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Consumption-investment problem with transaction costs

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History

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Outline

Classical Merton Problem

We are given a stochastic basis with an m-dimensional standard Wiener process w. The market contains a non-risky security which is the *numéraire*, i.e. its price is identically equal to unit, and m risky securities with the price evolution

$$dS^i_t = S^i_t(\mu^i dt + dM^i_t), \qquad i = 1, ..., m, \tag{1}$$

where $M = \Sigma w$ is a (deterministic) linear transform of w. Thus, M is a Gaussian martingale with $\langle M \rangle_t = At$; the covariance matrix $A = \Sigma\Sigma^*$ is assumed to be non-degenerated.
The dynamics of the value process :

$$dV_t = H_t dS_t - c_t dt, \tag{2}$$

where the m-dimensional predictable process H defines the number of shares in the portfolio, $c \geq 0$ is the consumption process.

Merton Problem : dynamics, constraints, and goal

It is convenient to choose as the control the process $\pi = (\alpha, c)$ with $\alpha^i_t := H^i_t S^i_t / V_t$ (the proportion of the wealth invested in the ith asset). Then the dynamics of the value process is :

$$dV_t = V_t \alpha_t (\mu dt + dM_t) - c_t dt, \qquad V_0 = x > 0, \qquad (3)$$

Constraints : α is bounded c is integrable, $V = V^{x,\pi} \geq 0$; $\pi = 0$ after the bankruptcy.
Infinite horizon. The investor's goal :

$$EJ^\pi_\infty \to \max, \qquad (4)$$

where

$$J^\pi_t := \int_0^t e^{-\beta s} u(c_s) ds. \qquad (5)$$

where u is increasing and concave. For simplicity : $u \geq 0$, $u(0) = 0$. A typical example : $u(c) = c^\gamma / \gamma$, $\gamma \in]0, 1[$. The parameter $\beta > 0$ shows that the agent prefers to consume sooner than later.

Merton Problem : the Bellman function

Define the Bellman function

$$W(x) := \sup_{\pi \in \mathcal{A}(x)} EJ^{\pi}_{\infty}, \quad x > 0. \qquad (6)$$

By convention, $\mathcal{A}(0) := \{0\}$ and $W(0) := 0$.
The Bellman function W inherits the properties of u. It is increasing (as $\mathcal{A}(\tilde{x}) \supseteq \mathcal{A}(x)$ when $\tilde{x} \geq x$) and concave (almost obvious in H-parametrization). The process $H = \lambda H_1 + (1-\lambda) H_2$ admits the representation via α with

$$\alpha^i = H^i S^i / V = \frac{\lambda V_1}{\lambda V_1 + (1-\lambda) V_2} \alpha^i_1 + \frac{(1-\lambda) V_2}{\lambda V_1 + (1-\lambda) V_2} \alpha^i_2;$$

α is bounded when α_j are bounded. Thus,
$\pi = (\alpha, \lambda c_1 + (1-\lambda) c_2) \in \mathcal{A}(x)$ with $x = \lambda x_1 + (1-\lambda) x_2$ and

$$W(\lambda x_1 + (1-\lambda) x_2) \geq EJ^{\pi}_{\infty} \geq \lambda EJ^{\pi_1}_{\infty} + (1-\lambda) EJ^{\pi_2}_{\infty}$$

due to concavity of u. We obtain the concavity of W by taking supremum over π_i.

Merton Problem : the result

Theorem

Let $u(c) = c^\gamma/\gamma$, $\gamma \in]0,1[$. *Assume that*

$$\kappa_M := \frac{1}{1-\gamma}\left(\beta - \frac{1}{2}\frac{\gamma}{1-\gamma}|A^{-1/2}\mu|^2\right) > 0. \tag{7}$$

Then the optimal strategy $\pi^o = (\alpha^o, c^o)$ *is given by the formulae*

$$\alpha^o = \theta := \frac{1}{1-\gamma}A^{-1}\mu, \qquad c^o_t = \kappa_M V^o_t, \tag{8}$$

where V^o *is the solution of the linear stochastic equation*

$$dV^o = V^o_t\theta(\mu dt + dM_t) - \kappa_M V^o_t dt, \qquad V^o_0 = x. \tag{9}$$

The process V^o *is optimal and the Bellman function is*

$$W(x) = \left(\kappa_M^{\gamma-1}/\gamma\right)x^\gamma = \mathbf{m}x^\gamma. \tag{10}$$

Merton Problem - comments

For the two-asset model

$$\kappa_M := \frac{1}{1-\gamma}\left(\beta - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2}\right) > 0.$$

Notice that we cannot guarantee without additional assumptions that W is finite. If the latter property holds, then, due to the concavity, $W(x)$ is continuous for $x > 0$, but the question whether it is continuous at zero should be investigated specially.
At last, when the utility u is a power function, the Bellman function W, if finite, is proportional to u. Indeed, the linear dynamics of the control system implies that $W(\nu x) = \nu^\gamma W(x)$ whatever is $\nu > 0$, i.e. the Bellman function is positive homogeneous of the same order as the utility function. In a scalar case this homotheticity property defines, up to a multiplicative constant, a unique finite function, namely x^γ.

HJB equation and verification theorem,1

For our infinite horizon problem the HJB is :

$$\sup_{(\alpha,c)}\left[\frac{1}{2}|A^{1/2}\alpha|^2x^2f''(x)+\alpha\mu xf'(x)-\beta f(x)-f'(x)c+u(c)\right]=0$$

where $x>0$ and sup is taken over $\alpha\in\mathbf{R}^d$ and $c\in\mathbf{R}_+$.
Simple observation : Let $f:\mathbf{R}_+\to\mathbf{R}_+$ and $\pi\in\mathcal{A}(x)$. Put

$$X^{f,x,\pi}=X^f_t=e^{-\beta t}f(V_t)+J^\pi_t$$

where $V=V^{x,\pi}$. If f is smooth, by the Ito formula

$$X^f_t=f(x)+D_t+N_s$$

where (with $L(x,\alpha,c)=[...]$ of the HJB equation)

$$D_t:=\int_0^t e^{-\beta s}L(V_s,\alpha_s,c_s)ds,\qquad N_t:=\int_0^t e^{-\beta s}f'(V_s)V_s\alpha_s dM_s.$$

The process N is a local martingale up to the bankruptcy time σ. That is, there are $\sigma_n\uparrow\sigma$ such that the stopped processes N^{σ_n} are uniformly integrable martingales. If $\sigma=\infty$ and N is a martingale we take $\sigma_n=n$.

HJB equation and verification theorem,2

If $\sup_{(\alpha,c)}[\ldots] \le 0$, then N and X_t^f are supermartingales. Hence,

$$EJ_t = EX_t^f - Ee^{-\beta t} f(V_t) \le EX_t^f \le f(x).$$

Proposition

If f is a supersolution of the HJB, then $W \le f$ and, hence, $W \in C(\mathbf{R}_+ \setminus \{0\})$. If, moreover, $f(0+) = 0$, then $W \in C(\mathbf{R}_+)$.

Theorem

Let $f \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be a positive concave function solving the HJB equation, $f(0) = 0$. Suppose that $\sup$ is attained on $\alpha(x)$ and $c(x)$ where that α is bounded measurable, $c \ge 0$ and the equation

$$dV_t^o = V_t^o \alpha(V_t^o)(\mu dt + dM_t) - c(V_t^o)dt, \qquad V_0^o = x,$$

admits a strong solution V_t^o. If $\lim Ee^{-\beta\sigma_n} f(V_{\sigma_n}^o) = 0$, then $W = f$ and the optimal control $\pi^o = (\alpha(V^o), c(V^o))$.

Proof of the Merton Theorem, 1

The verification theorem is very efficient if we have a guess about the solution. It is the case when the utility is a power function : the problem is to find the constant !

Put $u^*(p) := \sup_{c\geq 0}[u(c) - cp]$. For $u(c) = c^\gamma/\gamma$ we have

$$u^*(p) = \frac{1-\gamma}{\gamma} p^{\gamma/(\gamma-1)}.$$

Expecting that $f'' < 0$, we find that the maximum of the quadratic form over α is attained at

$$\alpha^o(x) = -A^{-1}\mu \frac{f'(x)}{xf''(x)} == A^{-1}\mu/(1-\gamma).$$

Thus, the HJB equation is :

$$-\frac{1}{2}|A^{-1/2}\mu|^2 \frac{(f'(x))^2}{f''(x)} - \beta f(x) + \frac{1-\gamma}{\gamma}(f'(x))^{\frac{\gamma}{\gamma-1}} = 0.$$

Its solution $f(x) = \mathbf{m}x^\gamma$ should have $\mathbf{m} = \kappa_M^{\gamma-1}/\gamma$.

The function $\alpha^o(x)$ is constant, $c^o(x) = \kappa_M x$, and the equation pretending to describe the optimal dynamics is linear :

Proof of the Merton Theorem, 2

$$\frac{dV_t^o}{V_t^o} = \left(\frac{1}{1-\gamma}|A^{-1/2}\mu|^2 - \kappa_M\right)dt + \frac{A^{-1}\mu}{1-\gamma}dM, \qquad V_0^o = x.$$

Its solution is the geometric Brownian motion which never hits zero. Noticing that $\langle A^{-1}\mu M\rangle_t = |A^{-1/2}\mu|^2 t$, we have that

$$V_t^o = x\exp\left\{\left(\frac{1}{1-\gamma} - \frac{1}{2}\frac{1}{(1-\gamma)^2}\right)|A^{-1/2}\mu|^2 t - \kappa_M t + \frac{A^{-1}\mu}{1-\gamma}M_t\right\}.$$

Since $E(V_t^o)^p = x^p e^{\kappa_p t}$ where κ_p is a constant, the process N for this control is a true martingale; we $\sigma_n = n$.
For $p=\gamma$ the corresponding constant

$$\kappa_\gamma = \frac{1}{2}\frac{\gamma}{1-\gamma} - \gamma\kappa_M = \beta - \kappa_M.$$

Thus,

$$e^{-\beta t}E(V_t^o)^\gamma = x^\gamma e^{-\kappa_M t} \to 0, \quad t\to\infty.$$

The Merton theorem is proven.

Merton Problem - discussion, 1

- The optimal strategy with the power utility prescribes to keep constant proportions of wealth in each position. E.g., for $m = 1$ where the quantities $V_t^{2o} := \alpha^o V_t^o$ and $V_t^{1o} = (1-\alpha^o)V_t^o$ the optimal holdings in the risky and non-risky assets,

$$\alpha^o = \theta = \frac{1}{1-\gamma}\frac{\mu}{\sigma^2}.$$

Thus,

$$V_t^{2o} := \frac{\alpha^o}{1-\alpha^o}V_t^{1o} = \frac{\theta}{1-\theta}V_t^{1o}.$$

The process (V_t^{1o}, V_t^{2o}) evolves on the plain (v^1, v^2) along the straight line with slope $\theta/(1-\theta)$, the *Merton line*.

- We consider the case where the non-risky asset pays no interest ($r = 0$). For the power utility function models with zero interest rate are not less general due to the identity $u(e^{rs}c_s) = e^{\gamma rs}u(c_s)$: the maximization problem where the consumption is measured in "money" is the same as that where the consumption is measured in "bonds", but with β replaced by $\tilde{\beta} := \beta - \gamma r$.

Merton Problem - discussion, 2

- An analysis of the proof shows that, with minor changes, it works also when $\gamma < 0$ and the same explicit formulae represent the optimal solution in this case. The HJB approach can be extended to the logarithmic utility function $u(c) = \ln c$ (corresponding to $\gamma = 0$). Of course, one needs to impose an additional constraint on the consumption to ensure the integrability of J^{π}_{∞}.
- Turning back to the multi-asset case, we define the scalar process $\tilde{M}$ with $d\tilde{M} = \theta(\mu dt + dM_t)$. Consider the same consumption-investment problem imposing the restriction that the investments should be shared between money and the risky asset which price follows the process $\tilde{M}$. Any value process and consumption process in this two-asset model are those of the original one. One can imagine a financial institution (a mutual fund) which offers such an artificial asset, called the *market portfolio*. This allows the agent to allocate his wealth only in the non-risky asset and the market portfolio. Due to this economical interpretation, the Merton result sometimes is referred to as the *mutual fund theorem*.

Robustness of the Merton solution, 1

The Merton solution is robust : a deviation of the order ε from the Merton proportion θ leads to losses in the expected utility only of order ε^2. Suppose that in the two-asset model the investor's strategy is to maintain the proportion $\alpha^o+\varepsilon$ and consume a constant part $(1+\delta)\kappa_M$ of the current wealth optimizing the expected utility in δ. Assume for simplicity that $x=1$. Now the dynamics is

$$\frac{dV_t}{V_t}=(\alpha^o+\varepsilon)(\mu dt+\sigma dw_t)-(1+\delta)\kappa_M dt,$$

and V is the geometric Brownian motion

$$V_t=\exp\left\{(\alpha^o+\varepsilon)\mu t-\frac{1}{2}(\alpha^o+\varepsilon)^2\sigma^2 t-(1+\delta)\kappa_M t+(\alpha^o+\varepsilon)\sigma w_t\right\}.$$

We have that

$$EV_t^\gamma=e^{\kappa_\gamma(\varepsilon,\delta)t}$$

where

$$\kappa_\gamma(\varepsilon,\delta)=\beta-\kappa_M-\frac{1}{2}\gamma(1-\gamma)\sigma^2\varepsilon^2-\gamma\kappa_M\delta$$

and, in particular, $\kappa_\gamma(0,0)=\kappa_\gamma=\beta-\kappa_M$.

Robustness of the Merton solution, 2

The coefficient at ε is zero and this is a crucial fact. It follows that

$$EJ_\infty = \frac{1}{\gamma}\kappa_M^\gamma (1+\delta)^\gamma \int_0^\infty e^{-\beta t} EV_t^\gamma \, dt = \frac{1}{\gamma}\kappa_M^{\gamma-1} \frac{(1+\delta)^\gamma}{1+\frac{1}{2\kappa_M}\gamma(1-\gamma)\sigma^2\varepsilon^2+\gamma\delta}.$$

Maximization over δ gives us the optimal value $\delta^o = \frac{1}{2\kappa_M}\gamma\sigma^2\varepsilon^2$ for which

$$EJ_\infty = \frac{1}{\gamma}\kappa_M^{\gamma-1}(1+\delta^o)^{\gamma-1} = \mathbf{m} - \frac{1}{2}(1-\gamma)\kappa_M^{\gamma-2}\sigma^2\,\varepsilon^2 + O(\varepsilon^4)$$

and we get the claimed asymptotic.

Outline

Basic model in discrete time, 1

The portfolio contains d assets (currencies). Their quotes are given in units of a *numéraire*, traded or not. security. At time t the quotes are expressed by the vector of prices $S_t = (S_t^1, \dots, S_t^d)$; its components are strictly positive.

The agent's positions can be described either nominally (in "physical" units) $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or as values invested in each asset $V = (V_t^1, \dots, V_t^d)$ with the obvious relation $\widehat{V}_t^i = V_t^i / S_t^i$. This suggests the notation $\widehat{V}_t = V_t / S_t$. More formally, introducing the diagonal operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d), \qquad (11)$$

we may write that $\widehat{V}_t = \phi_t V_t$. So, any asset can be exchanged to any other. At time t, the increase of the value of ith position in one unit of the numéraire by changing the value of jth position requires diminishing the value of the latter in $1 + \lambda^{ji}$ units of the numéraire. The matrix of transaction cost coefficients $\Lambda = (\lambda^{ij})$ has non-negative entries and the zero diagonal.

Basic model in discrete time, 2

In the dynamical multiperiod setting $S = (S_t)$ is an adapted process ; it is convenient to choose the scales to have all $S_0^i = 1$ and assume as a convention that $S_{0-}^i = 1$.
The portfolio evolution can be described by the initial condition $V_{-0} = v$ (the endowments of the agent entering the market) and the increments at dates $t \geq 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i - c_t^i,$$

with

$$\Delta B_t^i := \sum_{j \leq d} \Delta L_t^{ji} - \sum_{j \leq d} (1 + \lambda^{ij}) \Delta L_t^{ij},$$

where $\Delta L_t^{ji} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ represents the net amount transferred from the position j to the position i at the date t. The 1st term in the rhs comes from the price movements. The 2nd corresponds to the agent's actions at the date t (made after the instant when the new prices were announced), $c_t^i \geq 0$ is the wealth taken for consumption. The matrix (ΔL^{ij}) is the investor order immediately executed by the trader.

Basic model in discrete time, 3

Introducing the process Y ("stochastic logarithm") with

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1,$$

we rewrite the dynamics of the value process as the linear controlled difference equation of a very simple structure with the components connected only via controls :

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i - c_t^i, \quad V_{-1}^i = v^i.$$

We can diminish the dimension of controls and choose B as the control strategy. Indeed, any $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$ defines the $\mathcal{F}_t$-measurable r.v. ΔB_t with values in the set $-M$ where

$$M := \Big\{ x \in \mathbf{R}^d : \ \exists\, a \in \mathbf{M}_+^d \text{ such that } x^i = \sum_{j \le d} [(1+\lambda^{ij})a^{ij} - a^{ji}], \ i \le d \Big\}.$$

Vice versa, a simple measurable selection arguments show that any portfolio increment $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$ is generated by a certain (in general, not unique) "order" $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$.

Basic model in discrete time, 3

We put $K = M + \mathbf{R}^d_+$. It is easy to see that K is the *solvency region*. It coincides with M if $\Lambda \neq 0$. One can check that K is a polyhedral cone

$$K = \mathrm{cone}\,\{(1+\lambda^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}.$$

Thus, in discrete time the dynamics of the vector-valued portfolio processes is given by a linear difference equation with conic constraints on the control. Of course, one can easily imagine other interesting models falling in the scope of this scheme, e.g., one where all transactions charge the money account. Mathematically, it is interesting to consider general conic constraints, not only polyhedral (also, depending on t, price levels etc.).

Apparently, such a model should be easily extended to the continuous time setting as a controlled linear stochastic differential equations ...

Easily ?

Continuous-time Wiener-driven model, 1

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued semimartingale on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the trivial initial σ-algebra. Let K and $\mathcal{C}$ be **proper** cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. Define the set $\mathcal{A}$ of controls $\pi = (B, C)$ as the set of adapted càdlàg processes of bounded variation such that

$$\dot{B} \in -K, \qquad \dot{C} \in \mathcal{C}.$$

Let $\mathcal{A}_a$ be the set of controls with absolutely continuous C and $\Delta C_0 = 0$. For the elements of $\mathcal{A}_a$ we have $c := dC/dt \in \mathcal{C}$.
The controlled process $V = V^{x,\pi}$ is the solution of the linear system

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_{0-}^i = x^i, \quad i = 1, ..., d.$$

For $x \in \operatorname{int} K$ we consider the subsets $\mathcal{A}^x$ and $\mathcal{A}_a^x$ of “admissible” controls for which the processes $V^{x,\pi}$ never leave the set $\operatorname{int} K \cup \{0\}$ and has the origin as an absorbing point.

Continuous-time Wiener-driven model, 2

Let $G := (-K) \cap \partial\mathcal{O}_1(0)$ where $\partial\mathcal{O}_1(0) = \{x \in \mathbf{R}^d : \ |x| = 1\}$ in accordance with the notation for the open ball $\mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x - y| < r\}$.
The set G is a compact and $-K = \mathrm{cone}\, G$. We denote by Σ_G the *support function* of G, given by the relation $\Sigma_G(p) := \sup_{x \in G} px$.

We shall work using the following assumption :

H. The process Y is a continuous process with independent increments with mean $EY_t = \mu t$, $\mu \in \mathbf{R}^d$, and the covariance $DY_t = At$.

In our proof of the dynamic programming principle (needed to derive the HJB equation) we shall assume that the stochastic basis is a canonical one, that is the space of continuous functions with the Wiener measure.

Continuous-time Wiener-driven model, 3

Proposition

There is a constant $\kappa > 0$ *such that*

$$E \sup_{t \le T} |V_t|^2 \le \kappa |x|^2 e^{\kappa T^2} \qquad \forall\ V = V^{x,\pi},\ x \in \operatorname{int} K,\ \ T \ge 0.$$

Proof. The constant κ is "generic". Take arbitrary $p \in \operatorname{int} K^*$ with $|p| = 1$. Since $pdB \le 0$ and $pdC \ge 0$ we get that

$$pV_s \le px + \int_0^s \tilde{p} V_r dr + \int_0^s V_r d\tilde{M}_r,$$

where $\tilde{p}^i := p^i \mu^i$ and $\tilde{M}^i = p^i M^i$, M is the martingale part of Y. The crucial observation is that there is $\kappa > 0$ such that $\kappa^{-1}|y| \le py$ for any $y \in K$. Since $|py| \le |y|$ for any $y \in \mathbf{R}^d$, we obtain that

$$|V_s| \le \kappa |x| + \kappa \int_0^s |V_r| dr + \kappa \left| \int_0^s V_r d\tilde{M}_r \right|.$$

The rest is standard : localization and Gronwall–Bellman.

Goal functionals, 1

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. With every $\pi = (B, C) \in \mathcal{A}_a^x$ we associate the "utility process"

$$J_t^\pi := \int_0^t e^{-\beta s} U(c_s) ds \, , \quad t \geq 0 \, ,$$

where $\beta > 0$. We consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π and define its *Bellman function*

$$W(x) := \sup_{\pi \in \mathcal{A}_a^x} EJ_\infty^\pi \, , \quad x \in \operatorname{int} K \, , \quad W(x) = 0, \; x \in \partial K.$$

If π_i, $i = 1, 2$, are admissible strategies for the initial points x_i, then the strategy $\lambda\pi_1 + (1 - \lambda)\pi_2$ is an admissible strategy for the initial point $\lambda x_1 + (1 - \lambda)x_2$ for any $\lambda \in [0, 1]$, and the corresponding absorbing time dominates the maximum of the absorbing times for both π_i. It follows that the function W is **concave** on $\operatorname{int} K$. Since $A_a^{x_1} \subseteq A_a^{x_2}$ when $x_2 - x_1 \in K$, the function W is **increasing** with respect to the partial ordering $\geq_K$ generated by the cone K.

Goal functionals, 2

Remark 1. Usually, $\mathcal{C} = \mathbf{R}_+ e_1$ and $\sigma^0 = 0$, i.e. the only first (non-risky) asset is consumed. Our presentation is oriented to the scalar power utility function $u(c) = c^\gamma/\gamma$, $\gamma \in]0, 1[$. As was already discussed, in this case there is no need to consider a non-zero interest rate for the non-risky asset which can be chosen as the numéraire.

Remark 2. We consider here a model with mixed "regular-singular" controls. The assumption that the consumption has an intensity c and the agent's utility depends on this intensity is not very satisfactory from the economical point of view. One can consider models with an intertemporal substitution and consumption by "gulps", i.e. dealing with "singular" controls of the class $\mathcal{A}^x$ and the utility processes

$$J_t^\pi := \int_0^t e^{-\beta s} U(\bar{C}_s) ds \,,$$

where

$$\bar{C}_s = \int_0^s K(s,r) dC_r$$

with a suitable kernel $K(s,r)$, e.g., $e^{-\gamma(s-r)}$.

The Hamilton–Jacobi–Bellman equation

We introduce a continuous function of four variables by putting

$$F(X,p,W,x) := \max\{F_0(X,p,W,x) + U^*(p), \Sigma_G(p)\},$$

$X \in \mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$,

$$F_0(X,p,W,x) := (1/2)\mathrm{tr}\, A(x)X + \mu(x)p - \beta W$$

where $A^{ij}(x) := a^{ij}x^i x^j$, $\mu^i(x) := \mu^i x^i$, $1 \le i,j \le d$. In the detailed form

$$F_0(X,p,W,x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}x^i x^j X^{ij} + \sum_{i=1}^{d} \mu^i x^i p^i - \beta W.$$

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.
We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$F(W''(x), W'(x), W(x), x) = 0, \quad x \in \mathrm{int}\, K, \qquad W(x) = 0, \quad x \in \partial K.$$

Viscosity solutions, 1. Semijets.

The idea of viscosity solutions is to plug into F the derivatives and Hessians of quadratic functions touching W from above and below. Let f and g be functions defined in a neighborhood of zero. We shall write $f(.) \lessapprox g(.)$ if $f(h) \le g(h) + o(|h|^2)$ as $|h| \to 0$. The notations $f(.) \gtrapprox g(.)$ and $f(.) \approx g(.)$ have the obvious meaning.
For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we consider the quadratic function

$$Q_{p,X}(z) := pz + (1/2)\langle Xz, z\rangle\,, \quad z \in \mathbf{R}^d\,,$$

and define the *super-* and *subjets* of a function v at the point x :

$$\begin{aligned} J^+v(x) &:= \{(p,X):\ v(x+.) \lessapprox v(x) + Q_{p,X}(.)\},\\ J^-v(x) &:= \{(p,X):\ v(x+.) \gtrapprox v(x) + Q_{p,X}(.)\}. \end{aligned}$$

In other words, $J^+v(x)$ (resp. $J^-v(x)$) is the family of coefficients of quadratic functions $v(x) + Q_{p,X}(y - .)$ dominating the function $v(.)$ (resp., dominated by this function) in a neighborhood of x with precision up to the 2nd order included and coinciding with $v(x)$ at this point.

Viscosity solutions, 2. Basic definitions.

A function $v \in C(K)$ is called *viscosity supersolution* if

$$F(X, p, v(x), x) \leq 0 \qquad \forall (p, X) \in J^- v(x), \; x \in \operatorname{int} K.$$

A function $v \in C(K)$ is called *viscosity subsolution* of if

$$F(X, p, v(x), x) \geq 0 \qquad \forall (p, X) \in J^+ v(x), \; x \in \operatorname{int} K.$$

A function $v \in C(K)$ is a *viscosity solution* if v is simultaneously a viscosity super- and subsolution.

At last, a function $v \in C(K)$ is called *classical supersolution* if $v \in C^2(\operatorname{int} K)$ and $\mathcal{L}v \leq 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

The above notions can be formulated also for open subsets of K.

If v is smooth at a point x, then

$$\begin{aligned} J^+ v(x) &:= \{(p, X) : \; p = v'(x), X \geq v''(x)\}, \\ J^- v(x) &:= \{(p, X) : \; p = v'(x), X \leq v''(x)\}, \end{aligned}$$

where the inequality between matrices is understood in the sense of partial ordering induced by the cone of positive semidefinite matrices.

Viscosity solutions, 3

The pair $(v'(x), v''(x))$ is the unique element belonging to the intersection of $J^- v(x)$ and $J^+ v(x)$. Thus, any viscosity solution v which is in $C^2(\operatorname{int} K)$ is the classical solution. It is easy to check that a classical solution solves the HJB equation in the viscosity sense : the needed property that F is increasing in X with respect to the partial ordering holds.

Remark on a mnemonic rule. In the smooth case for the second order Taylor approximation, i.e. for the quadratic function $(v'(x), v''(x))$ we have the equality. Thus, if $X \geq v''(x)$, for the pair $(v'(x), X)$ which is an element of $J^+ v(x)$, we have obviously the inequality ≥ 0. Note that in the literature it is quite often the equation is written with the opposite sign and so its lhs is decreasing in X ...

For the sake of simplicity and having in mind the specific case we shall work on, the definitions includes the requirement that the viscosity super- and subsolutions are continuous on K including the boundary.

Viscosity solutions, 4. Alternative definitions.

Lemma

Let $v \in C(K)$. The following conditions are equivalent :
(a) v is a viscosity supersolution ;
(b) for any ball $\mathcal{O}_r(x) \subseteq \operatorname{int} K$ and $f \in C^2(\mathcal{O}_r(x))$ such that $v(x) = f(x)$ and $f \le v$ on $\mathcal{O}_r(x)$, the inequality $\mathcal{L}f(x) \le 0$ holds.

Proof. $(a) \Rightarrow (b)$ The pair $(f'(x), f''(x)) \in J^- v(x)$ (the Taylor formula).
$(b) \Rightarrow (a)$ Take (p, X) in $J^- v(x)$. We construct a smooth function f with $f'(x) = p$, $f''(x) = X$ satisfying the requirements of (b).
By definition,

$$v(x+h) - v(x) - Q_{p,X}(h) \ge |h|^2 \varphi(|h|),$$

where $\varphi(u) \to 0$ as $u \downarrow 0$. Consider on $]0, r[$ the function

$$\delta(u) := \sup_{\{h:\ |h| \le u\}} \frac{1}{|h|^2}(v(x+h) - v(x) - Q_{p,X}(h))^- \le \sup_{\{y:\ 0 \le y \le u\}} \varphi^-(y)$$

which is continuous, increasing and $\delta(u) \to 0$ as $u \downarrow 0$.

Viscosity solutions, 5

The function

$$\Delta(u) := \frac{2}{3}\int_u^{2u}\int_\eta^{2\eta}\delta(\xi)d\xi d\eta$$

vanishes at zero with its two right derivatives ;
$u^2\delta(u) \le \Delta(u) \le u^2\delta(4u)$. Thus the function $x \mapsto \Delta(|x|)$ belongs to $C^2(\mathcal{O}_r(0))$, its Hessian vanishes at zero, and

$$v(x+h) - v(x) - Q_{p,X}(h) \ge -|h|^2\delta(|h|) \ge -\Delta(|h|).$$

So, $f(y) := v(x) + Q_{p,X}(y-x) - \Delta(|y-x|)$ is the needed function. $\square$

For subsolutions we have a similar result with the inverse inequalities.

Viscosity solutions, 6

Lemma

Suppose that v is a viscosity solution. If v is twice differentiable at x_0, then it satisfies the HJB equation at x in the classical sense.

Proof. It is not assumed that v' is defined in a neighborhood of x_0. "Twice differentiable" means here that the Taylor formula at x_0 holds :

$$v(x) = v(x_0) + \langle v'(x_0), x - x_0 \rangle + \frac{1}{2}\langle v''(x_0)(x - x_0), x - x_0 \rangle + o(|x - x_0|^2).$$

Let us consider the C^2-function

$$f_\varepsilon(x) = v(x_0) + \langle v'(x_0), x - x_0 \rangle + \frac{1}{2}\langle v''(x_0)(x - x_0), x - x_0 \rangle + \varepsilon |x - x_0|^2,$$

with $f_\varepsilon(x_0) = v(x_0)$. If $\varepsilon < 0$, then $f_\varepsilon \le v$ in a small neighborhood of x_0. Thus, by the previous lemma $\mathcal{L} f_\varepsilon(x_0) \le 0$. Letting ε tend to zero, we obtain that $\mathcal{L} v(x_0) \le 0$. Taking $\varepsilon > 0$ we get the opposite inequality. □

Viscosity solutions, 7. “Modified inequality”.

Lemma

A function $v \in C(K)$ is a viscosity supersolution iff for every $x \in \operatorname{int} K$ the inequality $F(\phi''(x), \phi'(x), v(x), x) \le 0$ holds for any $\phi \in C^2(x)$ such that at x the difference $v - \phi$ attains its local minimum.

Proof. One needs to check only that for a supersolution the inequality holds when $v - \phi$ has a local minimum at x, i.e. when for all y from a certain neighborhood $\mathcal{O}_\varepsilon(x)$ we have the bound

$$v(y) - \phi(y) > v(x) - \phi(x), \qquad y \neq x.$$

Let $\bar{v}$ be a C^2-function dominated by v and let g be a smooth function on $\mathbf{R}_+$ with values in $[0,1]$ and such that $g(t) = 1$ when $t \le \varepsilon/2$ and $g(t) = 0$ when $t \ge \varepsilon$. Consider the C^2-function $\tilde{\phi} = \tilde{\phi}(y)$ with

$$\tilde{\phi}(y) = [\phi(y) + v(x) - \phi(x)]g(|x-y|) + (1 - g(|x-y|))\bar{v}(y).$$

The difference $v - \tilde{\phi}$ attains its minimal value, zero, x and, hence, by the supersolution property the inequality holds for $\tilde{\phi}$ as well as for ϕ because the two derivatives of both functions coincide at x. □

Viscosity solutions, 8. Scalar argument (ODE).

Lemma

Let $\psi \in C^1(a,b)$ be the viscosity solution of $\psi''(z) = G(\psi'(z), \psi(z), z)$. where G is a continuous function. Then $\psi \in C^2(a,b)$ and the equation holds in the classical sense.

Proof. Take $[z_1, z_2] \subset]a,b[$ and consider the C^2-function $\psi_\varepsilon(z)$ such that

$$\psi''_\varepsilon(z) = G(\psi'(z), \psi(z), z) + \varepsilon, \qquad \psi_\varepsilon(z_i) = \psi(z_i), \quad i = 1, 2.$$

We argue first with $\varepsilon > 0$. Suppose that $\psi - \psi_\varepsilon$ attains a local minimum at $z \in]z_1, z_2[$. Then, necessarily, $\psi'_\varepsilon(z) = \psi'(z)$. According to the above criterion for the supersolution,

$$\psi''_\varepsilon(z) \le G(\psi'_\varepsilon(z), \psi(z), z) = G(\psi'(z), \psi(z), z)$$

in contradiction with the definition of ψ_ε. Thus, the difference $\psi - \psi_\varepsilon$ is minimal at the extremities where it is equal to zero. I.e., $\psi(z) \ge \psi_\varepsilon(z)$ for all $z \in [z_1, z_2]$. Letting $\varepsilon \downarrow 0$ and noting that $\psi_\varepsilon(z) \to \psi_0(z)$ (even uniformly), we get that the inequality $\psi(z) \ge \psi_0(z)$. Arguing with $\varepsilon < 0$ and using the subsolution property, we obtain the reverse inequality. □

Viscosity solutions, 9. The Ishii lemma

Lemma

Let v and $\tilde v$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Put $\Delta(x,y) := v(x) - \tilde v(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that

$$(n(\widehat{x} - \widehat{y}), X) \in \bar{J}^+ v(\widehat{x}), \qquad (n(\widehat{x} - \widehat{y}), Y) \in \bar{J}^- \tilde v(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{12}$$

Here I is the identity matrix and $\bar{J}^+ v(x)$ and $\bar{J}^- v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of the set-value mappings $J^+ v$ and $J^- v$, respectively. If v is smooth, the claim follows directly from the necessary conditions of a local maximum (with $X = v''(\widehat{x})$, $Y = \tilde v''(\widehat{y})$ and the constant is 1 instead of 3).

Viscosity solutions, 9. Linear algebra.

Lemma

The inequality (21) implies that for any $d \times m$ matrices B and C

$$\operatorname{tr}(BB'X - CC'Y) \leq 3n|B - C|^2. \quad (13)$$

Notice that $A(x) = \operatorname{diag} xA \operatorname{diag} x$. We denote by $\operatorname{diag} x$ the diagonal matrix whose entries on the diagonal are the coordinates of the vector x. Applying the above lemma with the matrices $B = \operatorname{diag} xA^{1/2}$ and $C = \operatorname{diag} yA^{1/2}$ we obtain the following inequality which we need in the sequel :

$$\operatorname{tr}(A(x)X - A(y)Y) \leq 3n|A^{1/2}|^2|x - y|^2. \quad (14)$$

Uniqueness of the solution and Lyapunov functions

Definition. We say that a positive function $\ell \in C(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if the following properties are satisfied :
1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell(x) \le 0$ for all $x \in \operatorname{int} K$,
2) $\ell(x) \to \infty$ as $|x| \to \infty$.

Theorem

Suppose that there exists a Lyapunov function ℓ. Then the Dirichlet problem for the HJB equation has at most one viscosity solution in the class of continuous functions satisfying the growth condition

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty. \tag{15}$$

Uniqueness. Proof, 1.

Let W and $\tilde{W}$ be two viscosity solutions of (**??**) coinciding on ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0.$$

We introduce continuous functions $\Delta_n : K \times K \to \mathbf{R}$ by putting

$$\Delta_n(x,y) := W(x) - \tilde{W}(y) - \frac{1}{2}n|x-y|^2 - \varepsilon[\ell(x) + \ell(y)], \qquad n \geq 0.$$

Note that $\Delta_n(x,x) = \Delta_0(x,x)$ for all $x \in K$ and $\Delta_0(x,x) \leq 0$ when $x \in \partial K$. From the assumption that ℓ has a higher growth rate than W we deduce that $\Delta_n(x,y) \to -\infty$ as $|x| + |y| \to \infty$. It follows that the sets $\{\Delta_n \geq a\}$ are compacts and the function Δ_n attains its maximum. I.e., there is $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n, y_n) = \bar{\Delta}_n := \sup_{(x,y)\in K\times K} \Delta_n(x,y) \geq \bar{\Delta} := \sup_{x \in K} \Delta_0(x,x) > 0.$$

All (x_n, y_n) belong to the compact set $\{(x,y) : \ \Delta_0(x,y) \geq 0\}$. It follows that the sequence $n|x_n - y_n|^2$ is bounded.

Uniqueness. Proof, 2.

We continue to argue with a subsequence along which (x_n, y_n) converge to some limit $(\widehat{x}, \widehat{x})$. Necessarily, $n|x_n - y_n|^2 \to 0$ (otherwise $\Delta_0(\widehat{x}, \widehat{x}) > \bar{\Delta}$). It is easily seen that $\bar{\Delta}_n \to \Delta_0(\widehat{x}, \widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x} \in \operatorname{int} K$ as well as x_n and y_n for sufficiently large n.
By the Ishii lemma applied to $v := W - \varepsilon\ell$ and $\tilde{v} := \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X^n and Y^n satisfying (21) and such that

$$(n(x_n - y_n), X^n) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in \bar{J}^- \tilde{v}(y_n).$$

Putting $p_n := n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n := n(x_n - y_n) - \varepsilon\ell'(y_n)$, $X_n := X^n + \varepsilon\ell''(x_n)$, $Y_n := Y^n - \varepsilon\ell''(y_n)$, we rewrite this as :

$$(p_n, X_n) \in \bar{J}^+ W(x_n), \qquad (q_n, Y_n) \in \bar{J}^- \tilde{W}(y_n). \tag{16}$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions,

$$F(X_n, p_n, W(x_n), x_n) \geq 0 \geq F(Y_n, q_n, \tilde{W}(y_n), y_n).$$

The 2nd inequality implies that $mq_n \leq 0$ for each $m \in G = (-K) \cap \partial\mathcal{O}_1(0)$. But $\ell'(x) \in \operatorname{int} K^*$ when $x \in \operatorname{int} K$. So,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Uniqueness. Proof, 3.

Since G is a compact, $\Sigma_G(p_n) < 0$. It follows that

$$F_0(X_n, p_n, W(x_n), x_n) + U^*(p_n) \geq 0 \geq F_0(Y_n, q_n, \tilde{W}(y_n), y_n) + U^*(q_n).$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ hence also by K^*. Thus, $U^*(p_n) \leq U^*(q_n)$ and

$$b_n := F_0(X_n, p_n, W(x_n), x_n) - F_0(Y_n, q_n, \tilde{W}(y_n), y_n) \geq 0.$$

Clearly,

$$\begin{aligned} b_n &= \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\ &\quad -\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)). \end{aligned}$$

By virtue of (14) the first sum is dominated by $\mathrm{const} \times n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative according to the definition of Lyapunov function. It follows that $\limsup b_n \leq -\beta\bar{\Delta} < 0$. □

Lyapunov functions and classical supersolutions, 1

Let $u \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be an increasing strictly concave function with $u(0) = 0$ and $u(\infty) = \infty$. Introduce the function $R := -u'^2/(u'' u)$. Assume that $\bar{R} := \sup_{z>0} R(z) < \infty$.

For $p \in K^* \setminus \{0\}$ we define the function $f(x) = f_p(x) := u(px)$ on K. If $y \in K$ and $x \neq 0$, then $yf'(x) = (py)u'(px) \geq 0$. The inequality is strict when $p \in \operatorname{int} K^*$.

Recall that $A(x)$ is the matrix with $A^{ij}(x) = A^{ij} x^i x^j$ and the vector $\mu(x)$ has the components $\mu^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Putting $z := px$ for brevity, we isolate the full square :

$$\begin{aligned}\mathcal{L}_0 f(x) \quad = \quad & \frac{1}{2}\left[\langle A(x)p, p\rangle u''(z) + 2\langle \mu(x), p\rangle u'(z) + \frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle}\frac{u'^2(z)}{u''(z)}\right] \\ & + \frac{1}{2}\frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle} R(z)u(z) - \beta u(z).\end{aligned}$$

Since $u'' \leq 0$, the expression [...] is negative. So, the rhs is negative if $\beta \geq \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2}\sup_{x\in K}\frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle}.$$

Lyapunov functions and classical supersolutions, 2

If $\langle A(x)p, p\rangle = 0$ we cannot argue in this way, but if in such a case also $\langle \mu(x), p\rangle = 0$, then $\mathcal{L}_0 f(x) = -\beta u(z) \le 0$ for any $\beta \ge 0$.

Proposition

Let $p \in \operatorname{int} K^*$. Suppose that $\langle \mu(x), p\rangle$ vanishes on the set $\{x \in \operatorname{int} K : \ \langle A(x)p, p\rangle = 0\}$. If $\beta \ge \eta(p)\bar{R}$, then f_p is a Lyapunov function.

Proposition

Assume $\langle A(x)p, p\rangle \ne 0$ for all $x \in \operatorname{int} K$ and $p \in K^* \setminus \{0\}$. Suppose that $u^*(au'(z)) \le g(a)u(z)$ for every $a, z > 0$ with $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar{\eta}\bar{R}$, then there is a_0 such that for every $a \ge a_0$ the function af_p is a classical supersolution, whatever is $p \in K^*$ with $p^1 \ne 0$. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of $\operatorname{int} K$.

Lyapunov functions and classical supersolutions, 3

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have

$$R(z) = \gamma/(1-\gamma) = \bar{R},$$

and $u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z)$.
If Y is such that $\sigma^1 = 0$, $\mu^1 = 0$ (i.e. the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$, then, by the Cauchy–Schwarz inequality applied to $\langle \mu(x), p \rangle$,

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2.$$

The inequality

$$\beta > \frac{\gamma}{1-\gamma}\frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2$$

(implying the relation $\beta > \bar{\eta}\bar{R}$) is a standing assumption in many studies on the consumption-investment problem under transaction costs.

Supersolutions and the Bellman function, 1

Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to the partial ordering $\geq_K$ and such that for every $x \in \operatorname{int} K$ and $\pi \in \mathcal{A}_a^x$ the positive process $X^f = X^{f,x,\pi}$ given by the formula

$$X_t^f := e^{-\beta t} f(V_t) + J_t^\pi, \tag{17}$$

where $V = V^{x,\pi}$, is a supermartingale.

The set Φ of f with this property is convex and stable under the operation $\wedge$ (recall that the minimum of two supermartingales is a supermartingale). Any continuous function which is a monotone limit (increasing or decreasing) of functions from Φ also belongs to Φ.

Supersolutions and the Bellman function, 1

Lemma

(a) If $f \in \Phi$, then $W \le f$;
(b) if for any $y \in \partial K$ there exists $f \in \Phi$ such that $f(y) = 0$, then W is continuous on K.

Proof. (a) Using the positivity of f, the supermartingale property of X^f, and, finally, the monotonicity of f we get the following chain of inequalities leading to the required property :

$$EJ_t^\pi \le EX_t^f \le f(V_0) \le f(V_{0-}) = f(x).$$

(b) Recall that a concave function is locally Lipschitz continuous on the interior of its domain, i.e. on the interior of the set where it is finite. Hence, if Φ is not empty, then W is continuous (and even locally Lipschitz continuous) on $\mathrm{int}\, K$. The continuity at a point $y \in \partial K$ follows from the assumed property because $0 \le W \le f$. □

Supersolutions and the Bellman function, 2

Lemma

If $f : K \to \mathbf{R}_+$ is a classical supersolution, then $f \in \Phi$.

Proof. A classical supersolution is increasing with respect to $\geq_K$. Indeed,

$$f(x+h) - f(x) = f'(x + \vartheta h)h \qquad \forall\, x, h \in \operatorname{int} K$$

for some $\vartheta \in [0,1]$. The rhs is ≥ 0 because for the supersolution f we have $\Sigma_G(f'(y)) \leq 0$ whatever is $y \in \operatorname{int} K$, or, equivalently, $f'(y)h \geq 0$ for every $h \in K$. By continuity, $f(x+h) - f(x) \geq 0$ for every $x, h \in K$. In order to apply the Itô formula we introduce the process $\tilde{V} = V^{\sigma-} = VI_{[0,\sigma[} + V_{\sigma-}I_{[\sigma,\infty[}$, where σ is the 1st hitting time of zero by V. It coincides with V on $[0,\sigma[$ but either always remains in $\operatorname{int} K$ (due to the stopping at σ if $V_{\sigma-} \in \operatorname{int} K$) or exits to the boundary in a continuous way and stops there. Let $\tilde{X}^f$ correspond to $\tilde{V}$. Since

$$X^f = \tilde{X}^f + e^{-\beta\sigma}(f(V_{\sigma-} + \Delta B_\sigma) - f(V_{\sigma-}))I_{[\sigma,\infty[},$$

by the monotonicity it is suffices to check that $\tilde{X}^f$ is a supermartingale.

Supersolutions and the Bellman function, 3

Applying the Itô formula to $e^{-\beta t}f(\tilde{V}_t)$ we obtain on $[0,\sigma[$:

$$\tilde{X}_t^f = f(x) + \int_0^t e^{-\beta s}[\mathcal{L}_0 f(V_s) - c_s f'(V_s) + U(c_s)]ds + R_t + m_t, \quad (18)$$

where m is a process such that $m^{\sigma_n} = (m_{t\wedge\sigma_n})$ are continuous martingales for some σ_n increasing to σ, and

$$R_t := \int_0^t e^{-\beta s} f'(\tilde{V}_{s-})dB_s^c + \sum_{s\le t} e^{-\beta s}[f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})]. \quad (19)$$

By definition of a supersolution, for any $x \in \operatorname{int} K$,

$$\mathcal{L}_0 f(x) \le -U^*(f'(x)) \le cf'(x) - U(c) \quad \forall\, c \in K.$$

Thus, the integral in (18) is a decreasing process. The process R is also decreasing because the terms in the sum are negative by monotonicity of f while the integral is negative because

$$f'(\tilde{V}_{s-})dB_s^c = I_{\{\Delta B_s = 0\}} f'(\tilde{V}_{s-})\dot{B}_s d||B||_s$$

where $f'(\tilde{V}_{s-})\dot{B}_s \le 0$ since $\dot{B} \in K$.

Supersolutions and the Bellman function, 4

Taking into account that $\tilde{X}^f \geq 0$, we obtain from (18) that for each n the negative decreasing process $R_{t\wedge\sigma_n}$ dominates an integrable process and so it is integrable. The same holds for the stopped integral. Being a sum of integrable decreasing process and a martingale, the process $\tilde{X}^f_{t\wedge\sigma_n}$ is a positive supermartingale and, by the Fatou lemma, $\tilde{X}^f$ is a supermartingale as well. $\square$

Strict local supersolutions

The next result is of great importance. It plays the crucial role in deducing from the Dynamic Programming Principle the property W to be a subsolution of the HJB equation.
We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ and define τ^π as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi := \inf\{t \geq 0 : \ |V_t^{\pi,x} - x| \geq r\}.$$

For simplicity we assume that f is smooth in a neighborhood of $\bar{\mathcal{O}}_r(x)$.

Lemma

Let $f \in C^2(\bar{\mathcal{O}}_r(x))$ be such that $\mathcal{L}f \leq -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta > 0$ and an interval $]0, t_0]$ such that

$$\sup_{\pi \in \mathcal{A}_a^x} EX_{t\wedge\tau^\pi}^{f,x,\pi} \leq f(x) - \eta t \qquad \forall\, t \in]0, t_0].$$

Dynamic Programming Principle, 1

Let $\mathcal{T}_f$ and $\mathcal{T}_b$ be, respectively, the sets of all finite and bounded stopping times.

Lemma

We have

$$W(x) \leq \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{20}$$

If $W(x) < \infty$ *for all* $x \in \operatorname{int} K$, *then*

$$W(x) \leq \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_b} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau}^{x,\pi})\right). \tag{21}$$

Lemma

Assume that $W(x) < \infty$ *for all* $x \in \operatorname{int} K$. *Then for any* $\tau \in \mathcal{T}_f$

$$W(x) \geq \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{22}$$

Dynamic Programming Principle, 2

The following property of the Bellman function is usually referred to as the (weak) "dynamic programming principle" :

Theorem

Assume that $W(x) < \infty$ *for* $x \in \operatorname{int} K$. *Then for any* $\tau \in \mathcal{T}_f$

$$W(x) = \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{23}$$

However, it seems that this nicely looking formulation is not sufficient...

The Bellman function and the HJB equation, 1

Lemma

If (22) holds then W is a viscosity supersolution of the HJB equation.

Proof. Let $x \in \mathcal{O} \subseteq \operatorname{int} K$. We choose a test function $\phi \in C^2(\mathcal{O})$ such that $\phi(x) = W(x)$ and $W \geq \phi$ in $\mathcal{O}$.
At first, we fix $m \in K$ and argue with $\varepsilon > 0$ small enough to ensure that $x - \varepsilon m \in \mathcal{O}$. The function W is increasing with respect $\geq_K$. Thus,

$$\phi(x) = W(x) \geq W(x - \varepsilon m) \geq \phi(x - \varepsilon m).$$

It follows that $-m\phi'(x) \leq 0$ and, therefore, $\Sigma_G(\phi'(x)) \leq 0$.
Take now π with $B_t = 0$ and $c_t = c \in \mathcal{C}$. Let τ_r be the exit time of the continuous process $V = V^{x,\pi}$ from the ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$.

The Bellman function and the HJB equation, 2

The identity (22) implies that

$$W(x) \geq E\left(J^{\pi}_{t\wedge\tau_r} + e^{-\beta(t\wedge\tau_r)}W(V_{t\wedge\tau_r})\right)$$

and this inequality holds true if replace W by ϕ. Writing all terms of the latter in the rhs and applying the Itô formula we get that

$$\begin{aligned} 0 &\geq E\left(\int_0^{t\wedge\tau_r} e^{-\beta s}U(c_s)ds + e^{-\beta(t\wedge\tau_r)}\phi(V_{t\wedge\tau_r})\right) - \phi(x) \\ &\geq E\int_0^{t\wedge\tau_r} e^{-\beta s}[\mathcal{L}_0\phi(V_s) - c\phi'(V_s) + U(c)]ds \\ &\geq \min_{y\in\bar{\mathcal{O}}_r(x)}[\mathcal{L}_0\phi(y) - c\phi'(y) + U(c)]E\left[\frac{1}{\beta}\left(1 - e^{-\beta(t\wedge\tau_r)}\right)\right]. \end{aligned}$$

Dividing the resulting inequality by t and taking successively the limits as t and r converge to zero we infer that $\mathcal{L}_0\phi(x) - c\phi'(x) + U(c) \leq 0$. Maximizing over $c \in \mathcal{C}$ yields the bound $\mathcal{L}_0\phi(x) + U^*(\phi'(x)) \leq 0$. Hence, W is a supersolution. $\square$

The Bellman function and the HJB equation, 3

Lemma

If (20) holds then W is a viscosity subsolution of the HJB equation.

Proof. Let $x \in \mathcal{O} \subseteq \operatorname{int} K$. Let $\phi \in C^2(\mathcal{O})$ be a function such that $\phi(x) = W(x)$ and $W \le \phi$ on $\mathcal{O}$. Assume that the subsolution inequality for ϕ fails at x. Thus, there exists $\varepsilon > 0$ such that $\mathcal{L}\phi \le -\varepsilon$ on some ball $\bar{\mathcal{O}}_r(x) \subseteq \mathcal{O}$. By virtue of Lemma 18 (applied to the function ϕ) there are $t_0 > 0$ and $\eta > 0$ such that on the interval $]0, t_0]$ for any strategy $\pi \in \mathcal{A}_a^x$

$$E\left(J^{\pi}_{t\wedge\tau^{\pi}} + e^{-\beta\tau^{\pi}}\phi(V^{x,\pi}_{t\wedge\tau^{\pi}})\right) \le \phi(x) - \eta t,$$

where τ^{π} is the exit time of the process $V^{x,\pi}$ from the ball $\bar{\mathcal{O}}_r(x)$. Fix $t \in]0, t_0]$. By the second claim of Lemma 19) there exists $\pi \in \mathcal{A}_a^x$ such that

$$W(x) \le E\left(J^{\pi}_{t\wedge\tau} + e^{-\beta\tau}W(V^{x,\pi}_{t\wedge\tau})\right) + (1/2)\eta t,$$

for every stopping time τ, in particular for τ^{π}.
Using the inequality $W \le \phi$ and applying Lemma 18 we obtain that $W(x) \le \phi(x) - (1/2)\eta t$. A contradiction because $W(x) = \phi(x)$. □

The Bellman function and the HJB equation, 4

Theorem

Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of the HJB equation.

Proof. The claim follows from the two lemmas above. □

Outline

Model

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued Lévy process modelling **relative** price movements (i.e. $dY^i_t = dS^i_t/S^i_{t-}$ or $S^i_t = S^i_0\mathcal{E}_t(Y^i)$) :

$$dY_t = \mu t + \Xi dw_t + \int z(p(dz, dt) - \Pi(dz)dt)$$

w is a Wiener process and $p(dt, dx)$ is a Poisson random measure with the compensator $\Pi(dz)dt$ where $\Pi(dz)$ is concentrated on $]-1, \infty]^d$. The matrix Ξ is such that $A = \Xi\Xi^*$ is non-degenerated,

$$\int (|z|^2 \wedge |z|)\Pi(dz) < \infty.$$

Let K and $\mathcal{C}$ be **proper** cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. The set $\mathcal{A}_a$ of controls $\pi = (B, C)$ is the set of **predictable** càdlàg processes of bounded variation such that $dC_t = c_t dt$ and

$$\dot{B} \in -K, \qquad c \in \mathcal{C}.$$

Dynamics

The process $V = V^{x,\pi}$ is the solution of the linear system

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_{0-}^i = x^i, \quad i = 1, ..., d.$$

This solution can be expressed explicitly using the Doléans-Dade exponentials $S_t^i = \mathcal{E}_t(Y^i)$ (we assume that $S_0 = \mathbf{1}$) :

$$V_t^i = S_t^i x^i + S_t^i \int_{[0,t]} \frac{1}{S_{s-}^i}(dB_s^i - dC_s^i), \quad i = 1, ..., d.$$

We introduce the stopping time

$$\theta = \theta^{x,\pi} := \inf\{t : \ V_t^{x,\pi} \notin \operatorname{int} K\}.$$

For $x \in \operatorname{int} K$ we consider the subset $\mathcal{A}_a^x$ of "admissible" controls for which $\pi = I_{[0,\theta^{x,\pi}]}\pi$, i.e. the process $V^{x,\pi}$ stops at the moment of ruin : no more consumption.

Goal Functional

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. For $\pi = (B, C) \in \mathcal{A}_a^x$ we put

$$J_t^\pi := \int_0^t e^{-\beta s} U(c_s) ds$$

and consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π. The *Bellman function*

$$W(x) := \sup_{\pi \in \mathcal{A}_a^x} EJ_\infty^\pi , \quad x \in \operatorname{int} K ,$$

is **increasing** with respect to the partial ordering $\geq_K$.
The process $V^{\lambda x_1 + (1-\lambda)x_2, \lambda\pi_1 + (1-\lambda)\pi_2}$ is the convex combination of V^{x_i,π_i} with the same coefficients. For **continuous** Y the ruin time is the maximum of θ^{x_i,π_i} and the concavity of u implies the concavity of W. But if Y has **jumps**, the ruin times are not related in this way and we cannot guarantee (at least, by the above argument) that the Bellman function is concave.

The Hamilton–Jacobi–Bellman Equation, I

Let $G := (-K) \cap \partial\mathcal{O}_1(0)$ where $\mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x - y| < r\}$
Then $-K = \mathrm{cone}\, G$. We denote by Σ_G the *support function* of G, i.e. $\Sigma_G(p) = \sup_{x \in G} px$. Put

$$F(X, p, \mathcal{I}(f, x), W, x) = \max\{F_0(X, p, \mathcal{I}(f, x), W, x) + U^*(p), \Sigma_G(p)\},$$

where $X \in \mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$, $f \in C_1(K) \cap C^2(x)$ and the function F_0 is given by

$$\begin{aligned} F_0(X, p, \mathcal{I}(f, x), W, x) &= \frac{1}{2}\mathrm{tr}\, A(x)X + \mu(x)p + \mathcal{I}(f, x) - \beta W(x) \\ &= \frac{1}{2}\sum_{i,j} a^{ij}x^i x^j X^{ij} + \sum_i \mu^i x^i p^i + \mathcal{I}(f, x) - \beta W(x) \end{aligned}$$

where $A(x)$ is the matrix with $A^{ij}(x) = a^{ij}x^i x^j$, $\mu^i(x) = \mu^i x^i$,

$$\mathcal{I}(f, x) = \int (f(x + \mathrm{diag}\, xz) - f(x) - \mathrm{diag}\, xzf'(x)) I(z, x)\Pi(dz), \quad x \in \mathrm{int}\, K,$$

$$I(z, x) = I_{\{z:\ x + \mathrm{diag}\, xz \in K\}} = I_K(x + \mathrm{diag}\, xz).$$

The Hamilton–Jacobi–Bellman Equation, II

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \mathcal{I}(\phi, x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.
We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$\begin{aligned} F(W''(x), W'(x), \mathcal{I}(W, x), W(x), x) &= 0, \qquad x \in \operatorname{int} K, \\ W(x) &= 0, \qquad x \in \partial K. \end{aligned}$$

In general, W has no derivatives at some points $x \in \operatorname{int} K$ and the notation above needs to be interpreted. The idea of viscosity solutions is to substitute W in F by suitable test functions.

Viscosity Solutions

- A function $v \in C(K)$ is called *viscosity supersolution* (of HJB) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \geq f$ the inequality $\mathcal{L}f(x) \leq 0$ holds.
- A function $v \in C(K)$ is called *viscosity subsolution* of if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \leq f$ the inequality $\mathcal{L}f(x) \geq 0$ holds.
- $v \in C(K)$ is *viscosity solution* of if v is simultaneously a viscosity super- and subsolution.
- $v \in C_1(K) \cap C^2(\operatorname{int} K)$ is *classical supersolution* of HJB if $\mathcal{L}v \leq 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

Lemma

Suppose that the function v is a viscosity solution. If v is twice differentiable at $x_0 \in \operatorname{int} K$, then it satisfies HJB at this point in the classical sense.

Jets

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we put $Q_{p,X}(z) = pz + (1/2)\langle Xz, z\rangle$ and define the *super-* and *subjets* of a function v at the point x :

$$\begin{aligned} J^+ v(x) &= \{(p,X):\ v(x+h) \le v(x) + Q_{p,X}(h) + o(|h|^2)\}, \\ J^- v(x) &= \{(p,X):\ v(x+h) \ge v(x) + Q_{p,X}(h) + o(|h|^2)\}. \end{aligned}$$

I.e. $J^+ v(x)$ (resp. $J^- v(x)$) is the family of coefficients of quadratic functions $v(x) + Q_{p,X}(y - .)$ dominating $v(.)$ (resp., dominated by $v(.)$) near x up to the 2nd order and coinciding with $v(.)$ at x.
For integro-differential operators viscosity solution does not admit an equivalent formulation in terms of jets.

Lemma

Let v be a viscosity supersolution, $x \in \operatorname{int} K$, and $(p,X) \in J^- v(x)$. Then there is a function $f \in C_1(K) \cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \ge v$ on K and, hence,

$$F(X, p, \mathcal{I}(f,x), W(x), x) \le 0.$$

Supermaringales and Majorants of W

Put $\tilde{V} = V^{\theta-} = VI_{[0,\theta[} + V_{\sigma-}I_{[\theta,\infty[}$ where θ is the ruin time. Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to $\geq_K$ and such that for each $x \in \operatorname{int} K$, $\pi \in \mathcal{A}_a^x$

$$X^f = X^{f,x,\pi} = e^{-\beta t} f(\tilde{V}) + J^\pi,$$

is a supermartingale. This set is convex and stable under the operation $\wedge$. Any continuous function which is a monotone limit of functions from Φ also belongs to Φ.

Lemma

(a) If $f \in \Phi$, then $W \leq f$;
(b) if a point $y \in \partial K$ is such that there is $f \in \Phi$ with $f(y) = 0$, then W is continuous at y.

Proof. Indeed : $EJ_t^\pi \leq EX_t^f \leq f(\tilde{V}_0) = f(V_0) \leq f(V_{0-}) = f(x)$.

Supermaringales and Supersolutions of HJB, I

Lemma

Let $f : K \to \mathbf{R}_+$ be a function in $C_1(K) \cap C^2(\mathrm{int}\, K)$. If f is a classical supersolution of HJB, then f is a monotone function and X^f is a supermartingale, i.e. $f \in \Phi$.

Proof. A classical supersolution is increasing with respect to $\geq_K$. Indeed, for any $x, h \in \mathrm{int}\, K$ there is $\vartheta \in [0,1]$ such that

$$f(x+h) - f(x) = f'(x + \vartheta h)h \geq 0$$

because for the supersolution $\Sigma_G(f'(y)) \leq 0$ when $y \in \mathrm{int}\, K$, or, equivalently, $f'(y)h \geq 0$ for every $h \in K$. By continuity, $f(x+h) - f(x) \geq 0$ for every $x, h \in K$.

Supermaringales and Supersolutions of HJB, II

Using the Itô formula we have :

$$X_t^f = f(x) + \int_0^{t\wedge\theta} e^{-\beta s}[\mathcal{L}_0 f(\tilde{V}_s) - c_s f'(\tilde{V}_s) + U(c_s)]ds + R_t + m_t,$$

where the integral is a decreasing process (since $[\ldots] \le \mathcal{L}f(\tilde{V}_s)$),

$$R_t = \int_0^{t\wedge\theta} e^{-\beta s} f'(V_{s-})dB_s^c + \sum_{s\le t} e^{-\beta s}[f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})]$$

is also decreasing and m is the local martingale with

$$\begin{aligned} m_t \quad = \quad & \int_0^{t\wedge\theta} e^{-\beta s} f'(\tilde{V}_{s-})\mathrm{diag}\, \tilde{V}_s \Xi dw_s \\ & + \int_0^t \int e^{-\beta s}[f(\tilde{V}_{s-} + \mathrm{diag}\, \tilde{V}_{s-}z) - f(\tilde{V}_{s-})]I(\tilde{V}_{s-}, z)\tilde{p}(dz, ds). \end{aligned}$$

$\tilde{p}(dz, ds) = p(dz, ds) - \Pi(dz)ds.$

Strict Local Supersolutions

We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ and define τ^π as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi = \inf\{t \geq 0 : \ |V_t^{\pi,x} - x| \geq r\}.$$

Lemma

Let $f \in C_1(K) \cap C^2(\bar{\mathcal{O}}_r(x))$ be such that $\mathcal{L}f \leq -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta > 0$ and an interval $]0, t_0]$ such that

$$\sup_{\pi \in \mathcal{A}_a^x} EX^{f,x,\pi}_{t\wedge\tau^\pi} \leq f(x) - \eta t \qquad \forall\, t \in]0, t_0].$$

Dynamic Programming Principle

For the following two assertions we need to assume that Ω is a path space.

Lemma

Let $\mathcal{T}_f$ and $\mathcal{T}_b$ be, respectively, the sets of all finite and bounded stopping times. Then

$$W(x) \le \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right).$$

Lemma

Assume that $W(x)$ is continuous on $\operatorname{int} K$. Then for any $\tau \in \mathcal{T}_f$

$$W(x) \ge \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right).$$

Bellman Function and HJB

Theorem

Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of the HJB equation).

Proof.

Uniqueness Theorem for HJB

Definition. We say that a positive function $\ell \in C(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if the following properties are satisfied :
1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell(x) \leq 0$ for all $x \in \operatorname{int} K$,
2) $\ell(x) \to \infty$ as $|x| \to \infty$.

Theorem

Assume that the jump measure Π *does not charge* $(d-1)$*-dimensional surfaces. Suppose that there exists a Lyapunov function* ℓ*. Then the Dirichlet problem for the HJB equation has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty.$$

Uniqueness Theorem for HJB. Idea of the proof, I

Let W and $\tilde{W}$ be two viscosity solutions of HJB coinciding on ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0.$$

Define continuous functions $\Delta_n : K \times K \to \mathbf{R}$

$$\Delta_n(x,y) := W(x) - \tilde{W}(y) - \frac{1}{2}n|x-y|^2 - \varepsilon[\ell(x) + \ell(y)], \qquad n \geq 0.$$

Note that $\Delta_n(x,x) = \Delta_0(x,x)$ for all $x \in K$ and $\Delta_0(x,x) \leq 0$ when $x \in \partial K$. Since ℓ has a higher growth rate than W we deduce that $\Delta_n(x,y) \to -\infty$ as $|x| + |y| \to \infty$. The sets $\{\Delta_n \geq a\}$ are compacts and Δ_n attains its maximum. I.e., there is $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n, y_n) = \bar{\Delta}_n := \sup_{(x,y)\in K\times K} \Delta_n(x,y) \geq \bar{\Delta} := \sup_{x \in K} \Delta_0(x,x) > 0.$$

All (x_n, y_n) belong to the compact $\{(x,y) : \ \Delta_0(x,y) \geq 0\}$. Thus, the sequence $n|x_n - y_n|^2$ is bounded. We assume wlg that (x_n, y_n) converge to $(\widehat{x}, \widehat{x})$. Also, $n|x_n - y_n|^2 \to 0$ (otherwise we $\Delta_0(\widehat{x}, \widehat{x}) > \bar{\Delta}$). Clearly, $\bar{\Delta}_n \to \Delta_0(\widehat{x}, \widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x}$ is in the interior of K and so are x_n and y_n.

Uniqueness Theorem for HJB. The Ishii Lemma.

Lemma

Let v and $\tilde{v}$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Consider the function $\Delta(x,y) = v(x) - \tilde{v}(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that

$$(n(\widehat{x} - \widehat{y}), X) \in \bar{J}^+ v(\widehat{x}), \qquad (n(\widehat{x} - \widehat{y}), Y) \in \bar{J}^- \tilde{v}(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Here $\bar{J}^+ v(x)$ and $\bar{J}^- v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of $J^+ v$ and $J^- v$.
The matrix inequality implies the bound

$$\mathrm{tr}\,(A(x)X - A(y)Y) \leq 3n|A|^{1/2}|x-y|^2.$$

Uniqueness Theorem for HJB. Idea of the proof, II

By the Ishii lemma applied to $v = W - \varepsilon\ell$ and $\tilde{v} = \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X^n and Y^n such that

$$(n(x_n - y_n), X^n) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in \bar{J}^- \tilde{v}(y_n).$$

Using the notations $p_n = n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n = n(x_n - y_n) - \varepsilon\ell'(y_n)$, $X_n = X^n + \varepsilon\ell''(x_n)$, $Y_n = Y^n - \varepsilon\ell''(y_n)$, we may rewrite the last relations in the following equivalent form :

$$(p_n, X_n) \in \bar{J}^+ W(x_n), \qquad (q_n, Y_n) \in \bar{J}^- \tilde{W}(y_n).$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions, one can find, the functions $f_n \in C_1(K) \cap C^2(x_n)$ and $\tilde{f}_n \in C_1(K) \cap C^2(y_n)$ such that $f_n'(x_n) = p_n$, $f_n''(x_n) = X_n$, $f_n(x_n) = W(x_n)$, $f_n \le W$ on K, and $\tilde{f}_n'(y_n) = q_n$, $\tilde{f}_n''(y_n) = Y_n$, $\tilde{f}_n(y_n) = \tilde{W}(y_n)$, $\tilde{f}_n \ge \tilde{W}$ on K,

$$F(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) \ge 0 \ge F(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n).$$

Uniqueness Theorem for HJB. Idea of the proof, III

The second inequality implies that $mq_n \leq 0$ for each $m \in G = (-K) \cap \partial\mathcal{O}_1(0)$. But for the Lyapunov function $\ell'(x) \in \operatorname{int} K^*$ when $x \in \operatorname{int} K$ and, therefore,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Since G is a compact, $\Sigma_G(p_n) < 0$. It follows that

$$\begin{aligned} F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) + U^*(p_n) &\geq 0, \\ F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) + U^*(q_n) &\leq 0. \end{aligned}$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ hence also by K^*. Thus, $U^*(p_n) \leq U^*(q_n)$ and we obtain the inequality

$$b_n = F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) - F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) \geq 0$$

Uniqueness Theorem for HJB. Idea of the proof, IV

Clearly,

$$\begin{aligned} b_n &= \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\ &\quad -\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) \\ &\quad +\varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)). \end{aligned}$$

The first term in the rhs is dominated by a constant multiplied by $n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative according to the definition of the Lyapunov function. To complete the proof, it remains to show that

$$\limsup_n \left(\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n)\right) \le 0.$$

Indeed, with this we have that $\limsup b_n \le -\beta\bar{\Delta} < 0$.

Uniqueness Theorem for HJB. Idea of the proof, V

Let

$$\begin{aligned} F_n(z) &= \big[(f_n-\varepsilon\ell)(x_n+\operatorname{diag} x_n z)-(f_n-\varepsilon\ell)(x_n) \\ &\quad -\operatorname{diag} x_n z(f_n'-\varepsilon\ell')(x_n)\big] I(z,x_n), \\ \tilde F_n(z) &= \big[(\tilde f_n+\varepsilon\ell)(y_n+\operatorname{diag} y_n z)-(\tilde f_n+\varepsilon\ell)(y_n) \\ &\quad -\operatorname{diag} y_n z(\tilde f_n'+\varepsilon\ell')(y_n)\big] I(z,y_n). \end{aligned}$$

and $H_n(z)=F_n(z)-\tilde F_n(z)$ With this notation

$$\mathcal{I}(f_n-\varepsilon\ell,x_n)-\mathcal{I}(\tilde f_n+\varepsilon\ell,y_n)=\int H_n(z)\Pi(dz)$$

and the needed inequality will follow from the Fatou lemma if we show that there is a constant C such that for all sufficiently large n

$$H_n(z)\le C(|z|\wedge|z|^2) \quad \text{for all } z\in K \tag{24}$$

and

$$\limsup_n H_n(z)\le 0 \quad \Pi\text{-a.s.} \tag{25}$$

Uniqueness Theorem for HJB. Idea of the proof, VI

Using the properties of f_n we get the bound :

$$F_n(z) \leq \Big[(W-\varepsilon\ell)(x_n+\operatorname{diag} x_n z)-(W-\varepsilon\ell)(x_n) \\ -\operatorname{diag} x_n z n(x_n-y_n)\Big] I(z,x_n)$$

Since the continuous function W and I are of sublinear growth and the sequences x_n and $n(x_n-y_n)$ are converging (hence bounded), absolute value of the function in the right-hand side of this inequality is dominated by a function $c(1+|z|)$. The arguments for $-\tilde{F}_n(z)$ are similar. So, the function H_n is of sublinear growth.
We have the following identity :

$$\begin{aligned} H_n(z) = & (\Delta_n(x_n+\operatorname{diag} x_n z, y_n+\operatorname{diag} y_n z)-\Delta_n(x_n,y_n) \\ & +(1/2)n|\operatorname{diag}(x_n-y_n)z|^2)I(z,x_n)I(z,y_n) \\ & +(f_n(x_n+\operatorname{diag} x_n z)-W(x_n+\operatorname{diag} x_n z))I(z,x_n)I(z,y_n) \\ & -(\tilde{f}_n(y_n+\operatorname{diag} y_n z)-\tilde{W}(y_n+\operatorname{diag} y_n z))I(z,x_n)I(z,y_n) \\ & +F_n(z)(1-I(z,y_n))-\tilde{F}_n(z)(1-I(z,x_n)). \end{aligned}$$

Uniqueness Theorem for HJB. Idea of the proof, VII

The function $\Delta(x,y)$ attains its maximum at (x_n, y_n) and $f_n \le W$, $\tilde f_n \ge \tilde W$. It follows that

$$H_n(z) \le (1/2)n|x_n - y_n|^2|z|^2 + F_n(z)(1 - I(z,y_n)) - \tilde F_n(z)(1 - I(z,x_n)).$$

Let $\delta > 0$ be the distance between $\widehat{x}$ from and ∂K. Then $x_n, y_n \in \mathcal{O}_{\delta/2}(\widehat{x})$ for large n and, hence, the second and the third terms in the rhs above are functions vanishing on $\mathcal{O}_1(0)$. So, for such n the function H_n is dominated from above on $\mathcal{O}_1(0)$ by $c_n|z|^2$ where $c_n := (1/2)n|x_n - y_n|^2 \to 0$ as $n \to \infty$. Therefore, (24) holds. The relation (24) also holds because the second and the first terms tends to zero (stationarily) for all z except the set $\{z :\ \widehat{x} + \mathrm{diag}\,\widehat{x}z \in \partial K\}$. The coordinates of points of $\partial K \setminus \{0\}$ are non-zero. So this set is empty if $\widehat{x}$ has a zero coordinate. If all components $\widehat{x}$ are nonzero, the operator $\widehat{x}$ is non-degenerated and the set in question is of zero measure Π in virtue of our assumption.

Lyapunov Functions, I

Let $u \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be increasing strictly concave, $u(0) = 0$, $u(\infty) = \infty$. Define $R = -u'^2/(u''u)$ and assume that $\bar{R} = \sup_{z>0} R(z) < \infty$.

For $p \in K^* \setminus \{0\}$ define the function $f(x) = f_p(x) := u(px)$. If $y \in K$ and $x \neq 0$, then $yf'(x) = (py)u'(px) \leq 0$. The inequality is strict when $p \in \operatorname{int} K^*$.

Recall that $A^{ij}(x) = A^{ij}x^i x^j$ and $\mu^i(x) = \mu^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Isolating the full square we get that $\mathcal{L}_0 f(x)$ is equal to

$$\frac{1}{2}\left[\langle A(x)p,p\rangle u''(px) + 2\langle \mu(x),p\rangle u'(px) + \frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}\frac{u'^2(px)}{u''(px)}\right]$$
$$+\frac{1}{2}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}R(px)u(px) + \mathcal{I}(f,x) - \beta u(px).$$

Note that

$$(f(x+\operatorname{diag} xz) - f(x) - \operatorname{diag} xzf'(x)) = (1/2)u''(\dots)(px)^2 \leq 0.$$

Lyapunov Functions, II

It follows that $\mathcal{L}_0 f(x) \le 0$ if $\beta \ge \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2} \sup_{x \in K} \frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle}.$$

If $\langle A(x)p, p\rangle = 0$ and $\langle \mu(x), p\rangle = 0$, then $\mathcal{L}_0 f(x) = -\beta u(px) \le 0$ for any $\beta \ge 0$.

Proposition

Let $p \in \operatorname{int} K^*$. Suppose that $\langle \mu(x), p\rangle$ vanishes on the set $\{x \in \operatorname{int} K : \ \langle A(x)p, p\rangle = 0\}$. If $\beta \ge \eta(p)\bar{R}$, then f_p is a Lyapunov function.

Existence of Classical Supersolutions

The same ideas are useful also in the search of supersolutions. Since $\mathcal{L}f = \mathcal{L}_0 f + U^*(f')$, it is natural to choose u related to U. For the case where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties and assuming, moreover, the inequality

$$u^*(au'(z)) \le g(a)u(z)$$

we get, using the homogeneity of $\mathcal{L}_0$, the following result.

Proposition

Assume $\langle A(x)p, p\rangle \neq 0$ for all $x \in \operatorname{int} K$ and $p \in K^* \setminus \{0\}$. Suppose that $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar{\eta}\bar{R}$, then there is a_0 such that for every $a \ge a_0$ the function af_p is a classical supersolution of HJB, whatever is $p \in K^*$ with $p^1 \neq 0$. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of $\operatorname{int} K$.

Power Utility Function

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have :

$$R(z) = \gamma/(1-\gamma) = \bar{R},$$

$$u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z) = g(a)u(z).$$

If $A = \operatorname{diag}\sigma$, $\sigma^1 = 0$, $\mu^1 = 0$ (the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$, then, by the Cauchy–Schwarz inequality,

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2 .$$

The inequality

$$\beta > \frac{\gamma}{1-\gamma}\frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2$$

(implying the bound $\beta > \bar{\eta}\bar{R}$) ensures the existence of a classical supersolution.