\omega) = 0$;
• $B_{t+s} - B_t$ *is* $N(0, s\Sigma)$*-distributed and independent of* $(B_{t_1}, \ldots, B_{t_n})$*, for each* $n \in \mathbb{N}$*,* $0 \leq t_1 \leq \ldots \leq t_n \leq t$*, and* $t, s \geq 0$.

Remark

We denote by $L^p_G(\mathscr{F}_T)$ *(resp.* $L^p_G(\mathscr{F})$*) the topological completion of* $L^0_{ip}(\mathscr{F}_T)$ *(resp.* $L^0_{ip}(\mathscr{F})$*) under the Banach norm* $\mathbb{E}[|\cdot|^p]^{\frac{1}{p}}$*,* $1 \leq p \leq \infty$.

Definition

The related conditional expectation of $X \in L_{ip}^0(\mathscr{F}_T)$ *under* $L_{ip}^0(\mathscr{F}_{t_j})$*:*

$$\begin{aligned}\mathbb{E}[X|\mathscr{F}_{t_j}] &= \mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})|\mathscr{F}_{t_j}] \\ &= \mathbb{E}[\psi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}})],\end{aligned}$$

where $\psi(x_1, \ldots, x_j) = \bar{\mathbb{E}}[\varphi(x_1, \ldots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)]$*, and* $0 \le t_1 \le \ldots \le t_n \le T$*,* $(\xi_1, \ldots, \xi_n)$ *are a sequence of G-normal distributed random vectors.*

Remark

Since, for X*,* $Y \in L_{ip}^0(\mathscr{F}_{t_j})$*,*

$$\mathbb{E}[|\mathbb{E}[X|\mathscr{F}_{t_j}] - \mathbb{E}[Y|\mathscr{F}_{t_j}]|] \le \mathbb{E}[|X - Y|],$$

the mapping $\mathbb{E}[\cdot|\mathscr{F}_{t_j}] : L_{ip}^0(\mathscr{F}_T) \to L_{ip}^0(\mathscr{F}_{t_j})$ *can be continuously extended to* $L_G^1(\mathscr{F}_T) \to L_G^1(\mathscr{F}_{t_j})$*.*

From the above definition, we know each G-expectation is determined by the parameter G, which coincides with Γ, so we have the following representation theorem according to Denis-Hu-Peng [2008].

Let $\mathscr{A}_{0,\infty}^{\Gamma}$ be the collection of all $\Gamma-$valued natural filtration $\{\mathscr{F}_t,\ t \geq 0\}$ adapted processes on the interval $[0,\infty)$. For each fixed $\theta \in \mathscr{A}_{0,\infty}^{\Gamma}$, set P_θ be the law of the process $(\int_0^t \theta_s dB_s)_{t\geq 0}$ under the Wiener measure P, and $\mathscr{P} := \{P_\theta : \theta \in \mathscr{A}_{0,\infty}^{\Gamma}\}$.

Definition (Quasi-Surely)

For each $A \in \mathscr{B}(\Omega)$, we define

$$\bar{C}(A) := \sup_{P \in \mathscr{P}} P(A),$$

then a set A is called polar if $\bar{C}(A) = 0$. Moreover, a property is said to hold "quasi-surely" (q.s.) if it holds outside a polar set.

Theorem (Denis-Hu-Peng [2008])

For each $X \in L^0(\Omega)$, *we define*

$$\bar{\mathbb{E}}[X] := \sup_{P \in \mathscr{P}} E_P(X),$$

then we have

$$L_G^1(\mathscr{F}) = \{X \in\in L^0(\Omega) : X \text{ is q.s.continuous and } \lim_{n \to +\infty} \bar{\mathbb{E}}[|X| I_{\{|X|>n\}}] = 0\}.$$

Furthermore for all $X \in L_G^1(\mathscr{F})$, $\mathbb{E}[X] = \bar{\mathbb{E}}[X]$.

From Denis-Hu-Peng [2008] and Gao [2009], we also have the following monotone convergence theorem:

Theorem

$$X_n \in L_G^1(\mathscr{F}),\ X_n \downarrow X,\ q.s. \Rightarrow \mathbb{E}[X_n] \downarrow \bar{\mathbb{E}}[X].$$

$$X_n \in \mathscr{B}(\Omega),\ X_n \uparrow X,\ q.s.,\ E_P(X_1) > -\infty \text{ for all } P \in \mathscr{P} \Rightarrow \bar{\mathbb{E}}[X_n] \uparrow \bar{\mathbb{E}}[X]. \quad (5)$$

In Gao [2009], a generalized Itô integral and a generalized Itô formula with respect to G-Brownian motion are established:

A partion of $[0,T]$:

$$\pi_T^N = \{t_0, t_1, \dots, t_N\}, 0 = t_0 < t_1 < \dots < t_N = T.$$

A collection of simple processes $M_G^{p,0}(0,T;\mathbb{R})$, $p \geq 1$:

$$M_G^{p,0}(0,T;\mathbb{R}) := \{\eta_t = \sum_{j=0}^{N-1} \xi_j I_{[t_j,t_{j+1})}(t); \xi_j \in L_G^p(\mathscr{F}_{t_j}), j = 0, 1, , ..., N-1\}.$$

Norm on $M_G^{p,0}(0,T;\mathbb{R})$:

$$\left(\int_0^T \mathbb{E}[|\eta_t|^p]dt\right)^{\frac{1}{p}} = \left(\sum_{j=0}^{N-1} \mathbb{E}[|\xi_j|^p](t_{j+1} - t_j)\right)^{\frac{1}{p}}.$$

We denote by $M_G^p(0,T;\mathbb{R})$ the topological completion of $M_G^{p,0}(0,T;\mathbb{R})$ under the Banach norm $(\int_0^T \mathbb{E}[|\eta_t|^p]dt)^{\frac{1}{p}}$.

Let $\mathbf{a} = (a_1, \ldots, a_d)^T$ be a given vector in $\mathbb{R}^d$, we set $(B_t^{\mathbf{a}})_{t\geq 0} = (\mathbf{a}, B_t)_{t\geq 0}$, where $(\mathbf{a}, B_t)$ denotes the scalar product of $\mathbf{a}$ and B_t.

Definition

For each $\eta \in M_G^{2,0}([0,T])$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t),$$

we define

$$\mathscr{I}(\eta) = \int_0^T \eta_s dB_s^{\mathbf{a}} := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}}),$$

the mapping can be continuously extended to $\mathscr{I} : M_G^2([0,T]) \to L_G^2(\mathscr{F}_T)$. Then, for each $\eta \in M_G^2([0,T])$, the stochastic integral is defined by

$$\int_0^T \eta_s dB_s^{\mathbf{a}} := \mathscr{I}(\eta).$$

Denote by $(\langle B^{\mathbf{a}}\rangle_t)_{t\geq 0}$ the quadratic variation process of process $(B_t^{\mathbf{a}})_{t\geq 0}$, we know from Peng [2007] that $(\langle B^{\mathbf{a}}\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B^{\mathbf{a}}\rangle_0 = 0$, and for each fixed $s \geq 0$,

$$\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s = \langle (B^s)^{\mathbf{a}}\rangle_t,$$

where $B_t^s = B_{t+s} - B_s, t \geq 0, (B^s)_t^{\mathbf{a}} = (\mathbf{a}, B_t^s)$.

The mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t := \frac{1}{4}(\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}}\rangle_t).$$

Definition

Define the mapping $M_G^{1,0}([0,T]) \to L_G^1(\mathscr{F}_T)$ *as follows:*

$$\mathscr{Q}(\eta) = \int_0^T \eta(s) d\langle B^{\mathbf{a}}\rangle_s := \sum_{k=0}^{N-1} \xi_k(\langle B^{\mathbf{a}}\rangle_{t_{k+1}} - \langle B^{\mathbf{a}}\rangle_{t_k}).$$

Then $\mathscr{Q}$ *can be uniquely extended to* $M_G^1([0,T])$. *We still use* $\mathscr{Q}(\eta)$ *to denote the mapping* $\int_0^T \eta(s) d\langle B^{\mathbf{a}}\rangle_s$, $\eta \in M_G^1([0,T])$.

We introduce two important inequalities for G-stochastic integrals which we will need in the sequel.

Theorem (Gao [2009], BDG Inequality)

Let $p \geq 2$ and $\eta = \{\eta_s, s \in [0,T]\} \in M_G^p([0,T])$. For $\mathbf{a} \in \mathbb{R}^d$, set $X_t = \int_0^t \eta_s dB_s^{\mathbf{a}}$. Then there exists a continuous modification $\tilde{X}$ of X, i.e., on some $\tilde{\Omega} \subset \Omega$, with $\bar{C}(\tilde{\Omega}^c) = 0$, $\tilde{X}_\cdot(\omega) \in C_0[0,T]$ and $\bar{C}(|X_t - \tilde{X}_t| \neq 0) = 0$ for all $t \in [0,T]$, such that

$$\bar{\mathbb{E}}[\sup_{s \leq u \leq t} |\tilde{X}_u - \tilde{X}_s|^p] \leq C_p \sigma_{\mathbf{a}\mathbf{a}^T}^{p/2} \mathbb{E}[\left(\int_s^t |\eta_u|^2 du \right)^{p/2}],$$

where $0 < C_p < \infty$ is a positive constant independent of $\mathbf{a}$, η and Γ.

Theorem (Gao [2009])

Let $p \geq 1$ *and* $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$. *Let* $\eta \in M_G^p([0,T])$. *Then there exists a continuous modification* $\tilde{X}_t^{\mathbf{a},\bar{\mathbf{a}}}$ *of* $X_t^{\mathbf{a},\bar{\mathbf{a}}} := \int_0^t \eta_s d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_s$ *such that for any* $0 \leq s \leq t \leq T$,

$$\mathbb{E}[\sup_{u\in[s,t]} |\tilde{X}_u^{\mathbf{a},\bar{\mathbf{a}}} - \tilde{X}_s^{\mathbf{a},\bar{\mathbf{a}}}|^p]$$
$$\leq \left(\frac{1}{4}\sigma_{(\mathbf{a}+\bar{\mathbf{a}})(\mathbf{a}+\bar{\mathbf{a}})^T} + \frac{1}{4}\sigma_{(\mathbf{a}-\bar{\mathbf{a}})(\mathbf{a}-\bar{\mathbf{a}})^T}\right)^p (t-s)^{p-1}\mathbb{E}[\int_s^t |\eta_u|^p du].$$

Remark

By the above two Theorems, we can assume that the stochastic integrals $\int_0^t \eta_s dB_s^{\mathbf{a}}$, $\int_0^t \eta_s d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_s$ *and* $\int_0^t \eta_s ds$ *are continuous in* t *for all* $\omega \in \Omega$.

Theorem (G-Itô's formula)

Let α^ν, $\eta^{\nu ij}$ and $\beta^{\nu j} \in M_G^2([0,T])$, $\nu = 1,\dots,n$, $i,j = 1,\dots,d$ be bounded processes and consider

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \sum_{i,j=1}^d \int_0^t \eta_s^{\nu ij} d\langle B^i, B^j\rangle_s + \sum_{j=1}^d \int_0^t \beta_s^{\nu j} dB_s^j,$$

where $X_0^\nu \in \mathbb{R}$, $\nu = 1,\dots,n$. Let $\Phi \in C^2(\mathbb{R}^n)$ be a real function with bounded derivatives such that $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ are uniformly Lipschitz. Then for each $s,t \in [0,T]$, in $L_G^2(\mathscr{F}_t)$

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = & \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du + \int_s^t \partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} d\langle B^i, B^j\rangle_u \\ & + \frac{1}{2}\int_s^t \partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j} d\langle B^i, B^j\rangle_u \\ & + \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} dB_u^j,\end{aligned}$$

where the repeated indices ν, μ, i and j imply the summation.

Consider the following stochastic differential equation (1) driven by a d-dimensional G-Brownian motion, and we rewrite it in an equivalent form:

$$X_t = x + \int_0^t b(s, X_s)ds + \sum_{i,j=1}^{d} \int_0^t h_{ij}(s, X_s)d\langle B^i, B^j\rangle_s + \sum_{j=1}^{d} \int_0^t \sigma_j(s, X_s)dB^j_s, \qquad (6)$$

- $b(\cdot, x),\ h_{ij}(\cdot, x),\ \sigma_j(\cdot, x) \in M_G^2([0,T];\mathbb{R}^n)$;
- $|b(t,x)|^2 + \sum_{i,j=1}^d |h_{ij}(t,x)|^2 + \sum_{j=1}^d |\sigma_j(t,x)|^2 \le \beta_1^2(t) + \beta_2^2(t)|x|^2$; $\qquad (H1)$
- $|b(t,x_1) - b(t,x_2)|^2 + \sum_{i,j=1}^d |h_{ij}(t,x_1) - h_{ij}(t,x_2)|^2 + \sum_{j=1}^d |\sigma_j(t,x_1) - \sigma_j(t,x_2)|^2 \le \beta^2(t)\rho(|x_1 - x_2|^2)$, $\qquad (H2)$

where $\beta_1 \in M_G^2([0,T])$, $\beta : [0,T] \to \mathbb{R}^+$, $\beta_2 : [0,T] \to \mathbb{R}^+$ are square integrable, and $\rho : (0,+\infty) \to (0,+\infty)$ is continuous, increasing, concave function satisfying

$$\rho(0+) = 0, \int_0^1 \frac{dr}{\rho(r)} = +\infty. \qquad (7)$$

Theorem 1

We suppose (H1) and (H2), then there exists a unique continuous process $X(\cdot;x) \in L_G^2([0,T];\mathbb{R}^n)$ *(for all* $t \geq 0$, $X(t;x) \in L_G^2(\mathscr{F}_t;\mathbb{R}^n)$*) which satisfies (6).*

Lemma 1 (Hu-Lerner [2002])

Let $\rho : (0,+\infty) \to (0,+\infty)$ *be a continuous, increasing function satisfying (7) and let* u *be a measurable, non-negative function defined on* $(0,+\infty)$ *satisfying*

$$u(t) \leq a + \int_0^t \beta(s)\rho(u(s))ds, \ t \in (0,+\infty),$$

where $a \in [0,+\infty)$, *and* $\beta : [0,T] \to \mathbb{R}^+$ *is Lebesgue integrable. We have:*

- *If* $a = 0$, *then* $u(t) = 0$, *for* $t \in [0,+\infty)$;
- *If* $a > 0$, *we define* $v(t) = \int_{t_0}^t (ds/\rho(s))$, $t \in [0,+\infty)$, *where* $t_0 \in (0,+\infty)$, *then*

$$u(t) \leq v^{-1}(v(a) + \int_0^t \beta(s)ds). \tag{8}$$

Lemma 2 (Jensen's Inequality)

Let $\rho : \mathbb{R} \to \mathbb{R}$ *be a continuous increasing, concave function defined on* $\mathbb{R}$*, then for each* $X \in L_G^1(\mathscr{F})$*, by a classical argument, the following inequality holds,*

$$\rho(\bar{\mathbb{E}}[X]) \geq \bar{\mathbb{E}}[\rho(X)].$$

Proof to Theorem 1: Uniqueness

$$\begin{aligned}|X(t;x_1) - X(t;x_2)|^2 \leq 4|x_1 - x_2|^2 + 4\left|\int_0^t b(s,X(s;x_1)) - b(s,X(s;x_2))ds\right|^2 \\ + 4\left|\sum_{i,j=1}^{d}\int_0^t h_{ij}(s,X(s;x_1)) - h_{ij}(s,X(s;x_2))d\langle B^i,B^j\rangle_s\right|^2 \\ + 4\left|\sum_{j=1}^{d}\int_0^t \sigma_j(s,X(s;x_1)) - \sigma_j(s,X(s;x_2))dB_s^j\right|^2.\end{aligned}$$

From the theorems in Gao [2009], we notice that

$$\bar{\mathbb{E}}[\sup_{0\le r\le t}|\int_0^r (b(s,X(s;x_1))-b(s,X(s;x_2)))ds|^2]$$
$$\le K_1 t\int_0^t \beta^2(s)\bar{\mathbb{E}}[\rho(|X(s,x_1)-X(s,x_2)|^2)]ds;$$

$$\bar{\mathbb{E}}[\sup_{0\le r\le t}|\int_0^r (h_{ij}(s,X(s;x_1))-h_{ij}(s,X(s;x_2)))d\langle B^i,B^j\rangle_s|^2]$$
$$\le K_2 t\int_0^t \beta^2(s)\bar{\mathbb{E}}[\rho(|X(s,x_1)-X(s,x_2)|^2)]ds;$$

$$\bar{\mathbb{E}}[\sup_{0\le r\le t}|\int_0^r (\sigma_j(s,X(s;x_1))-\sigma_j(s,X(s;x_2)))dB^j_s|^2]$$
$$\le K_3\int_0^t \beta^2(s)\bar{\mathbb{E}}[\rho(|X(s;x_1)-X(s;x_2)|^2)]ds.$$

Let

$$u(t) = \bar{\mathbb{E}}[\sup_{0\le r\le t} |X(r;x_1) - X(r;x_2)|^2],$$

due to the sub-additivity property of $\bar{\mathbb{E}}[\cdot]$ and Lemma 2, we have

$$\begin{aligned} u(t) &\le C_1|x_1 - x_2|^2 + C_2\int_0^t \beta^2(s)\bar{\mathbb{E}}[\rho(|X(s;x_1) - X(s;x_2)|^2)]ds \\ &\le C_1|x_1 - x_2|^2 + C_2\int_0^t \beta^2(s)\rho(\bar{\mathbb{E}}[|X(s;x_1) - X(s;x_2)|^2])ds \\ &\le C_1|x_1 - x_2|^2 + C_2\int_0^t \beta^2(s)\rho(u(s))ds. \end{aligned}$$

In particular, if $x_1 = x_2$, we obtain the uniqueness of the solution to (6).□

Proof to Theorem 1: Existence

Picard Iteration:

$$X^{m+1}(t) = x + \int_0^t b(s, X^m(s))ds + \sum_{i,j=1}^{d} \int_0^t h_{ij}(s, X^m(s))d\langle B^i, B^j\rangle_s$$
$$+ \sum_{j=1}^{d} \int_0^t \sigma_j(s, X^m(s))dB_s^j, \ t \in [0, T]. \tag{9}$$

Priori estimate for $\{\bar{\mathbb{E}}[|X^m(t)|^2], m \geq 0\}$:

$$\bar{\mathbb{E}}[|X^{m+1}(t)|^2] \leq C_1|x|^2 + C_2 \int_0^t \bar{\mathbb{E}}[\beta_1^2(s) + \beta_2^2(s)|X^m(s)|^2]ds.$$

Dominated by the solution p of:

$$p(t) = C_1|x|^2 + C_2 \int_0^t \bar{\mathbb{E}}[\beta_1^2(s)]ds + C_2 \int_0^t \beta_2^2(s)p(s)ds,$$

which is bounded from the explicit form.

Set

$$u_{k+1,m}(t) = \sup_{0\leq r\leq t} \bar{\mathbb{E}}[|X^{k+1+m}(r) - X^{k+1}(r)|^2].$$

From the definition of the sequence $\{X^m(\cdot), m \geq 0\}$, we have

$$u_{k+1,m}(t) \leq C\int_0^t \beta^2(s)\rho(u_{k,m}(s))ds.$$

Set

$$v_k(t) = \sup_m u_{k,m}(t),\ 0 \leq t \leq T,$$

then,

$$0 \leq v_{k+1}(t) \leq C\int_0^t \beta^2(s)\rho(v_k(s))ds.$$

Finally, we have:

$$\limsup_{k\to+\infty} v_k(t) = 0,\ 0 \leq t \leq T.$$

Hence, $\{X^m(\cdot), m \geq 0\}$ is a Cauchy sequence in $L^2_G([0,T];\mathbb{R}^n)$, set

$$X(t) = \sum_{m=1}^{\infty}(X^m(t) - X^{m-1}(t)),$$

after checking the right side of the iteration, the proof of the existence is complete. □

Theorem 2

We assume the following one-sided integral-Lipschitz conditions for b, h *and* σ, *i.e., for all* $x, x_1, x_2 \in \mathbb{R}^n$ *and* $i, j = 1, \ldots d$,

(H1') $b(\cdot, x)$, $h_{ij}(\cdot, x)$, $\sigma_j(\cdot, x) \in M_G^2([0,T];\mathbb{R}^n)$ *are uniformly bounded;*

(H2') $2\langle x_1 - x_2, b(t,x_1) - b(t,x_2)\rangle \leq \beta^2(t)\rho(|x_1 - x_2|^2)$;

$2\langle x_1 - x_2, h_{ij}(t,x_1) - h_{ij}(t,x_2)\rangle \leq \beta^2(t)\rho(|x_1 - x_2|^2)$;

$|\sigma_j(t,x_1) - \sigma_j(t,x_2)|^2 \leq \beta^2(t)\rho(|x_1 - x_2|^2)$

where $\beta : [0,T] \to \mathbb{R}^+$ *is square integrable. Then there exists at most one solution* $X(\cdot)$ *in* $L_G^2([0,T], \mathbb{R}^n)$ *to (6).*

Proof to Theorem 2: Applying G-Itô's formula to $|X^1(t) - X^2(t)|^2$, we obtain:

$$
\begin{aligned}
& d(|X^1(t) - X^2(t)|^2) \\
=& 2\langle X^1(t) - X^2(t), b(t, X^1(t)) - b(t, X^2(t))\rangle dt \\
+& 2\langle X^1(t) - X^2(t), h_{ij}(t, X^1(t)) - h_{ij}(t, X^2(t))\rangle d\langle B^i, B^j\rangle_t \\
+& (\sigma_i(t, X^1(t)) - \sigma_i(t, X^2(t)))_k(\sigma_j(t, X^1(t)) - \sigma_j(t, X^2(t)))_k d\langle B^i, B^j\rangle_t \\
+& 2\langle X^1(t) - X^2(t), \sigma_j(t, X^1(t)) - \sigma_j(t, X^2(t))\rangle dB_t^j,
\end{aligned}
$$

where the repeated indices k, i and j imply the summation and $\sigma_j = ((\sigma_j)_1, \cdots, (\sigma_j)_n)^T$.

From the theorems in Gao [2009], we can deduce that

$$
\bar{\mathbb{E}}[|X^1(t) - X^2(t)|^2] \leq C \int_0^t \beta^2(s)\rho(\bar{\mathbb{E}}[|X^1(s) - X^2(s)|^2])ds.
$$

Finally, Lemma 1 gives the uniqueness result. □

Theorem 3

We suppose (H1') and the following condition: for any $x_1,\, x_2 \in \mathbb{R}^n$

$$\text{(H2'')}\quad \begin{array}{l} |b(t,x_1)-b(t,x_2)| \le \beta(t)\rho_1(|x_1-x_2|); \\ |h_{ij}(t,x_1)-h_{ij}(t,x_2)| \le \beta(t)\rho_1(|x_1-x_2|); \\ |\sigma_j(t,x_1)-\sigma_j(t,x_2)|^2 \le \beta(t)\rho_2(|x_1-x_2|^2), \end{array}$$

where $\beta : [0,T] \to \mathbb{R}^+$ *is square integrable,* $\rho_1,\ \rho_2 : (0,+\infty) \to (0,+\infty)$ *are continuous, concave and increasing, and both of them satisfy (7). Furthermore, we assume that*

$$\rho_3(r) = \frac{\rho_2(r^2)}{r},\ r \in (0,+\infty)$$

is also continuous, concave and increasing, and

$$\rho_3(0+) = 0,\ \int_0^1 \frac{dr}{\rho_1(r)+\rho_3(r)} = +\infty.$$

Then there exists a unique solution to the equation (6).

Example

$$\rho_1(r) = r\ln\frac{1}{r};$$

$$\rho_2(r) = r\ln\frac{1}{r}.$$

As for existence, we need some stronger conditions, nevertheless, the example above satisfies the conditions of Theorem 3 but not Theorem 1.

Proof of Theorem 3:

Picard Iteration:

$$\begin{aligned} X^{m+1}(t) =& x + \int_0^t b(s, X^m(s))ds \\ &+ \sum_{i,j=1}^{d} \int_0^t h_{ij}(s, X^m(s))d\langle B^i, B^j\rangle_s + \sum_{j=1}^{d} \int_0^t \sigma_j(s, X^{m+1}(s))dB_s^j. \end{aligned}$$

Because of the assumptions of this theorem and thanks to Theorem 1, the sequence $\{X^m(\cdot), m \geq 0\}$ is well defined.

Note that as $|x|$ is not C^2, we approximate $|x|$ by $F_\varepsilon \in C^2$, where

$$F_\varepsilon(x) = (|x|^2 + \varepsilon)^{\frac{1}{2}},\ x \in \mathbb{R}^n.$$

We notice that

$$|F_\varepsilon'(x)| \leq 1,\ |F_\varepsilon''(x)| \leq \frac{2}{(|x|^2 + \varepsilon)^{\frac{1}{2}}},$$

and $F_\varepsilon'(x)$, $F_\varepsilon''(x)$ are bounded and uniformly Lipschitz.

Applying G-Itô formula to $F_\varepsilon(X^{k+1+m}(t)-X^{k+1}(t))$, and taking the G-expectation, we get from Theorem 16 that, for some positive constant K,

$$\begin{aligned}\bar{\mathbb{E}}[F_\varepsilon(X^{k+1+m}(t)-X^{k+1}(t))] \leq & \bar{\mathbb{E}}[\int_0^t |b(s,X^{k+m}(s))-b(s,X^k(s))|ds] \\ & +K\sum_{i,j=1}^{d}\bar{\mathbb{E}}[\int_0^t |h_{ij}(s,X^{k+m}(s))-h_{ij}(s,X^k(s))|ds] \\ & +K\sum_{j=1}^{d}\bar{\mathbb{E}}[\int_0^t \frac{|\sigma_j(s,X^{k+m+1}(s))-\sigma_j(s,X^{k+1}(s))|^2}{(|X^{k+m+1}(s)-X^{k+1}(s)|^2+\varepsilon)^{\frac{1}{2}}}ds].\end{aligned}$$

Letting $\varepsilon \to 0$, we deduce from Lemma 1 and (5) that, for some positive constant C,

$$\begin{aligned}u_{k+1,m}(t) &= \sup_{0\leq r\leq t}\bar{\mathbb{E}}[|X^{k+1+m}(r)-X^{k+1}(r)|] \\ &\leq C\int_0^t \beta(s)(\rho_1(u_{k,m}(s))+\rho_3(u_{k+1,m}(s)))ds.\end{aligned}$$

Set

$$v_k(t) = \sup_m u_{k,m}(t),\ 0 \le t \le T,$$

then,

$$0 \le v_{k+1}(t) \le C \int_0^t \beta(s)(\rho_1(v_k(s)) + \rho_3(v_{k+1}(s)))ds.$$

By Lemma 1, we deduce that

$$\limsup_{k \to +\infty} v_k(t) = 0,\ t \in [0,T].$$

Hence, $\{X^m(\cdot), m \ge 0\}$ is a Cauchy sequence in $L_G^1([0,T],\mathbb{R}^n)$. Then there exists $X(\cdot) \subset L_G^1([0,T],\mathbb{R}^n)$ and a subsequence $\{X^{m_l}(\cdot), l \ge 1\} \subset \{X^m(\cdot), m \ge 1\}$ such that

$$X^{m_l} \to X,\ as\ l \to +\infty,\ q.s..$$

Since b, h_{ij} and σ_j are bounded, it is easy to check that, for some positive constant $M > 0$,

$$\sup_{m\geq 0}\sup_{0\leq t\leq T} \bar{\mathbb{E}}[|X^m(t)|^p] \leq M, \text{ where } p > 2,$$

and for each $P_\in \mathscr{P}$,

$$\begin{aligned} E_P(|X(t)|^p) =& E_P(\liminf_{l\to+\infty}|X^{m_l}(t)|^p) \leq \liminf_{l\to+\infty} E_P(|X^{m_l}(t)|^p) \\ \leq& \liminf_{l\to+\infty}\sup_{P\in\mathscr{P}} E_P(|X^{m_l}(t)|^p) = \liminf_{l\to+\infty}\bar{\mathbb{E}}[|X^{m_l}(t)|^p] \\ \leq& M. \end{aligned}$$

Hence,

$$\sup_{0\leq t\leq T}\bar{\mathbb{E}}[|X(t)|^p] = \sup_{0\leq t\leq T}(\sup_{P_\in\mathscr{P}} E_P(|X(t)|^p)) \leq M$$

and

$$\sup_{0\leq t\leq T}\bar{\mathbb{E}}[|X^m(t)-X(t)|^p] \leq 2^p\sup_{0\leq t\leq T}\bar{\mathbb{E}}[|X^m(t)|^p] + 2^p\sup_{0\leq t\leq T}\bar{\mathbb{E}}[|X(t)|^p] \leq 2^{p+1}M.$$

Consequencely, for a fixed $\varepsilon > 0$,

$$
\begin{aligned}
&\limsup_{m\to+\infty} \bar{\mathbb{E}}[|X^m(t) - X(t)|^2] \\
\le&\limsup_{m\to+\infty} (\varepsilon^2 \bar{\mathbb{E}}[I_{\{|X^m(t)-X(t)|\le\varepsilon\}}] + \bar{\mathbb{E}}[|X^m(t) - X(t)|^2 I_{\{|X^m(t)-X(t)|\ge\varepsilon\}}]) \\
\le&\varepsilon^2 + \limsup_{m\to+\infty} (\bar{\mathbb{E}}[|X^m(t) - X(t)|^p])^{\frac{2}{p}} (\bar{\mathbb{E}}[|I_{\{|X^m(t)-X(t)|\ge\varepsilon\}}|^{\frac{p}{p-2}}])^{\frac{p-2}{p}} \\
\le&\varepsilon^2 + 8M^{\frac{2}{p}} \limsup_{m\to+\infty} (\bar{\mathbb{E}}[I_{\{|X^m(t)-X(t)|\ge\varepsilon\}}])^{\frac{p-2}{p}} \\
=&\varepsilon^2.
\end{aligned}
$$

The last step above can be easily deduced from $\lim_{m\to+\infty}(\sup_{0\le t\le T} \bar{\mathbb{E}}|X^m(t) - X(t)|) = 0$. Since ε can be arbitrary small, we have $\lim_{m\to+\infty} \bar{\mathbb{E}}[|X^m(t) - X(t)|^2] = 0$.

On the other hand, since $\rho_1 : (0,+\infty) \to (0,+\infty)$ are continuous, concave and increasing, then for arbitrary fixed $\varepsilon > 0$, there exists a constant K_ε, such that $|\rho_1(x)| \leq K_\varepsilon |x|$, for $x > \varepsilon$. Hence, for some positive constant C, we have

$$
\begin{aligned}
&\lim_{m\to\infty} \bar{\mathbb{E}}[\sup_{0\leq t\leq T} |\int_0^t (b(s,X^m(s)) - b(s,X(s)))ds|^2] \\
\leq &C \lim_{m\to\infty} \int_0^T \beta^2(s)\bar{\mathbb{E}}[\rho_1^2(|X^m(s) - X(s)|)]ds, \\
\leq &C \lim_{m\to\infty} \left(\int_0^T \beta^2(s)\bar{\mathbb{E}}[\rho_1^2(|X^m(s) - X(s)|)I_{\{|X^m(s)-X(s)|>\varepsilon\}}]ds \right. \\
&\left. +\rho_1^2(\varepsilon^2) \int_0^T \beta^2(s)\bar{\mathbb{E}}[I_{\{|X^m(s)-X(s)|\leq\varepsilon\}}]ds \right) \\
\leq &C \lim_{m\to\infty} \left(\int_0^T \beta^2(s)\bar{\mathbb{E}}[K_\varepsilon^2|X^m(s) - X(s)|^2]]ds \right) + C\rho_1^2(\varepsilon^2) \int_0^T \beta^2(s)ds \\
= &C\rho_1^2(\varepsilon^2) \int_0^T \beta^2(s)ds
\end{aligned}
$$

Notice that ρ_1 is continuous, $\rho_1(0+) = 0$ and ε can arbitrary small, then we have

$$\lim_{m\to\infty} \bar{\mathbb{E}}[\sup_{0\le t\le T} |\int_0^t (b(s,X^m(s)) - b(s,X(s)))ds|^2] = 0.$$

Similarly we get

$$\lim_{m\to\infty} \bar{\mathbb{E}}[\sup_{0\le t\le T} |\int_0^t (h_{ij}(s,X^m(s)) - h_{ij}(s,X(s)))d\langle B^i, B^j\rangle_s|^2] = 0$$

and

$$\begin{aligned}
&\lim_{m\to+\infty} \bar{\mathbb{E}}[\sup_{0\le t\le T} |\int_0^t (\sigma_j(s,X^m(s)) - \sigma_j(s,X(s)))dB_s^j|^2] \\
\le &\lim_{m\to+\infty} C\int_0^T \beta^2(s)\rho_2(\bar{\mathbb{E}}[|X^m(s) - X(s)|^2])ds \\
= &0.
\end{aligned}$$

Then the proof of the existence of the solution to (6) is complete. $\square$

Consider the following type of G-backward stochastic differential equation (G-BSDE):

$$Y_t = \mathbb{E}[\xi + \int_t^T f(s, Y_s)ds + \sum_{i,j=1}^{d} \int_t^T g_{ij}(s, Y_s)d\langle B^i, B^j\rangle_s | \mathscr{F}_t], \ t \in [0, T]. \quad (10)$$

- $f(\cdot, x), g_{ij}(\cdot, x) \in M_G^1([0, T]; \mathbb{R}^n)$;
- $|g(s, y)| + |f(s, y)| \leq \beta(t) + c|y|$;
- $|g(s, y_1) - g(s, y_2)| + |f(s, y_1) - f(s, y_2)| \leq \rho(|y_1 - y_2|)$.

where $c > 0$, $\beta \in M_G^1([0, T]; \mathbb{R}_+)$ and $\rho : (0, +\infty) \to (0, +\infty)$ is a continuous, concave, increasing function satisfying (7).

Theorem 4

Under the assumptions above, (10) admits a unique solution $Y \in L_G^1([0, T], \mathbb{R}^n)$.

Proof to Theorem 4: Due to the sub-additivity property of $\mathbb{E}[\cdot|\mathscr{F}_t]$, we obtain:

$$
\begin{aligned}
|Y_t^1 - Y_t^2| \leq \mathbb{E}[| & \int_t^T (f(s,Y_s^1) - f(s,Y_s^2))ds| \\
& + \sum_{i,j=1}^{d} |\int_t^T (g_{ij}(s,Y_s^1) - g_{ij}(s,Y_s^2))d\langle B^i, B^j\rangle_s||\mathscr{F}_t]
\end{aligned}
$$

Taking the G-expectation on both sides, we have from theorem in Gao [2009] and Lemma 2 that

$$
\begin{aligned}
\mathbb{E}[|Y_t^1 - Y_t^2|] \leq (\mathbb{E}[| & \int_t^T (f(s,Y_s^1) - f(s,Y_s^2))ds| \\
& + \sum_{i,j=1}^{d} \mathbb{E}[|\int_t^T (g_{ij}(s,Y_s^1) - g_{ij}(s,Y_s^2))d\langle B^i, B^j\rangle_s|]) \\
& \leq K\mathbb{E}\int_t^T \rho(|Y_s^1 - Y_s^2|)ds \\
& \leq K\int_t^T \rho(\mathbb{E}[|Y_s^1 - Y_s^2|])ds.
\end{aligned}
$$

Set

$$u(t) = \mathbb{E}[|Y_t^1 - Y_t^2|],$$

then

$$u(t) \leq K \int_t^T \rho(u(s))ds,$$

and we deduce from Lemma 1 that $u(t) = 0$.

Then the uniqueness of the solution can be now easily proved.

As for the existence of solution, we proceed as Theorem 1: define a sequence of $(Y^m, m \geq 0)$, as follows:

$$Y_t^{m+1} = \mathbb{E}[\xi + \int_t^T f(s, Y_s^m)ds + \sum_{i,j=1}^{d} \int_t^T g_{ij}(s, Y_s^m)d\langle B^i, B^j\rangle_s|\mathscr{F}_t], \quad Y^0 = 0.$$

Then the rest of the proof goes in a similar way as that in Theorem 1. □

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