Maximum Principle of State-Constraint Optimal Control Governed by Navier-Stokes Equations in 2-D

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INTRODUCTION

In this work we consider the optimal problem:

$$\mathbf{Min}\frac{1}{2}\int_0^T \left(|y(t)-y^0(t)|^2+|u(t)|_U^2\right)dt; \tag{1}$$

$y(t)\in K, K$ is a closed convex subset in H. Here $y^0(t)\in L^2(0,T;H)$, and $(y(t),u(t))$ is the solution to the following equation:

$$\begin{cases} y'(t)+\nu Ay(t)+By(t)=Du(t)+f, \\ y(0)=y_0 \end{cases} \tag{2}$$

$f(t)\in L^2(0,T;H), u(t)\in L^2(0,T;U), y_0\in V$

INTRODUCTION

$$H = \{y(t); y(t) \in (L^2(\Omega))^2, \nabla \cdot y(t) = 0, y(t) \cdot \mathbf{n} = 0 \ on \ \partial\Omega\}$$
$$V = \{y(t); y(t) \in (H_0^1(\Omega))^2, \nabla \cdot y(t) = 0\}$$

and $V'$is the dual space of V, $D(A) = (H^2(\Omega))^2 \cap V$, Ω is a bounded open subset with smooth boundary in $\mathbf{R^2}$, $\mathbf{n}$ is the outward vector to $\partial\Omega$ and

$$A = -P\triangle, By = P[(\nabla \cdot y)y]$$

where P is the projection to H. We shall denote by the symbol $|\cdot|$ the norm in $\mathbf{R^2}$, H and $(L^2(\Omega))^2$, and $\|\cdot\|$ the norm of the space V. Define the trilinear function $b(y, z, w)$ by

$$b(y, z, w) = \int_\Omega \sum_{i,j=1}^{2} y_i D_i z_j w_j dx, \ \forall y, z, w \in V$$

U is a Hilbert space and $D \in L(U, H)$. We denote by $|\cdot|_U$ the norm in U, and $(\cdot, \cdot)_U$ the scalar product in U.

INTRODUCTION

Lemma

(1) $b(y,z,w) = -b(y,w,z)$ *and there exists a positive constant* C*, s.t.*

$$|b(y,z,w)| \leq C\|y\|_{m_1}\|z\|_{m_2+1}\|w\|_{m_3}$$

where m_1, m_2, m_3 *are positive number, satisfy the inequality:*

$$m_1 + m_2 + m_3 \geq 1, m_i \neq 1$$

$$m_1 + m_2 + m_3 > 1, \exists m_i = 1$$

(2)there exists a positive constant C*, s.t.*

$$\|y\|_m \leq C\|y\|_l^{1-\alpha}\|y\|_{l+1}^{\alpha}$$

where $\alpha = m - l \in (0,1)$*. Here* $\|\cdot\|_{m_i}$ *denotes the norm of the space* $H^{m_i}(\Omega)$*.*

INTRODUCTION

Definition

Let E be a Banach space, and E^* is it's dual space.
$\forall \omega(t) \in BV(0, T; E^*)$, we define the continuous functional μ_ω on $C([0, T]; E)$ by

$$\mu_\omega(z(t)) = \int_0^T (z(t), d\omega(t))_{(E,E^*)}$$

$(\cdot, \cdot)_{(E,E^*)}$ denotes the dual product between E and E^*, the integral is the Riemann-Steiljes integral. Denote $M(0, T; E^*)$ the dual space of $C([0, T]; E)$. For the closed convex subset K in E, denote $\mathcal{K}$ by $\mathcal{K} = \{y(t) \in C([0, T]; E); y(t) \in E, \forall t \in [0, T]\}$, and the normal cone of $\mathcal{K}$ on $y(t)$ is

$$\mathcal{N}_{\mathcal{K}}(y(t)) = \{\mu \in M(0, T; E^*); \mu(y(t) - z(t)) \geq 0, \forall z(t) \in \mathcal{K}\}$$

INTRODUCTION

The main results of this work is about the maximum principle of the optimal control problem governed by Navier-Stokes equations with state constraint in 2-D. To get the results, we make some assumptions as following:
(A) $\exists\ \tilde{z}(t), \tilde{u}(t)$ such that $\tilde{z}(t) \in \mathrm{int}K$, for t in a dense subset of $[0, T]$, where $\tilde{z}(t), \tilde{u}(t)$ satisfies the following equation:

$$\begin{cases} \tilde{z}'(t) + \nu A\tilde{z}(t) + (B'(y^*(t)))\tilde{z}(t) = B(y^*(t)) + D\tilde{u}(t) + f(t), \\ \tilde{z}(0) = y_0 \end{cases} \tag{3}$$

Here $y^*(t)$ is the optimal state function for the optimal control problem (1),(2).
(A') $\exists\ \tilde{z}(t), \tilde{u}(t)$ such that $\tilde{z}(t) \in \mathrm{int}_V K$, for t in a dense subset of $[0, T]$, where $\tilde{z}(t), \tilde{u}(t)$ satisfies the equation (3)

MAIN RESULTS

Theorem

Suppose that the pair $(y^*(t), u^*(t))$ is solution for optimal control problem (1),(2). Then under the assumption (A), there are $p(t) \in L^\infty(0, T; H)$ and $\omega(t) \in BV(0, T; H)$, such that:

$$D^* p(t) = u^*(t) \quad a.e.[0, T] \tag{4}$$

where $p(t)$ satisfies the following equation

$$\begin{cases} p'(t) = \nu A p(t) + (B'(y^*(t))^*) p(t) + y^*(t) - y^0(t) + d\omega(t), \\ p(T) = 0 \end{cases} \tag{5}$$

MAIN RESULTS

Theorem

The latter equation holds in the sense of

$$\int_t^T \langle p'(s) - \nu Ap(s) - (B'(y^*(s))^*)p(s), \psi(s)\rangle ds$$

$$= \int_t^T \langle y^*(s) - y^0(s), \psi(s)\rangle ds + \int_t^T \langle d\omega(s), \psi(s)\rangle$$

$\forall \psi(t) \in C^1([0, T]; D(A))$. *Moreover,*

$$\mu_\omega \in \mathcal{N}_{\mathcal{K}}(y^*(t)) \tag{6}$$

where μ_ω *and* $\mathcal{N}_{\mathcal{K}}(y^*(t))$ *are defined as in definition 1 in the case that* $E = H$. *Here* $B'(y)$ *is the operator defined by*

$$\langle B'(y)z, w\rangle = b(y, z, w) + b(z, y, w), \ \forall \ z, w \in V$$

MAIN RESULTS

Theorem

Suppose the pair $(y^*(t), u^*(t))$ is the solution for optimal control problem (1),(2), then under (A') there are $p(t) \in L^\infty(0, T; V')$, $\omega(t) \in BV(0, T; V')$, such that (4) holds, and (5) holds in the sense of

$$\int_t^T (p'(s) - \nu A p(s) - (B'(y^*(s))^*) p(s), \psi(s))_{(V',V)} ds$$

$$= \int_t^T (y^*(s) - y^0(s), \psi(s))_{(V',V)} ds + \int_t^T (d\omega(s), \psi(s))_{(V',V)}$$

$\forall \psi(t) \in C^1([0, T]; D(A))$, *here* $(\cdot, \cdot)_{(V',V)}$ *is the dual product between* V' *and* V. *Moreover,(6) also holds, where* μ_ω *and* $\mathcal{N}_{\mathcal{K}}(y^*(t))$ *are defined as in definition 1 in the case that* $E = V$

PROOF

Before the proof of the two theorems we define the approximating cost function to the original one $F(y,u)$ which is defined by (1) as

$$F_\varepsilon(y,u)=\int_0^T \frac{1}{2}[|y(t)-y^0(t)|^2+|u(t)|^2+|u(t)-u^*(t)|_U^2]+\varphi_\varepsilon(y_\varepsilon(t))dt \tag{7}$$

where $\varphi_\varepsilon(y)$ is the regularization of φ, which is the characteristic function of K, and the function $\varphi_\varepsilon(y)$ is defined by

$$\varphi_\varepsilon(y)=inf\{\frac{|y-x|^2}{2\varepsilon}+\varphi(x);x\in H\} \tag{8}$$

Define

$$\mathscr{C}=\{(y,u)\in C([0,T];H)\times L^2(0,T;U);(y(t),u(t))\text{is the solution to (2)}\}$$

.

PROOF

Lemma

There exists at least one optimal pair for the optimal control problem:

$$\textbf{Min}\{F_\varepsilon(y,u);(y,u)\in\mathscr{C}\} \tag{9}$$

Lemma

Suppose $z_\varepsilon(t)$ *is the solution to the equation:*

$$\begin{cases} z_\varepsilon'(t)+\nu Az_\varepsilon(t)+(B'(y_\varepsilon(t)))z_\varepsilon(t)=B(y_\varepsilon(t))+D\tilde{u}(t)+f(t), \\ z_\varepsilon(0)=y_0 \end{cases} \tag{10}$$

then $z_\varepsilon(t)\longrightarrow\tilde{z}(t)$ *strongly in* $C([0,T];H)\cap L^2(0,T;V)$*, where* $\tilde{z}((t),\tilde{u}((t)$ *is defined in equation (3), and* $y_\varepsilon(t)$ *is the the optimal solution in lemma 2.*

PROOF

Proof of theorem 1:
step 1:(first order necessary condition for approximate problem)
Since $(y_\varepsilon, u_\varepsilon)$ minimize the functional $F_\varepsilon(y,u)$, we know that

$$\lim_{h\to 0} \frac{F_\varepsilon(u_\varepsilon + hu) - F_\varepsilon(u_\varepsilon)}{h} = 0, \quad \forall u \in U$$

and this yields

$$\langle y_\varepsilon - y^0, w_\varepsilon\rangle + (u_\varepsilon, u)_U + (u_\varepsilon - u^*, u)_U + \langle \partial\varphi_\varepsilon(y_\varepsilon), w_\varepsilon\rangle = 0 \quad (11)$$

where $w_\varepsilon = \lim_{h\to 0} \frac{y_\varepsilon^h - y_\varepsilon}{h}$, $(y_\varepsilon^h, u_\varepsilon + hu) \in \mathscr{C}$ and $w_\varepsilon(t)$ is the solution to the equation

$$w_\varepsilon'(t) + \nu A w_\varepsilon(t) + B'(y_\varepsilon(t))w_\varepsilon(t) = Du, \;\; w_\varepsilon(0) = 0 \qquad (12)$$

PROOF

suppose $p_\varepsilon(t)$ is the solution to the backward equation

$$\begin{cases} p'_\varepsilon(t) = \nu A p_\varepsilon(t) + (B'(y_\varepsilon(t))^*) p_\varepsilon(t) + y_\varepsilon(t) - y^0(t) + \partial\varphi_\varepsilon(y_\varepsilon(t)) \\ p_\varepsilon(T) = 0 \end{cases} \tag{13}$$

By (11) together with (12),(13), we get by calculation that

$$\langle p'_\varepsilon(t), w_\varepsilon(t)\rangle + \langle -Ap_\varepsilon(t) - (B'(y_\varepsilon(t))^*) p_\varepsilon(t), w_\varepsilon(t)\rangle + (u_\varepsilon - u^*, u)_U = 0.$$

Hence we have

$$(-D^* p_\varepsilon(t) + 2u_\varepsilon - u^*, u)_U = 0, \quad \forall u \in U$$

so, we get

$$D^* p_\varepsilon(t) = 2u_\varepsilon(t) - u^*(t), \ \ a.e. \ t \in [0, T] \tag{14}$$

PROOF

step 2: (pass $(y_\varepsilon, u_\varepsilon)$ to limit) By lemma 2,
$\exists (y_\varepsilon, u_\varepsilon) \in \mathscr{C}, s.t.\ F_\varepsilon(u_\varepsilon, y_\varepsilon) = \inf F_\varepsilon(u, y) = d_\varepsilon.$ since

$$F_\varepsilon(y_\varepsilon, u_\varepsilon) \leq F_\varepsilon(y^*, u^*) = F(y^*, u^*) = d$$

so $|d_\varepsilon| \leq C, \forall \varepsilon > 0$,hence

$$\|u_\varepsilon\|_{L^2(0,T;H)} \leq C \tag{15}$$

Multiply the equation

$$y_\varepsilon'(t) + \nu A y_\varepsilon(t) + B y_\varepsilon(t) = D u_\varepsilon(t) + f(t) \tag{16}$$

by $y_\varepsilon(t)$, $Ay_\varepsilon(t)$, integrate from 0 to t, we get

PROOF

$$\|y_\varepsilon(t)\|^2+\int_0^T |Ay_\varepsilon(t)|^2dt+\int_0^T |By_\varepsilon(t)|^2dt+\int_0^T |(y_\varepsilon(t))'|^2dt \leq C \tag{17}$$

hence, on a subsequence convergent to 0, again denoted by λ, we have

$$\begin{aligned} y_\varepsilon(t) \to y_1(t) \ strongly \ in \ C([0,T;H]) \cap L^2(0,T;V) \\ Ay_\varepsilon(t) \to Ay_1(t), (y_\varepsilon(t))' \to y_1'(t) \ weakly \ in \ L^2(0,T;H) \\ u_\varepsilon(t) \to u_1(t) \ weakly \ in \ L^2(0,T;U) \\ By_\varepsilon(t) \to By_1(t) \ strongly \ in \ L^2(0,T;H) \end{aligned}$$

so $(y_1(t), u_1(t))$ is a solution to equation (2)

PROOF

$$\varphi_(y_\varepsilon) = \frac{\varepsilon}{2}|\partial\varphi_\varepsilon(y_\varepsilon)|^2 + \varphi(J_\varepsilon^\varphi(y_\varepsilon)) \geq \frac{\varepsilon}{2}|\partial\varphi_\varepsilon(y_\varepsilon)|^2$$

so $\{\varepsilon|\partial\varphi_\varepsilon(y_\varepsilon)|^2\}$ is bounded in $L^1(0,T)$ and since $\partial\varphi_\varepsilon(y_\varepsilon){=}\frac{1}{\varepsilon}(y_\varepsilon - J_\varepsilon^\varphi(y_\varepsilon))$, where $J_\varepsilon^\varphi(y_\varepsilon)$ is the function satisfies $J_\varepsilon^\varphi(y_\varepsilon) - y_\varepsilon + \partial\varphi_\varepsilon(J_\varepsilon^\varphi(y_\varepsilon)) \ni 0$, we have

$$\int_0^T |y_\varepsilon - J_\varepsilon^\varphi(y_\varepsilon)|dt \leq \varepsilon T \int_0^T \varepsilon|\partial\varphi_\varepsilon(y_\varepsilon)|^2 dt \longrightarrow 0 \ as \ \varepsilon \longrightarrow 0$$

so $y_\varepsilon - J_\varepsilon^\varphi(y_\varepsilon) \longrightarrow 0 \ a.e. \ (0,T)$.since $J_\varepsilon^\varphi(y_\varepsilon) \in K, \ \forall t \in [0,T]$, so $y_1(t) \in K. \ \forall t \in [0,T]$, Inasmuch as

$$\liminf_{\varepsilon\longrightarrow 0} F_\varepsilon(y_\varepsilon, u_\varepsilon) \leq \lim_{\varepsilon\longrightarrow 0} F_\varepsilon(y^*, u^*) = F(y^*, u^*)$$

we have $u_1 = u^*, y_1 = y^*$ and $u_\varepsilon \longrightarrow u^*$ strongly in $L^2(0,T;H)$.

PROOF

step3: (pass $\partial\varphi_\varepsilon(y_\varepsilon), p_\varepsilon$ to limit) by assumption (A) and lemma 3, we know, $\exists\rho > 0, \varepsilon_0 > 0 \ s.t. \ z_\varepsilon(t) + \rho h \in K$, for t in a dense subset of [0,T], $\forall |h| = 1, \forall\varepsilon < \varepsilon_0$. For ε fixed, $z_\varepsilon(t)$ is continuous in $[0, T]$, so there exists a partition $\{t_i\}_{i=1}^N$ of $[0, T]$, $s.t.$ $|z_\varepsilon(t_i) - z_\varepsilon(t_{i-1})| < \frac{\rho}{2}, \ z_\varepsilon(t_i) + \rho h \in K, \forall 1 \le i \le N$. Since

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t_i) - \rho h\rangle dt$$

$$\geq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \varphi_\varepsilon(y_\varepsilon(t)) - \varphi_\varepsilon(z_\varepsilon(t_i) + \rho h) dt \geq 0$$

so $\rho \int_0^T |\partial\varphi_\varepsilon(y_\varepsilon)| dt \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t_i)\rangle dt$

$$= \int_0^T \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t)\rangle dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), z_\varepsilon(t) - z_\varepsilon(t_i)\rangle dt$$

PROOF

$$\frac{\rho}{2}\int_0^T |\partial\varphi_\varepsilon(y_\varepsilon)|dt \le \int_0^T \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t)\rangle dt$$

$$= \int_0^T \langle 2u_\varepsilon(t) - u^*, \tilde{u}(t) - u_\varepsilon(t)\rangle - \langle y_\varepsilon(t) - y^0(t), y_\varepsilon(t) - z_\varepsilon(t)\rangle dt \le C \tag{18}$$

we set $\omega_\varepsilon(t) = \int_0^t \partial\varphi_\varepsilon(y_\varepsilon(s))ds,\ \ t \in [0, T]$, by (18) we see that there exists a function $\omega(t) \in BV([0, t]; H)$, and a sequence convergent to 0, again denoted by λ, $s.t.$ $\omega_\varepsilon(t) \to \omega(t)$ weakly in H for every $t \in [0, T]$, and $\forall y(s) \in C([t, T]; H), \forall t \in [0, T]$.

$$\int_t^T \langle \partial\varphi_\varepsilon(y_\varepsilon(s)), y(s)\rangle ds = \int_t^T \langle d\omega(s), y(s)\rangle. \tag{19}$$

PROOF

Multiply equation 13) by $signp_\varepsilon(t) = \frac{p_\varepsilon(t)}{|p_\varepsilon(t)|}$, we have

$$\frac{d}{dt}|p_\varepsilon(t)| = \frac{\nu\|p_\varepsilon(t)\|^2}{|p_\varepsilon(t)|}$$

$$+\frac{b(p_\varepsilon(t), y_\varepsilon(t), p_\varepsilon(t))}{|p_\varepsilon(t)|} + \frac{\langle y_\varepsilon(t) - y^0(t), p_\varepsilon(t)\rangle}{|p_\varepsilon(t)|} + \frac{\langle \partial\varphi_\varepsilon(y_\varepsilon), p_\varepsilon(t)\rangle}{|p_\varepsilon(t)|}$$

since$|b(p_\varepsilon(t), y_\varepsilon(t), p_\varepsilon(t))| \leq C|p_\varepsilon(t)|\|p_\varepsilon(t)\|\|y_\varepsilon(t)\|$, we get

$$\frac{d}{dt}|p_\varepsilon(t)| \geq \frac{\nu\|p_\varepsilon(t)\|^2}{|p_\varepsilon(t)|} - C\frac{\|p_\varepsilon(t)\|\sqrt{|p_\varepsilon(t)|}}{\sqrt{|p_\varepsilon(t)|}} - |y_\varepsilon(t) - y^0(t)| - |\partial\varphi_\varepsilon(y_\varepsilon)|$$

integrate from 0 to t, by (18) and using Young's inequality

$$|p_\varepsilon(t)| + \frac{\nu}{2}\int_t^T \|p_\varepsilon(s)\|ds \leq C_1 + C_2\int_t^T |p_\varepsilon(s)|ds$$

By Gronwall's inequality, we know that $\|p_\varepsilon(t)\|_{L^\infty(0,T;H)} < C$, by Alaoglu's theorem,

$$p_\varepsilon(t) \to p(t) \quad w^* - L^\infty(0, T; H)$$

PROOF

so $\forall \psi(t) \in C^1(0, T; D(A))$, multiply the equation (13) by $\psi(t)$, letting ε pass to 0, we have

$$\int_t^T \langle p(s), -\psi'(s) - \nu A\psi(s) - B'(y^*(s))\psi(s)\rangle ds - \langle p(t), -\psi(t)\rangle$$

$$= \int_t^T \langle \psi(s), y^*(s) - y^0(s)\rangle ds + \int_t^T \langle \psi(s), d\omega(s)\rangle$$

so $p(t)$ satisfies the equation (4), and (5) also holds by passing ε to 0. Since

$$\int_0^T \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z(t)\rangle dt \geq \varphi_\varepsilon(y_\varepsilon(t)) - \varphi_\varepsilon(z(t)) \geq 0$$

$\forall z(t) \in \mathcal{K}$, by (19), pass ε to 0, we get

$$\int_0^T \langle d\omega(t), y^*(t) - z(t)\rangle \geq 0$$

i.e. $\mu_\omega \in \mathcal{N}_{\mathcal{K}}(y^*(t))$. the proof is completed. ♯

PROOF

Lemma

The solution to equation (10) $z_\varepsilon(t)$ *convergent to the solution to equation (3)* $\tilde{z}(t)$ *in* $C([0,T];V)$. $y_\varepsilon(t) \longrightarrow y^*(t)$ *strongly in* $C([0,T];V)$

PROOF

Proof of Th.2: By (A') and lemma 4, $\exists \rho, \varepsilon_0, s.t. z_\varepsilon(t) + \rho h \in K$, for t in a dense subset of [0,T], $\forall \varepsilon < \varepsilon_0, \|h\| = 1$. For ε fixed, $\exists$ a partition of $[0, T]$, $s.t.$ $\|z_\varepsilon(t_i) - z_\varepsilon(t_{i-1})\| < \frac{\rho}{2}$, $z_\varepsilon(t_i) + \rho h \in K$.

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t_i) - \rho h \rangle_{(V',V)} dt$$

$$\geq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \varphi_\varepsilon(y_\varepsilon(t)) - \varphi_\varepsilon(z_\varepsilon(t_i) + \rho h) dt \geq 0$$

$$\rho \int_0^T \|\partial\varphi_\varepsilon(y_\varepsilon)\|_{V'} dt \leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t_i) \rangle_{(V',V)} dt$$

$$= \int_0^T \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t) \rangle_{(V',V)} dt$$

$$+ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \partial\varphi_\varepsilon(y_\varepsilon(t)), z_\varepsilon(t) - z_\varepsilon(t_i) \rangle_{(V',V)} dt$$

PROOF

so

$$\frac{\rho}{2}\int_0^T \|\partial\varphi_\varepsilon(y_\varepsilon)\|_{V'}dt \le \int_0^T \langle\partial\varphi_\varepsilon(y_\varepsilon(t)), y_\varepsilon(t) - z_\varepsilon(t)\rangle_{(V',V)}dt$$

$$= \int_0^T \langle 2u_\varepsilon(t) - u^*, \tilde{u}(t) - u_\varepsilon(t)\rangle - \langle y_\varepsilon(t) - y^0(t), y_\varepsilon - z_\varepsilon\rangle dt \le C \tag{20}$$

we set $\omega_\varepsilon(t) = \int_0^t \partial\varphi_\varepsilon(y_\varepsilon(s))ds,\ t \in [0,T]$, by (20) we see that there exists a function $\omega(t) \in BV([0,t];V')$, and a sequence convergent to 0, again denoted by $\varepsilon\, s.t.\ \omega_\varepsilon(t) \to \omega(t)$ weakly in V' for every $t \in [0,T]$, and $\forall y(s) \in C([t,T];V), \forall t \in [0,T]$

$$\int_t^T (\partial\varphi_\varepsilon(y_\varepsilon(s)), y(s))_{(V',V)}ds \to \int_t^T (dw(s), y(s))_{(V',V)}. \tag{21}$$

PROOF

Multiply equation (13) by $\frac{A^{-1}p_\lambda(t)}{\|p_\varepsilon(t)\|_{V'}}$ in the sense of the dual product between V' and V, denote $q_\varepsilon(t) = A^{-1}p_\varepsilon(t)$, we have

$$\frac{d}{dt}\|p_\varepsilon(t)\|_{V'} = \frac{\nu|p_\varepsilon(t)|^2}{\|p_\varepsilon(t)\|_{V'}} + \frac{b(q_\varepsilon, y_\varepsilon, p_\varepsilon) + b(y_\varepsilon, q_\varepsilon, p_\varepsilon)}{\|p_\varepsilon(t)\|_{V'}}$$

$$+ \frac{\langle y_\varepsilon(t) - y^0(t), q_\varepsilon(t)\rangle}{\|p_\varepsilon(t)\|_{V'}} + \frac{\langle \partial\varphi_\varepsilon(y_\varepsilon), q_\varepsilon(t)\rangle}{\|p_\varepsilon(t)\|_{V'}}$$

integrate from 0 to t

$$\|p_\lambda(t)\|_{V'} + \frac{\nu}{2}\int_t^T \frac{|p_\varepsilon(s)|^2}{\|p_\varepsilon(s)\|_{V'}} ds \le C_1 + C_2 \int_t^T \|p_\varepsilon(s)\|_{V'} ds$$

By Gronwall's inequality, we get

$$\|p_\varepsilon(t)\|_{L^\infty(0,T;V')} \le C$$

by Alaogu's theorem,

$$p_\varepsilon(t) \to p(t) \quad w^* - L^\infty(0, T; V')$$

PROOF

so $\forall\psi(t)\in C^1(0,T;D(A))$, multiply the equation (13) by $\psi(t)$, letting ε pass to 0, we have

$$\int_t^T \langle p(s), -\psi'(s) - \nu A\psi(s) - B'(y^*(s))\psi(s)\rangle_{(V',V)} ds - \langle p(t), -\psi(t)\rangle_{(V',V)}$$

$$= \int_t^T \langle \psi(s), y^*(s) - y^0(s)\rangle_{(V,V')} ds + \int_t^T \langle \psi(s), dw(s)\rangle_{(V,V')}$$

so $p(t)$ satisfies the equation in theorem 2, (5) also holds by passing ε to 0. (6) follows by the same arguments in the proof of theoerm 1. the proof is completed. $\sharp$

EXAMPLE

Example 1. Let K be the set $K = \{y \in H; |y| \leq \rho\}$, then K is a closed convex set in H, since

$$\|\tilde{z}(t)\|_{C([0,T];H)} \leq C(\|B(y^*(t)) + D\tilde{u}(t) + f(t)\|_{L(0,T;H))})$$

so it is feasible to apply theorem 1 to get the necessary condition of the optimal control pair after checking whether condition (A) is satisfied or not.

EXAMPLE

Example 2. Let K be so called the Enstrophy set

$$K = \{y \in V; |\nabla \times y| \leq \varphi(|y|^2) + \rho\}$$

where $\nabla \times y = curl\ y(x)$, and it is true that $|\nabla \times y| = |\nabla y| = \|y\|$. Enstrophy set plays an important role in fluid mechanics. Since

$$\|\tilde{z}(t)\|_{C([0,T];V)} \leq C(\|B(y^*(t)) + D\tilde{u}(t) + f(t)\|_{L(0,T;H))})$$

so it is feasible to apply theorem 2 to get the necessary condition of the optimal control pair after checking whether condition (A') is satisfied or not.

EXAMPLE

Example 3. Let K be the so called Helicity set,

$$K = \{y \in V; \langle y, curl\ y \rangle^2 + \lambda \|y\|^2 \leq \rho^2\}$$

where λ, ρ are positive constants. The helicity set plays an important role in fluid mechanics and in particular, it is an invariant set of Euler's equation. By the same argument as in Example 2, we know that it is feasible to apply theorem 2 to get the necessary condition of the optimal pair when the state constrained set is Helicity set.

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