Backward stochastic partial differential equations driven by infinite dimensional martingales and applications

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Outline

Aims

- To derive the existence and uniqueness of the solutions to:

$$(BSPDE)\begin{cases} -\,dY(t) = \big(\,A(t)\;Y(t) + F(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\,\big)\;dt \\ \qquad\qquad\qquad -Z(t)\;dM(t) - dN(t), \\ Y(T) = \xi. \end{cases}$$

- to provide some applications to the maximum principle for a controlled stochastic evolution system.

Outline

- H is a separable Hilbert space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
- $M \in \mathcal{M}^{2,c}_{[0,T]}(H)$, i.e M is a continuous square integrable martingale in H.
- $< M >$ is the predictable quadratic variation of M.
- $\tilde{\mathcal{Q}}_M$ is the predictable process taking values in the space $L_1(H)$, which is associated with the Doléans measure of $M \otimes M$.
- $<< M >>_t = \int_0^t \tilde{\mathcal{Q}}_M(s)\, d < M >_s$.
- Assume: $\exists$ a predictable process $\mathcal{Q}$ satisfying $\mathcal{Q}(t,\omega)$ is symmetric, positive definite nuclear operator on H and

$$<< M >>_t = \int_0^t \mathcal{Q}(s)\, ds.$$

- ξ $(\xi(\omega) \in H)$ is the terminal value.

- $F : [0, T] \times \Omega \times H \times L_2(H) \to H$ is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$ - measurable.

  $L_2(H)$ is the space of Hilbert-Schmidt operators on H, inner product $\langle \cdot, \cdot \rangle_2$, norm $|| \cdot ||_2$.

- $A(t, \omega)$ is a predictable unbounded linear operator on H.

- The stochastic integral $\int_0^{\cdot} \Phi(s) \, dM(s)$ is defined for Φ s.t. for $(\Phi \circ \tilde{\mathcal{Q}}_M^{1/2})(t, \omega)(H) \in L_2(H)$, for every $h \in H : \Phi \circ \tilde{\mathcal{Q}}_M^{1/2}(h)$ is predictable and

$$\mathbb{E} \left[\int_0^T ||\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}||_2^2 \, d < M >_t \right] < \infty .$$

  The space of integrands $\dashrightarrow \Lambda^2(H; \mathcal{P}, M)$.

Let m be a 1-dimensional, continuous, square integrable martingale with respect to $\{\mathcal{F}_t\}_t$ s.t. $< m >_t = \int_0^t h(s)ds \; \forall \; 0 \leq t \leq T$, some cts $h : [0, T] \to (0, \infty)$.
$M(t) = \beta \, m(t)(= \int_0^t \beta \, dm(s))$, a fixed element $\beta \neq 0$ of H.

$$\Downarrow$$

- $M \in \mathcal{M}^{2,c}(H)$
- $<< M >>_t = \widetilde{\beta \otimes \beta} \; \int_0^t h(s)ds$, where $\widetilde{\beta \otimes \beta}$ is the identification of $\beta \otimes \beta$ in $L_1(H)$: $(\widetilde{\beta \otimes \beta})(k) = \langle \beta, k \rangle \, \beta, \;\; k \in H$.
- $< M >_t = |\beta|^2 \int_0^t h(s) \, ds$.
- $\tilde{\mathcal{Q}}_M = \frac{\widetilde{\beta \otimes \beta}}{|\beta|^2}$.
- Let $\mathcal{Q}(t) = \widetilde{\beta \otimes \beta} \; h(t) \Rightarrow \;\; << M >>_t = \int_0^t \mathcal{Q}(s) \, ds$.
- $\mathcal{Q}(\cdot)$ is bounded since $\mathcal{Q}(t) \leq \mathcal{Q} := \widetilde{\beta \otimes \beta} \; \max_{0 \leq t \leq T} h(t)$.
- $\mathcal{Q}^{1/2}(t)(k) = \frac{\langle \beta, k \rangle \, \beta}{|\beta|} \; h^{1/2}(t)$. In particular $\beta \in \mathcal{Q}^{1/2}(t)(H)$.

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- $L^2_{\mathcal{F}}(0,T;H) := \{\phi : [0,T]\times\Omega \to H,$ predictable, $\mathbb{E}\,[\int_0^T |\phi(t)|^2_H dt\,] < \infty\,\}.$

- $\mathcal{B}^2(H) := L^2_{\mathcal{F}}(0,T;H) \times \Lambda^2(H;\mathcal{P},M).$

  This is a separable Hilbert space, the norm:

$$||(\phi_1,\phi_2)||_{\mathcal{B}^2(H)} = \Big(\mathbb{E}\Big[\int_0^T |\phi_1(t)|^2_H\,dt\Big] + \mathbb{E}\Big[\int_0^T ||\phi_2(t)\tilde{\mathcal{Q}}_M^{1/2}(t)||_2^2\,d<M>_t\Big]\Big)^{1/2}.$$

- (V,H,V') is a rigged Hilbert space:
  ▷ V is a separable Hilbert space embedded continuously and densely in H.

  ▷ By identifying H with its dual $\Rightarrow$ get continuous and dense two inclusions: $V \subseteq H \subseteq V'$, V' is the dual space of V.

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Definition 1

Two elements M and N of $\mathcal{M}^{2,c}_{[0,T]}(H)$ are very strongly orthogonal (VSO) if

$$\mathbb{E}\,[M(u) \otimes N(u)] = \mathbb{E}\,[M(0) \otimes N(0)],$$

for all $[0, T]$ - valued stopping times u.

In fact: M and N are VSO $\Leftrightarrow << M, N >> = 0$.

Definition 2

A solution of the:

$$(BSPDE)\begin{cases} -\,dY(t) = (\,A(t)\;Y(t) + F(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\,)\;dt \\ \qquad\qquad -Z(t)\,dM(t) - dN(t), \quad 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases}$$

is a triple $(Y, Z, N) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$ s.t. $\forall\; t \in [0, T]$:

$$\begin{aligned} Y(t) \;&=\; \xi + \int_t^T (\,A(s)\,Y(s) + F(s, Y(s), Z(s)\mathcal{Q}^{1/2}(s))\,)\;ds \\ &\quad - \int_t^T Z(s)\,dM(s) - \int_t^T dN(s), \end{aligned}$$

$N(0) = 0$ and N is VSO to M.

Outline

- (A1) $F : [0, T] \times \Omega \times H \times L_2(H) \to H$ is a mapping such that the following properties are verified.

  (i) F is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$ - measurable.

  (ii) $\mathbb{E}\,[\int_0^T |F(t,0,0)|^2\, dt\,] < \infty$, where $F(t,0,0) = F(t,\omega,0,0)$.

  (iii) $\exists\, k_1 > 0$ such that $\forall\, y, y' \in H,\ \forall\, z, z' \in L_2(H)$

  $$|F(t,\omega,y,z) - F(t,\omega,y',z')|^2 \leq k_1\, (\,|y-y'|^2 + ||z-z'||_2^2\,),$$

  uniformly in (t,ω).

- (A2) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$.

- (A3) There exists a predictable process $\mathcal{Q}$ satisfying $\mathcal{Q}(t,\omega)$ is symmetric, positive definite nuclear operator on H and

$$<< M >>_t = \int_0^t \mathcal{Q}(s)\, ds.$$

- (A4) Every square integrable H-valued martingale with respect $\{\mathcal{F}_t\,,\ 0 \le t \le T\}$ has a continuous version.

- (A5) $A(t,\omega)$ is a linear operator on H, $\mathcal{P}$ - measurable, belongs to $L(V;V')$ uniformly in (t,ω) and satisfies the following conditions:

  (i) $A(t,\omega)$ satisfies the coercivity condition in the sense that

  $$2\,[A(t,\omega)\,y\,,y] + \ \alpha\,|y|_V^2 \le \lambda\,|y|_H^2 \quad a.e.\ t \in [0,T]\,, \ \ a.s.\ \forall\, y \in V,$$

  for some $\alpha, \lambda > 0$.

  (ii) $A(t,\omega)$ is uniformly continuous, i.e. $\exists\ k_3 \ge 0$ such that for all (t,ω)

  $$|\,A(t,\omega)\,y\,|_{V'} \le k_3\,|y|_V\,,$$

  for every $y \in V$.

Outline

Theorem 1

Assume (A1)–(A5). Then there exists a unique solution (Y, Z, N) of the

$$(BSPDE)\begin{cases} -\,dY(t) = \big(A(t)\;Y(t) + F(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\big)\,dt \\ \qquad\qquad -Z(t)\,dM(t) - dN(t), \quad 0 \le t \le T, \\ Y(T) = \xi \end{cases}$$

in $L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$.

Outline

We shall divide the proof of Theorem 1 into different cases:

Lemma 1

Suppose that $F \in L^2_{\mathcal{F}}(0, T; H)$ and (A2)–(A5) hold. Then

$$\begin{aligned} Y(t) &= \xi + \int_t^T (\, A(s)\, Y(s) + F(s)\,)\, ds \\ &\quad - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s) \end{aligned}$$

attains a unique solution:

$$(Y, Z, N) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H).$$

Proof of Lemma 1

▷ The proof of Lemma 1 is achieved through the method of Galerkin's finite dimensional approximation, e.g. by following Pardoux and Rozovskii.

▷ It can be found in

Al-Hussein, A., Backward stochastic partial differential equations driven by infinite dimensional martingales and applications, Stochastics, Vol. 81, No. 6, 2009, 601-626.

Lemma 2

Assume that (A2)–(A5) hold and F satisfies:

- (A1)′ $F : [0, T] \times \Omega \times L_2(H) \to H$ is a mapping s.t.

  (i) F is $\mathcal{P} \otimes \mathcal{B}(L_2(H))\mathcal{B}(H)$ - measurable.

  (ii) $\mathbb{E}\,[\,\int_0^T |F(t, 0, 0)|^2\, dt\,] < \infty$.

  (iii) $\exists\ k_2 > 0$ such that $\forall\ z, z' \in L_2(H)$

  $$|F(t, \omega, z) - F(t, \omega, z')|^2 \leq k_2\,(\,|y - y'|^2 + |z - z'||_2^2\,),$$

  uniformly in (t, ω).

Then there exists a unique solution (Y, Z, N) of the BSPDE:

continued Lemma 2

$$
\begin{aligned}
Y(t) &= \xi + \int_t^T (\, A(s)\, Y(s) + F(s, Z(s)\mathcal{Q}^{1/2}(s))\,)\, ds \\
&\quad - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s)
\end{aligned}
$$

in $L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}(H)$.

Final step:

- We establish the existence of solutions to our original (BSPDE).
- Let $Y_0 \equiv 0$, define recursively using Lemma 2 the BSPDE:

$$\begin{aligned} Y_n(t) &= \xi + \int_t^T \big(A(s)\, Y_n(s) + F(s, Y_{n-1}(s), Z_n(s)\mathcal{Q}^{1/2}(s)) \big)\, ds \\ &\quad - \int_t^T Z_n(s)\, dM(s) - \int_t^T dN_n(s), \;\; 0 \leq t \leq T, \end{aligned}$$

for $n \geq 1$.

- The solutions (Y_n, Z_n, N_n) lie in

$$L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}(H) \;\text{ for each }\; n \geq 1.$$

continued final step:

▷ We show $\{Y_n\}$, $\{Z_n\}$ and $\{N_n\}$ are Cauchy sequences in $L^2_{\mathcal{F}}(0,T;V)$, $\Lambda^2(H;\mathcal{P},M)$ and $\mathcal{M}^{2,c}(H)$, respectively.

▷ Let Y, Z and N denote the limits of these sequences.

▷ Then we show the very strong orthogonality between N and M.

▷ Now this convergence together with (A1) and (A5)(ii) allows us to let $n \to \infty$ in the previous sequence of BSPDEs to obtain:

$$\begin{aligned} Y(t) &= \xi + \int_t^T (\, A(s)\, Y(s) + F(s, Y(s), Z(s))\,)\, ds \\ &- \int_t^T Z(s)\, dM(s) - \int_t^T dN(s), \quad 0 \leq t \leq T. \end{aligned}$$

▷ Hence (Y, Z, N) is a solution to (BSPDE).

Outline

- These types of BSPDEs seem to be fresh!

- Yong, J. and Zhou, X. Y. [Springer 1999] insist on the condition:
  " $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $W(t)$, argumented by all the $\mathbb{P}$ - null sets in $\mathcal{F}$ "
  to study the adjoint equation of SDEs.

- Useful for studying the stochastic maximum principle for infinite dimensional controlled stochastic evolution systems.

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Outline

Consider the following stochastic evolution equation (SEE):

$$\begin{cases} dX^{v(\cdot)}(t) = (A(t)\, X^{v(\cdot)}(t) + \Psi(X^{v(\cdot)}(t), v(t))\,)\, dt + G(X^{v(\cdot)}(t))\, dM(t), \\ X^{v(\cdot)}(0) = x \in H. \end{cases} \quad (1)$$

- $\nu : [0, T] \times \Omega \to U$ (U is a sep. Hilbert space) is admissible if $\nu \in L^2_{\mathcal{F}}(0, T; U)$.

  The set of admissible controls $\mathcal{U}_{ad}$.

- The cost functional:

$$J(x, \nu(\cdot)) := \mathbb{E}\, [\, \int_0^T g(X^{\nu(\cdot)}(t), \nu(t)\,)\, dt + \phi(X^{\nu(\cdot)}(T))\,]. \quad (2)$$

- Define

$$J^*(x) := \inf\{J(x, \nu(\cdot)) : \ \nu(\cdot) \in \mathcal{U}_{ad}\}. \quad (3)$$

The control problem for this (SEE) is to find a control $\nu^*(\cdot)$ and the corresponding solution $X^{\nu^*(\cdot)}$ s.t.

$$J^*(x) = J(x, \nu^*(\cdot)). \tag{4}$$

- $\nu^*(\cdot)$ is an optimal control,
- $X^{\nu^*(\cdot)}$ is an optimal solution,
- J^* is the value function,
- the pair $(X^{\nu^*(\cdot)}, \nu^*(\cdot))$ is called an optimal pair of the stochastic control problem (1)-(4).

The adjoint equation

Define the Hamiltonian

$$\mathcal{H} : [0,T] \times H \times U \times H \times L_2(H) \to \mathbb{R},$$

$$\mathcal{H}(t,x,\nu,y,z) := -g(x,\nu) + \left\langle \Psi(x,\nu)\,, y \right\rangle + \left\langle G(x)\mathcal{Q}^{1/2}(t)\,, z \right\rangle_2 . \quad (5)$$

The adjoint equation:

$$\begin{cases} -\,dY^{\nu(\cdot)}(t) = & [\,A^*(t)\,Y^{\nu(\cdot)}(t) + \nabla_x \mathcal{H}(X^{\nu(\cdot)}(t), \nu(t), Y^{\nu(\cdot)}(t), Z^{\nu(\cdot)}(t))\,]\,dt \\ & \qquad -Z^{\nu(\cdot)}(t)\,dM(t) - dN^{\nu(\cdot)}(t), \\ Y^{\nu(\cdot)}(T) = & -\nabla\phi(X^{\nu(\cdot)}(T)). \end{cases} \quad (6)$$

$A^*(t)$ is the adjoint operator of $A(t)$.

Theorem 2

Given $\nu^*(\cdot) \in \mathcal{U}_{ad}$ assume $\exists$ unique solutions $X^{\nu^*(\cdot)}$, $(Y^{\nu^*(\cdot)}, Z^{\nu^*(\cdot)}, N^{\nu^*(\cdot)})$ of the corresponding SEE (1) and its adjoint equation (6).

Suppose that

(i) Ψ, G, g and ϕ are given C_b^1 mappings and ϕ is convex,

(ii) $\mathcal{H}(t, \cdot, \cdot, Y^{\nu^*(\cdot)}(t), Z^{\nu^*(\cdot)}(t))$ is concave for all $t \in [0, T]$ - a.s.,

(iii)
$$\mathcal{H}(t, X^{\nu^*(\cdot)}(t), \nu^*(t), Y^{\nu^*(\cdot)}(t), Z^{\nu^*(\cdot)}(t)) = \max_{\nu \in U} \mathcal{H}(t, X^{\nu^*(\cdot)}(t), \nu, Y^{\nu^*(\cdot)}(t), Z^{\nu^*(\cdot)}(t))$$

for a.e. $t \in [0, T]$ - a.s.

Then $(X^{\nu^*(\cdot)}, \nu^*(\cdot))$ is an optimal pair for the problem (1)-(4).

Proof.

The proof of Theorem 2 can be found in the paper:

Al-Hussein, A., Maximum principle for controlled stochastic evolution equations, Warwick-preprint, 2009.

An example

▷ Let $H = L^2(\mathbb{R}^n)$, $V = \mathbb{H}^1(\mathbb{R}^n)$ and $V' = \mathbb{H}^{-1}(\mathbb{R}^n)$.

▷ M is the continuous martingale given in Example 1, $\beta \neq 0$ a fixed element of H.

▷ U is a separable Hilbert space (space of controls).

▷ Assume $\tilde{F} : U \to H$ is a bounded linear operator.

▷ Consider the SEE:

$$\begin{cases} dX(t) = (A\,X(t) + \tilde{F}\,\nu(t))\,dt + \langle X(t)\,,\beta\rangle\,dM(t), \\ X(0) = x \in H. \end{cases} \tag{7}$$

$A = \frac{1}{2}\Delta.$

▷ Given a function $c : \mathbb{R}^n \to \mathbb{R}$ of H assume the cost functional:

$$J(x, \nu(\cdot)) := \mathbb{E}\,[\int_0^T |\nu(t)|_U^2\, dt\,] + \mathbb{E}\,[\,\langle c\,, X(T)\rangle_H\,] \tag{8}$$

and the value function:

$$J^*(x) := \inf\{J(x, \nu(\cdot)) :\ \nu \in \mathcal{U}_{ad}\}. \tag{9}$$

▷ Take for $(x, \nu) \in H \times U$:

$$\Psi(x, \nu) = \tilde{F}\,\nu,\ \ G(x) = \langle x\,, \beta\rangle,$$

$$g(x, \nu) = |\nu|^2,\ \ \phi(x) = \langle c\,, x\rangle.$$

- The Hamiltonian is:

$$\mathcal{H} : [0,T] \times H \times U \times H \times L_2(H) \to \mathbb{R},$$

$$\mathcal{H}(t,x,\nu,y,z) = -|\nu|_U^2 + \langle \tilde{F}\,\nu\,,y\rangle + \langle x\,,\beta\rangle\,\langle \mathcal{Q}^{1/2}(t)\,,z\rangle_2\,,$$
$$(t,x,\nu,y,z) \in [0,T] \times H \times U \times H \times L_2(H).$$

- Consider the adjoint BSPDE:

$$\begin{cases} -\,dY(t) = [\,\tfrac{1}{2}\Delta\,Y(t) + \langle \mathcal{Q}^{1/2}(t)\,,Z(t)\,\mathcal{Q}^{1/2}(t)\rangle_2\,\beta\,]\,dt \\ \qquad\qquad -Z(t)\,dM(t) - dN(t), \\ Y(T) = -\,c. \end{cases} \quad (10)$$

- The assumptions of Theorem 1 are satisfied for this BSPDE.

- Consequently it has a unique solution (Y, Z, N).

- Since $Y(T)$ is non-random we can choose $Z(t) = 0$ and $N(t) = 0$ for each $t \in [0, T]$. So:

$$\begin{cases} \frac{\partial}{\partial t} Y(t) = -\frac{1}{2} \Delta\, Y(t) \\ Y(T) = -c. \end{cases}$$

- Thus $Y(t) = -S(T-t)\, c$, where

$$(S(r)\, c)(\sigma) = \frac{1}{(2\pi r)^{n/2}} \int_{\mathbb{R}^n} c(x)\, e^{\left(\frac{-|\sigma - x|^2}{2r}\right)}\, dx\,, \quad \sigma \in \mathbb{R}^n, r > 0.$$

- By uniqueness of solutions of the BSPDE (10) this triple $(Y, 0, 0)$ is actually its unique solution.

▷ Now for fixed (t, x, y, z),
$U \ni \nu \mapsto H(t, x, \nu, y, z) \in \mathbb{R}$ takes its maximum at $\nu = -\frac{1}{2}\tilde{F}^* y$.

▷ Then we can let

$$\nu^*(t,\omega) = -\frac{1}{2}\tilde{F}^*\, Y(t,\omega)\ (\in U), \tag{11}$$

as a candidate for an optimal control.

▷ With this choice all the requirements in Theorem 2 are verified.
$\Rightarrow$ this candidate (11) is an optimal control for the problem (7)-(9).

▷ An optimal solution $\hat{X}$ is given by the solution of the equation:

$$\begin{cases} d\hat{X}(t) = (A\,\hat{X}(t) - \frac{1}{2}\tilde{F}\,\tilde{F}^*\,Y(t))\,dt + \langle \hat{X}(t)\,,\beta\rangle\, dM(t), \\ \hat{X}(0) = x \in H. \end{cases}$$

▷ The value function takes then the formula:

$$J^*(x) = \mathbb{E}\,[\int_0^T |-\frac{1}{2}\tilde{F}^*\,Y(t,\omega)|_U^2\, dt\,] + \mathbb{E}\,[\,\langle c\,,\hat{X}(T)\rangle\,].$$