Existence and uniqueness of solution for multidimensional BSDE with local conditions on the coefficient

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1. BSDEs–Introduction

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq 1}, P)$ be a complete probability space
  $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t) \vee \mathcal{N}$ be a filtration
- Consider the following ODE

$$\begin{cases} dY_t = 0, t \in [0, T], \\ Y_T = \xi \in R. \end{cases} \tag{1}$$

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which is not $\mathcal{F}_t-$adapted.
A natural way of making (2) $\mathcal{F}_t-$adapted is to redefine $Y_.$ as follows

$$Y_t = E(\xi|\mathcal{F}_t), \; t \in [0, T]. \tag{3}$$

Then $Y_.$ is $\mathcal{F}_t-$adapted and satisfies $Y_T = \xi$, but not equation (1).

1. BSDEs–Introduction

MRT $\Longrightarrow$ there exists an $\mathcal{F}_t-$adapted process Z square integrable s.t

$$Y_t = Y_0 + \int_0^t Z_s dB_s. \tag{4}$$

It follows that

$$Y_T = \xi = Y_0 + \int_0^T Z_s dB_s. \tag{5}$$

Combining (4) and (5), one has

$$Y_t = \xi - \int_t^T Z_s dB_s, \tag{6}$$

whose differential form is

$$\begin{cases} dY_t = Z_t dB_t, t \in [0, T], \\ Y_T = \xi. \end{cases} \tag{7}$$

Comparing (1) and (7), the term "$Z_t dB_t$" has been added.

1. BSDEs–Introduction

- BSDE is an equation of the following type:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \tag{8}$$

- T : TERMINAL TIME
- $f : \Omega \times [0, T] \times R^d \times R^{d \times n} \longrightarrow R^d$: GENERATOR or COEFFICIENT
- ξ : TERMINAL CONDITION $\mathcal{F}_T$−adapted process with value in R^d.
- UNKNOWNS ARE : $Y \in R^d$ and $Z \in R^{d \times n}$.

1. BSDEs–Introduction

Denote by $\mathbb{L}$ the set of $R^d \times R^{d\times n}$–valued processes (Y, Z) defined on $R_+ \times \Omega$ which are $\mathcal{F}_t$–adapted and such that:

$$\|(Y, Z)\|^2 = E\left(\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds\right) < +\infty.$$

The couple $(\mathbb{L}, \|.\|)$ is then a Banach space.

Definition

A solution of equation (8) is a pair of processes (Y, Z) which belongs to the space $(\mathbb{L}, \|.\|)$ and satisfies equation (8).

2. BSDEs with Lipshitz coefficient

Consider the following assumptions:

- For all $(y, z) \in R^d \times R^{d\times n}$: $(\omega, t) \longrightarrow f(\omega, t, y, z)$ is $\mathcal{F}_t-$ progressively measurable
- $f(., 0, 0) \in L^2([0, T] \times \Omega, R^d)$
- f is Lipschitz : $\exists K > 0$ and $\forall y, y' \in R^d, z, z' \in R^{d\times n}$ and $(\omega, t) \in \Omega \times [0, T]$ s.t

$$| \ f\,(\omega, t, y, z) - f(\omega, t, y', z') \ | \leq K\left(| \ y - y' \ | + | \ z - z' \ |\right).$$

- $\xi \in L^2(\Omega, \mathcal{F}_T; R^d)$

Theorem : Pardoux and Peng 1990

Suppose that the above assumptions hold true. Then, there exists a unique solution for BSDE (15).

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3. APPLICATIONS OF BSDE : FINANCE & PDE

Consider a market where only two basic assets are traded.

- BOND :
- STOCK :

Consider a European call option whose payoff is

$$(X_T - K)^+.$$

The option pricing problem is : fair price of this option at time $t = 0$?
Suppose that this option has a price y at time $t = 0$. Then the fair price for the option at time $t = 0$ should be such a y that the corresponding optimal investment would result in a wealth process Y_t satisfying

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3. APPLICATIONS OF BSDE : FINANCE & PDE

Denote by

- R_t : the amount that the writer invests in the stock
- $Y_t - R_t$: the remaining amount which is invested in the bond

R_t determines a strategy of the investment which is called a portfolio.
By setting $Z_t = \sigma R_t$, we obtain the following BSDE

$$\begin{cases} dX_t = bX_t dt + \sigma X_t dB_t \\ dY_t = \underbrace{\left(rY_t + \frac{b-r}{\sigma} Z_t\right)}_{f(t,Y_t,Z_t)} dt + Z_t dB_t, t \in [0,T], \\ X_0 = x, \;\; Y_T = \underbrace{(X_T - K)^+}_{\xi}. \end{cases} \tag{9}$$

Pardoux & Peng result $\Longrightarrow$ there exits a unique solution (Y_t, Z_t). The option price at time $t = 0$ is given by Y_0, and the portfolio is given by $R_t = \frac{Z_t}{\sigma}$.

3. APPLICATIONS OF BSDE : FINANCE & PDE

Let u be the solution of the following system of semi-linear parabolic PDE's:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{1}{2} Tr(\sigma\sigma^* \Delta u)(t,x) + b\nabla u(t,x) + f(t,x,u(t,x),\nabla u\sigma(t,x)) = 0 \\ u(T,x) = g(x). \end{cases} \tag{10}$$

Introducing $\{(Y^{s,x}, Z^{s,x})\,;\ s \leq t \leq T\}$ the adapted solution of the backward stochastic differential equation

$$\begin{cases} -dY_t = f(t, X_t^{s,x}, Y_t, Z_t)ds - Z_t^* dB_t \\ Y_T = g(X_T^{t,x}), \end{cases} \tag{11}$$

where $(X^{s,x})$ denotes the solution of the following stochastic differential equation

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_s = x. \end{cases} \tag{12}$$

3. APPLICATIONS OF BSDE : FINANCE & PDE

Then we have :

- u is a classical solution of PDE (10) $\Longrightarrow$

$$(Y_t^{s,x} = u(t, X_t^{s,x}),\ Z_t^{s,x} = \nabla u(t, X_t^{s,x})\sigma(s, X_t^{s,x}))$$

  is a solution the BSDE (11).
- There exists a solution to the BSDE (11)$\Longrightarrow u(t,x) = Y_t^{t,x}$, is a viscosity solution of PDE (10). This formula is a generalization of Feynman-Kac formula.

Suppose that $f(t,x,y,z) = c(t,x)y + h(t,x)$, we obtain

$$\begin{aligned} Y_t^{t,x} =& E\left[g(X_1^{t,x}) \exp\left(\int_t^1 c(r, X_r^{t,x})dr\right)\right. \\ &\left. + \int_t^1 h(s, X_s^{t,x}) \exp\left(\int_t^s c(r, X_r^{t,x})dr\right) ds\right], \end{aligned}$$

which is the classical Feynman-Kac formula.

4. BSDEs with locally Lipschitz coefficient

Consider the following assumptions:

(A1) f is continuous in (y, z) for almost all (t, ω).

(A2) There exist $K > 0$ and $0 \leq \alpha \leq 1$ such that

$$| f(t, \omega, y, z) | \leq K(1 + | y |^{\alpha} + | z |^{\alpha}).$$

(A3) For each $N > 0$, there exists L_N such that:

$$| f(t, y, z) - f(t, y', z') | \leq L_N(| y - y' | + | z - z' |)$$
$$| y |, | y' |, | z |, | z' | \leq N.$$

(A4) $\xi \in L^2(\Omega, \mathcal{F}_T; R^d)$

Theorem : Bahlali 2002

Assume moreover that there exists a positive constant L such that $L_N = L + \sqrt{\log N}$ then there exists a unique solution for the BSDE (1).

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5. BSDEs with locally monotone coefficient–Question

Question

Let $f(y) := -y \log |\ y\ |$. Suppose that $\xi \in \mathbb{L}^2(\mathcal{F}_T)$ or $\xi \in \mathbb{L}^p(\mathcal{F}_T), p > 1$ and consider the following BSDE with logarithmic nonlinearity

$$Y_t = \xi - \int_t^T Y_s \log |\ Y_s\ | ds - \int_t^T Z_s dW_s.$$

Does this equation has a unique solution?

5. BSDEs with locally monotone coefficient-Assumptions

Consider the following assumptions:

(H1) f is continuous in (y, z) for almost all (t, ω),

(H2) There exist $M > 0, \gamma < \frac{1}{2}$ and $\eta \in \mathbb{L}^1([0, T] \times \Omega)$ such that,

$$\langle y, f(t, \omega, y, z) \rangle \leq \eta + M|y|^2 + \gamma|z|^2 \quad P - a.s., \ a.e.\ t \in [0, T].$$

(H3) "Almost" quadratic growth : $\exists M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\overline{\eta} \in \mathbb{L}^{\alpha'}([0, T] \times \Omega)$ s.t :

$$| f(t, \omega, y, z) | \leq \overline{\eta} + M_1(| y |^{\alpha} + | z |^{\alpha}).$$

5. BSDEs with locally monotone coefficient

(H4) There exists a real valued sequence $(A_N)_{N>1}$ and constants $M > 1$, $r > 1$ such that:
i) $\forall N > 1, \quad 1 < A_N \leq N^r$.
ii) $\lim\limits_{N\to\infty} A_N = \infty$.
iii) Locally monotone condition : For every
$N \in N, \forall y, y', z, z'$such that $| y |, | y' |, | z |, | z' | \leq N$, we have

$$\langle y - y', f(t, y, z) - f(t, y', z')\rangle$$
$$\leq M \mid y - y' \mid^2 logA_N + M \mid y - y' \mid\mid z - z' \mid \sqrt{\log A_N} + MA_N^{-1}.$$

Theorem : Bahlali-Essaky-Hassani-Pardoux, 2002

Let ξ be a square integrable random variable. Assume that (H1)–(H4) are satisfied. Then the BSDE has a unique solution.

1. L^p-solutions to BSDEs with super-Motivation

Let's mention some considerations which have motivated the present work.

- The growth conditions on the nonlinearity constitute a critical case. Indeed, it is known that for any $\varepsilon > 0$, the solutions of the ordinary differential equation $X_t = x + \int_0^t X_s^{1+\varepsilon} ds$ explode at a finite time.
- The logarithmic nonlinearities appear in some PDEs arising in physics.
- In terms of continuous-state branching processes, the logarithmic nonlinearity $u \log u$ corresponds to the Neveu branching mechanism. This process was introduced by Neveu. For instance, the super-process with Neveu's branching mechanism is related to the Cauchy problem,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \log u = 0 \;\text{ on }\; (0,\; \infty) \times R^d \\ u(0^+) = \varphi > 0 \end{cases} \tag{13}$$

  Hence, our result can be seen as an alternative approach to the PDEs.
- It is worth noting that our condition on the coefficient f is new even for the classical Itô's forward SDEs.

$$X_s = x + \int_0^s X_r \log |X_r| dr + \int_0^s X_r \sqrt{|\log |X_r||} dW_r, \qquad 0 \leq s \leq T. \tag{14}$$

  For instance, the problem to establish the existence of a pathwise unique solution to the following équation (14) still remains open.

5. BSDEs with locally monotone coefficient–Example

Example

Let $f(y) := -y \log \mid y \mid$ then for all $\xi \in \mathbb{L}^2(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi - \int_t^T Y_s \log \mid Y_s \mid ds - \int_t^T Z_s dW_s.$$

Indeed, f satisfies **(H.1)-(H.3)** since $\langle y, f(y) \rangle \leq 1$ and $\mid f(y) \mid \leq 1 + \frac{1}{\varepsilon} \mid y \mid^{1+\varepsilon}$ for all $\varepsilon > 0$.
(H.4) is satisfied for every $N > e$ and $A_N = N$.

5. BSDEs with locally monotone coefficient–Idea of the proof

- We define a family of semi-norms $\Big(\rho_n(f)\Big)_{n\in \boldsymbol{N}}$ by,

$$\rho_n(f) = \boldsymbol{E}\int_0^T \sup_{|y|,|z|\leq n} |f(s,y,z)|ds.$$

- We Approximate f by a sequence $(f_n)_{n>1}$ of Lipschitz functions :

Lemma

Let f be a process which satisfies **(H.1)–(H.3)**. *Then* there exists a sequence of processes (f_n) such that,

(a) For each n, f_n is bounded and globally Lipschitz in (y,z) *a.e.* t and P-$a.s.\omega$.

There exists $M' > 0$, such that:

(b) $\sup_n |f_n(t,\omega,y,z)| \leq \overline{\eta} + M' + M_1(\mid y\mid^{\alpha} + \mid z\mid^{\alpha})$. P-$a.s.$, $a.e.$ $t\in[0,T]$.

(c)

$$\sup_n < y, f_n(t,\omega,y,z) >\leq \eta + M' + M|y|^2 + \gamma|z|^2$$

(d) For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.

5. BSDEs with locally monotone coefficient–Key steps of the proof

- We consider the following BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s, \qquad 0 \leq t \leq T.$$

Lemma

There exits a universal constant ℓ such that

a)

$$E\int_0^T e^{2Ms}|Z_s^{f_n}|^2 ds \leq \frac{1}{1-2\gamma}\left[e^{2MT}E \mid \xi \mid^2 + 2E\int_0^T e^{2Ms}(\eta + M')ds\right] = K_1$$

b) $E \sup_{0\leq t\leq T}(e^{2Mt} \mid Y_t^{f_n} \mid^2) \leq \ell K_1 = K_2$

c)

$$E\int_0^T e^{2Ms}|f_n(s, Y_s^{f_n}, Z_s^{f_n})|^{\overline{\alpha}} ds \leq$$

$$4^{\overline{\alpha}-1}\left[E\int_0^T e^{2Ms}((\overline{\eta} + M')^{\overline{\alpha}} + 4)ds + M_1^{\overline{\alpha}}K_1 + TM_1^{\overline{\alpha}}K_2\right] = K_3$$

d) $E\int_0^T e^{2Ms}|f(s, Y_s^{f_n}, Z_s^{f_n})|^{\overline{\alpha}} ds \leq K_3$, where $\overline{\alpha} = \min(\alpha', \frac{2}{\alpha})$.

- Hence the following convergences hold true

$$
\begin{aligned}
& Y^n \rightharpoonup Y, \text{ weakly star in } \mathbb{L}^2(\Omega, L^\infty[0,T]) \\
& Z^n \rightharpoonup Z, \text{ weakly in } \mathbb{L}^2(\Omega \times [0,T]) \\
& f_n(., Y^n, Z^n) \rightharpoonup \Gamma_. \text{ weakly in } \mathbb{L}^{\overline{\alpha}}(\Omega \times [0,T]),
\end{aligned}
$$

Moreover

$$Y_t = \xi + \int_t^T \Gamma_s ds - \int_t^T Z_s dW_s, \ \forall t \in [0,T].$$

- We Apply Itô's formula to $(|Y^n - Y^m|^2 + \varepsilon)^p$ for some $0 < p < 1$, instead of $|Y^n - Y^m|^2$:

$$\lim_{n \to +\infty} \left(E \sup_{0 \le t \le T} |Y_t^n - Y_t|^\beta + E \int_0^T |Z_s^n - Z_s| ds \right) = 0, \ 1 < \beta < 2.$$

- We Identify Γ_s by proving that : $\lim_n E \int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds = 0.$

6. L^p-solutions to BSDEs with locally monotone coefficient-Definition

Let $p > 1$ is an arbitrary fixed real number and all the considered processes are $(\mathcal{F}_t)$-predictable.

Definition

A solution of equation (8 is an $(\mathcal{F}_t)$-adapted and R^{d+dr}-valued process (Y, Z) such that

$$E \sup_{t \leq T} |Y_t|^p + E \left[\int_0^T |Z_s|^2 ds \right]^{\frac{p}{2}} + E \int_0^T |f(s, Y_s, Z_s)| ds < +\infty$$

and satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \tag{15}$$

6. L^p-solutions to BSDEs with locally monotone coefficient-Assumptions

We consider the following assumptions on (ξ, f):

(H.0)

$$\begin{cases} \text{There are } M \in \mathbb{L}^0(\Omega; \mathbb{L}^1([0,T]; R_+)),\ K \in \mathbb{L}^0(\Omega; \mathbb{L}^2([0,T]; R_+)) \text{ and } \gamma \in]0, \dfrac{1 \wedge (p-1)}{2}[\\ \text{such that: } \quad E \mid \xi \mid^p e^{\frac{p}{2}\int_0^T \lambda_s ds} < \infty, \text{ where } \lambda_s := 2M_s + \dfrac{K_s^2}{2\gamma} \end{cases}$$

(H.1) f is continuous in (y, z) for almost all (t, ω).

(H.2)

$$\begin{cases} \text{There are } \eta \text{ and } f^0 \in \mathbb{L}^0(\Omega \times [0,T]; R_+) \text{ satisfying } E\left(\displaystyle\int_0^T e^{\int_0^s \lambda_r dr} \eta_s ds\right)^{\frac{p}{2}} < \infty \\ \text{and } E\left(\displaystyle\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p < \infty, \text{ where } \lambda \text{ is defined in assumption } \textbf{(H.0)}, \\ \text{such that:} \\ \langle y, f(t,y,z) \rangle \quad \leq \quad \eta_t + f_t^0 |y| + M_t |y|^2 + K_t |y||z|. \end{cases}$$

6. L^p-solutions to BSDEs with locally monotone coefficient-Assumptions

(H.3) $\left\{ \begin{array}{l} \text{There are } \overline{\eta} \in \mathbb{L}^q(\Omega \times [0,T]; R_+)) \text{ (for some } q > 1) \text{ and } \alpha \in]1, p[, \alpha' \in]1, p \wedge 2[\\ \text{such that:} \\ \mid f(t,\omega,y,z) \mid \ \leq \ \overline{\eta}_t + \mid y \mid^{\alpha} + \mid z \mid^{\alpha'} . \end{array} \right.$

(H.4) $\left\{ \begin{array}{l} \text{There are } v \in \mathbb{L}^{q'}(\Omega \times [0,T]; R_+)) \text{ (for some } q' > 0) \text{ and } K' \in R_+ \text{ such that} \\ \text{for every } N \in N \text{ and every } y,\ y'\ z,\ z' \text{ satisfying } \mid y \mid, \mid y' \mid, \mid z \mid, \mid z' \mid \leq N \\ \\ 1_{v_t(\omega) \leq N} \langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z') \rangle \\ \leq K' \log A_N \mid y - y' \mid^2 + \sqrt{K' \log A_N} \mid y - y' \mid\mid z - z' \mid + K' \dfrac{\log A_N}{A_N} \\ \text{where } A_N \text{ is a increasing sequence and satisfies } A_N > 1, \lim_{N \to \infty} A_N = \infty \\ \text{and } A_N \leq N^{\mu} \text{ for some } \mu > 0. \end{array} \right.$

6. L^p-solutions to BSDEs with locally monotone coefficient-Existence and Uniqueness

Theorem : Bahlali-Essaky-Hassani

If **(H.0)-(H.4)** hold then (8) has a unique solution (Y, Z). Moreover we have

$$E\sup_t \mid Y_t \mid^p e^{\frac{p}{2}\int_0^t \lambda_s ds} + E\left[\int_0^T e^{\int_0^s \lambda_r dr} \mid Z_s \mid^2 ds\right]^{\frac{p}{2}}$$
$$\leq C\left\{E \mid \xi \mid^p e^{\frac{p}{2}\int_0^T \lambda_s ds} + E\left(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + E\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p\right\}.$$

for some constant C depending only on p and γ.

Example 1

Let $f(y) := -y\log|y|$ then for all $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi - \int_t^T Y_s \log|Y_s|ds - \int_t^T Z_s dW_s.$$

Indeed, f satisfies **(H.1)-(H.3)** since $\langle y, f(y)\rangle \leq 1$ and $|f(y)| \leq 1 + \frac{1}{\varepsilon}|y|^{1+\varepsilon}$ for all $\varepsilon > 0$. **(H.4)** is satisfied for every $N > e$ with $v_s = 0$ and $A_N = N$.

Example 2

Let $g(y) := y \log \frac{|\, y \,|}{1+ |\, y \,|}$ and $h \in \mathcal{C}(R^{dr}; R_+) \bigcap \mathcal{C}^1(R^{dr} - \{0\}; R_+)$ be such that :

$$h(z) = \begin{cases} |z|\sqrt{-\log|z|} & if |z| < 1 - \varepsilon_0 \\ |z|\sqrt{\log|z|} & if |z| > 1 + \varepsilon_0, \end{cases}$$

where $\varepsilon_0 \in]0, 1[$. Finally, we put $f(y, z) := g(y)h(z)$. Then for every $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

(H.4) is satisfied for every $N > \sqrt{e}$ with $v_s = 0$ and $A_N = N$

6. L^p-solutions to BSDEs with locally monotone coefficient-Examples

Example 3

Let $(X_t)_{t\leq T}$ be an $(\mathcal{F}_t)-$adapted and R^k-valued process satisfying :

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

where $X_0 \in R^k$ and $\sigma, b : [0, T] \times R^k \longrightarrow R^{kr} \times R^k$ are measurable functions such that $\|\sigma(s,x)\| \leq c$ and $|b(s,x)| \leq c(1+|x|)$, for some constant c.

Consider the following BSDE $Y_t = g(X_T) + \int_t^T (| X_s |^{\overline{q}} Y_s - Y_s \log | Y_s |)ds - \int_t^T Z_s dW_s.$

where $\overline{q} \in]0, 2[$ and g is a measurable function satisfying $| g(x) | \leq c \exp c | x |^{\overline{q}'}$, for some constants $c > 0$, $\overline{q}' \in [0, 2[$.

The previous BSDE has a unique solution (Y, Z) such that

$$E \sup_t | Y_t |^p + E\left[\int_0^T | Z_s |^2 ds\right]^{\frac{p}{2}} \leq C \exp\left(C | X_0 |^2\right).$$

(H.4) is satisfied with $v_s = \exp | X_s |^{\overline{q}}$ and $A_N = N$.

6. L^p-solutions to BSDEs with locally monotone coefficient-Examples

Example 4

Let (ξ, f) satisfying **(H.0)-(H.3)** and
(H'.4)

$$\begin{cases} \text{There are a positive process } C \text{ satisfying } E\int_0^T e^{q' C_s} ds < \infty \text{ and } K' \in R_+ \text{ such that:} \\ \langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z')\rangle \le K' \mid y - y' \mid^2 \left(C_t(\omega) + \mid \log(\mid y - y' \mid) \mid \right) \\ \qquad + K' \mid y - y' \mid\mid z - z' \mid \sqrt{C_t(\omega) + \mid \log \mid z - z' \mid \mid}. \end{cases}$$

Then the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

(H.4) is satisfied with $v_s = \exp(C_s)$ and $A_N = N$.

6. L^p-solutions to BSDEs with locally monotone coefficient-Idea of the proof

- We Approximate f by a sequence $(f_n)_{n>1}$ of Lipschitz functions :

$$f_n(t,y,z) = (c_1 e)^2 1_{\{\overline{\Lambda}_t \leq n\}} \psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)\times m^{(d+dr)} \int_{R^d}\int_{R^{dr}} f(t,y-u,z-v)\Pi_{i=1}^d \psi(mu_i)\Pi_{i=1}^d \Pi_{j=1}^r \psi(mv_{ij})dudv,$$

with $m := \frac{n^{2p}}{h_t}$ and $\overline{\Lambda}_t := \eta_t + \overline{\eta}_t + f_t^0 + M_t + K_t + \frac{1}{h_t}$ where h_t is a predictable process such that $0 < h_t \leq 1$.

- We consider the following BSDE

$$Y_t^n = \xi 1_{\{|\xi|\leq n\}} + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s, \qquad 0 \leq t \leq T.$$

- We Apply Itô's formula to $(\{|Y^n - Y^m|^2 + \varepsilon\}(\frac{1}{\varepsilon})^{2Ct})^{\frac{\beta}{2}}$ for some $1 < \beta < p\wedge 2$, instead of $|Y^n - Y^m|^2$ we have:
  For every $p' < p$,
  $(Y^n, Z^n) \longrightarrow (Y, Z)$ strongly in $\mathbb{L}^{p'}(\Omega; \mathcal{C}([0,T]; R^d)) \times \mathbb{L}^{p'}(\Omega; \mathbb{L}^2([0,T]; R^{dr}))$.
- For every $\hat{\beta} < \frac{2}{\alpha'} \wedge \frac{p}{\alpha} \wedge \frac{p}{\alpha'} \wedge q$

$$\lim_{n\rightarrow\infty} E\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^{\hat{\beta}} ds = 0.$$

1. L^p-solutions to BSDEs with super-linear growth coefficient-Application to PDEs

Consider the following system of semilinear PDE $(\mathcal{P}^{(g,F)})$

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + F(t,x,u(t,x),\sigma^*\nabla u(t,x)) = 0 \;\; t \in]0,T[, x \in R^k \\ u(T,x) = g(x) \quad x \in R^k \end{cases}$$

where $\mathcal{L} := \dfrac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial^2_{ij} + \sum_i b_i\partial_i, \quad \sigma \in \mathcal{C}^3_b(R^k, R^{kr}), \quad b \in \mathcal{C}^2_b(R^k, R^k).$

Let

$$\mathcal{H}^{1+} := \bigcup_{\delta\geq 0,\beta>1} \left\{ v \in \mathcal{C}([0,T];\mathbb{L}^\beta(R^k, e^{-\delta|x|}dx; R^d)) : \int_0^T \int_{R^k} |\sigma^*\nabla v(s,x)|^\beta e^{-\delta|x|} dxds < \infty \right\}.$$

Definition

A (weak) solution of $(\mathcal{P}^{(g,F)})$ is a function $u \in \mathcal{H}^{1+}$ such that for every $\varphi \in C^1_c$

$$\int_t^T < u(s), \frac{\partial\varphi(s)}{\partial s} > ds + < u(t), \varphi(t) >$$
$$= < g, \varphi(T) > + \int_t^T < F(s,.,u(s),\sigma^*\nabla u(s)), \varphi(s) > ds + \int_t^T < Lu(s), \varphi(s) > ds,$$

where $< f(s), h(s) > = \int_{R^k} f(s,x)h(s,x)dx$.

(A.0) $g(x) \in \mathbb{L}^{\overline{p}}(R^k, e^{-\delta|x|}dx; R^d)$.

(A.1) $F(t,x,.,.)$ is continuous a.e. (t,x)

(A.2) There are $\eta' \in \mathbb{L}^{\frac{\overline{p}}{2}\vee 1}([0,T] \times R^k, e^{-\delta|x|}dtdx; R_+))$, $f^{0'} \in \mathbb{L}^{\overline{p}}([0,T] \times R^k, e^{-\delta|x|}dtdx; R_+))$, and $M, M' \in R_+$ such that

$$\langle y, F(t,x,y,z)\rangle \leq \eta'(t,x) + f^{0'}(t,x)|y| + (M + M'|x|)|y|^2 + \sqrt{M + M'|x|}|y||z|.$$

(A.3) There are $\overline{\eta}' \in \mathbb{L}^q([0,T] \times R^k, e^{-\delta|x|}dtdx; R_+))$ (for some $q > 1$), $\alpha \in]1, \overline{p}[$ and $\alpha' \in]1, \overline{p} \wedge 2[$ such that

$$|F(t,x,y,z)| \leq \overline{\eta}'(t,x) + |y|^\alpha + |z|^{\alpha'}.$$

(A.4) There are $K, r \in R_+$ such that $\forall N \in N$ and every $e^{r|x|}$, $|y|, |y'|, |z|, |z'| \leq N$,

$$\langle y - y'; F(t,x,y,z) - F(t,x,y',z')\rangle \leq K \log N \left(\frac{1}{N} + |y-y'|^2\right) + \sqrt{K \log N}|y-y'||z-z'|.$$

1. Existence and uniqueness of solutions to PDE

Consider the diffusion process with infinitesimal operator $\mathcal{L}$

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \qquad t \leq s \leq T$$

Theorem : Bahlali-Essaky-Hassani

Under assumption **(A.0)**-**(A.4)** we have

1) The PDE $(\mathcal{P}^{(g,F)})$ has a unique solution u on $[0, T]$

2) For all $t \in [0, T]$ there exists $D_t \subset R^k$ such that

$i)$ $\displaystyle\int_{D_t^c} 1 \, dx = 0$

$ii)$ For all $t \in [0, T]$ and all $x \in D_t$ $(E^{(\xi^{t,x}, f^{t,x})})$ has a unique solution $(Y^{t,x}, Z^{t,x})$ on $[t, T]$ where $\xi^{t,x} := g(X_T^{t,x})$ and $f^{t,x}(s, y, z) := 1_{\{s>t\}} F(s, X_s^{t,x}, y, z)$

3) For all $t \in [0, T]$

$$\left(u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x})\right) = \left(Y_s^{t,x}, Z_s^{t,x}\right) \qquad \text{a.e.}(s, x, \omega)$$

1. Existence and uniqueness of solutions to PDE-Idea of the proof

- We Approximate F by a sequence $(F_n)_{n>1}$ of Lipschitz functions :

$$F_n(t,x,y,z) = (n^{2p}e^{|x|})^{(d+dr)}(c_1 e)^2 1_{\{\eta'(t,x)+\overline{\eta}'(t,x)+f^{0'}(t,x)+|x|\leq n\}}\psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)$$
$$\int_{R^d}\int_{R^{dr}} F(t,x,y-u,z-v)\Pi_{i=1}^d \psi(n^{2p}e^{|x|}u_i)\Pi_{i=1}^d\Pi_{j=1}^r \psi(n^{2p}e^{|x|}v_{ij})dudv,$$

- We consider $(Y^{t,x,n}, Z^{t,x,n})$ be the unique solution of BSDE (1) avec $\xi_n^{t,x} := g_n(X_T^{t,x})$ and $f_n^{t,x}(s,y,z) := 1_{\{s>t\}}F_n(s, X_s^{t,x}, y, z)$, with $g_n(x) := g(x)1_{\{|g(x)|\leq n\}}$.
- There exists a unique solution u^n of PDE $(\mathcal{P}^{(g_n,F_n)})$

$$\begin{cases} \dfrac{\partial u^n(t,x)}{\partial t} + \mathcal{L}u^n(t,x) + F_n(t,x,u^n(t,x),\sigma^*\nabla u^n(t,x)) = 0 t \in]0,T[, x \in R^k \\ u^n(T,x) = g_n(x) \quad x \in R^k \end{cases}$$

such that for all t

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n} \quad \text{and} \quad \sigma^*\nabla u^n(s, X_s^{t,x}) = Z_s^{t,x,n} \qquad \text{a.e } (s,\omega,x).$$

- We have the following convergence :

$$\lim_{n,m} \sup_{0\leq t\leq T} \int_{R^k} | u^n(t,x) - u^m(t,x) |^{p'} e^{-\delta'|x|}dx = 0$$
$$\lim_{n,m} \int_0^T \int_{R^k} | \sigma^*\nabla u^n(t,x) - \sigma^*\nabla u^m(t,x) |^{p'\wedge 2} e^{-\delta'|x|}dtdx = 0.$$

1. Existence and uniqueness of solutions to PDE-Idea of the proof

For uniqueness we prove that the system of semilinear PDEs

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)) = 0, & t \in]0,T[,\ x \in R^k \\ u(T,x) = g(x), & x \in R^k \end{cases}$$

has a unique solution if and only if 0 is the unique solution of the linear system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) = 0, & t \in]0,T[,\ x \in R^k \\ u(T,x) = 0, & x \in R^k \end{cases}$$

For Further Reading

K. Bahlali
Existence and uniqueness of solutions for BSDEs with locally Lipschitz coefficient.
Electron. Comm. Probab., **7**, 169–179, 2002.

K. Bahlali, E.H. Essaky, M. Hassani, E. Pardoux
Existence, uniqueness and stability of backward stochastic differential equations with locally monotone coefficient.
C. R. Acad. Sci., Paris **3**35, no. 9, 757–762, 2002.

K. Bahlali, E.H. Essaky, M. Hassani, E. Pardoux
p-integrable solutions to multidimensional BSDEs
and degenerate systems of PDEs with logarithmic nonlinearities.
Preprint.