Nonlinear Expectations and Stochastic Calculus under Uncertainty

—with Robust Central Limit Theorem and G-Brownian Motion

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- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $B_b(\Omega)$: all bounded functions in $L^0(\Omega)$;
- $C_b(\Omega)$: all continuous functions in $B_b(\Omega)$.

All along this section, we consider a given subset $\mathcal{P} \subseteq \mathcal{M}$.
We denote

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

One can easily verify the following theorem.

Theorem

*The set function $c(\cdot)$ is a Choquet capacity, i.e. (see [**?**, **?**]),*

1. $0 \leq c(A) \leq 1, \quad \forall A \subset \Omega.$
2. *If* $A \subset B$, *then* $c(A) \leq c(B)$.
3. *If* $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\Omega)$, *then* $c(\cup A_n) \leq \sum c(A_n)$.
4. *If* $(A_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{B}(\Omega)$: $A_n \uparrow A = \cup A_n$, *then* $c(\cup A_n) = \lim_{n\to\infty} c(A_n)$.

Furthermore, we have

Theorem

For each $A \in \mathcal{B}(\Omega)$, we have

$$c(A) = \sup\{c(K) : \ K \textit{ compact } \ K \subset A\}.$$

Proof.

It is simply because

$$c(A) = \sup_{P \in \mathcal{P}} \sup_{\substack{K \text{ compact} \\ K \subset A}} P(K) = \sup_{\substack{K \text{ compact} \\ K \subset A}} \sup_{P \in \mathcal{P}} P(K) = \sup_{\substack{K \text{ compact} \\ K \subset A}} c(K).$$

□

Definition

We use the standard capacity-related vocabulary: a set A is **polar** if $c(A) = 0$ and a property holds "**quasi-surely**" (q.s.)"qs if it holds outside a polar set.

We also have in a trivial way a Borel-Cantelli Lemma.

Lemma

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Borel sets such that

$$\sum_{n=1}^{\infty} c(A_n) < \infty.$$

Then $\limsup_{n\to\infty} A_n$ is polar .

Proof.

Applying the Borel-Cantelli Lemma under each probability $P \in \mathcal{P}$. □

The following theorem is Prokhorov's theorem.

Theorem

$\mathcal{P}$ is relatively compact if and only if for each $\varepsilon > 0$, there exists a compact set K such that $c(K^c) < \varepsilon$.

The following two lemmas can be found in [?].

Lemma

$\mathcal{P}$ is relatively compact if and only if for each sequence of closed sets $F_n \downarrow \emptyset$, we have $c(F_n) \downarrow 0$.

Proof.

We outline the proof for the convenience of readers.
"$\Longrightarrow$" part: It follows from Theorem –newth6 that for each fixed $\varepsilon > 0$, there exists a compact set K such that $c(K^c) < \varepsilon$. Note that $F_n \cap K \downarrow \emptyset$, then there exists an $N > 0$ such that $F_n \cap K = \emptyset$ for $n \geq N$, which implies $\lim_n c(F_n) < \varepsilon$. Since ε can be arbitrarily small, we obtain $c(F_n) \downarrow 0$.
"$\Longleftarrow$" part: For each $\varepsilon > 0$, let $(A_i^k)_{i=1}^{\infty}$ be a sequence of open balls of radius $1/k$ covering Ω. Observe that $(\cup_{i=1}^{n} A_i^k)^c \downarrow \emptyset$, then there exists an n_k such that $c((\cup_{i=1}^{n_k} A_i^k)^c) < \varepsilon 2^{-k}$. Set $K = \overline{\cap_{k=1}^{\infty} \cup_{i=1}^{n_k} A_i^k}$. It is easy to check that K is compact and $c(K^c) < \varepsilon$. Thus by Theorem –newth6, $\mathcal{P}$ is relatively compact. □

Lemma

Let $\mathcal{P}$ be weakly compact. Then for each sequence of closed sets $F_n \downarrow F$, we have $c(F_n) \downarrow c(F)$.

Proof.

We outline the proof for the convenience of readers. For each fixed $\varepsilon > 0$, by the definition of $c(F_n)$, there exists a $P_n \in \mathcal{P}$ such that $P_n(F_n) \geq c(F_n) - \varepsilon$. Since $\mathcal{P}$ is weakly compact, there exist P_{n_k} and $P \in \mathcal{P}$ such that P_{n_k} converge weakly to P. Thus

$$P(F_m) \geq \limsup_{k\to\infty} P_{n_k}(F_m) \geq \limsup_{k\to\infty} P_{n_k}(F_{n_k}) \geq \lim_{n\to\infty} c(F_n) - \varepsilon.$$

Letting $m \to \infty$, we get $P(F) \geq \lim_{n\to\infty} c(F_n) - \varepsilon$, which yields $c(F_n) \downarrow c(F)$. □

Following [**?**] (see also [**?**, **?**]) the upper expectation of $\mathcal{P}$ is defined as follows: for each $X \in L^0(\Omega)$ such that $E_P[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] = \mathbb{E}^{\mathcal{P}}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$

It is easy to verify

Theorem

The upper expectation $\mathbb{E}[\cdot]$ of the family $\mathcal{P}$ is a sublinear expectation on $B_b(\Omega)$ as well as on $C_b(\Omega)$, i.e.,

1. *for all X, Y in $B_b(\Omega)$, $X \geq Y \Longrightarrow \mathbb{E}[X] \geq \mathbb{E}[Y]$.*
2. *for all X, Y in $B_b(\Omega)$, $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.*
3. *for all $\lambda \geq 0$, $X \in B_b(\Omega)$, $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.*
4. *for all $c \in \mathbb{R}$, $X \in B_b(\Omega)$, $\mathbb{E}[X+c] = \mathbb{E}[X] + c$.*

Moreover, it is also easy to check

Theorem

We have

1. *Let* $\mathbb{E}[X_n]$ *and* $\mathbb{E}[\sum_{n=1}^{\infty} X_n]$ *be finite. Then* $\mathbb{E}[\sum_{n=1}^{\infty} X_n] \leq \sum_{n=1}^{\infty} \mathbb{E}[X_n]$.
2. *Let* $X_n \uparrow X$ *and* $\mathbb{E}[X_n]$, $\mathbb{E}[X]$ *be finite. Then* $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$.

Definition

The functional $\mathbb{E}[\cdot]$ is said to be **regular** if for each $\{X_n\}_{n=1}^{\infty}$ in $C_b(\Omega)$ such that $X_n \downarrow 0$ on Ω, we have $\mathbb{E}[X_n] \downarrow 0$.

Similar to Lemma –Lemma1 we have:

Theorem

$\mathbb{E}[\cdot]$ *is regular if and only if* $\mathcal{P}$ *is relatively compact.*

Proof.

"$\Longrightarrow$" part: For each sequence of closed subsets $F_n \downarrow \emptyset$ such that F_n, $n = 1, 2, \cdots$, are non-empty (otherwise the proof is trivial), there exists $\{g_n\}_{n=1}^{\infty} \subset C_b(\Omega)$ satisfying

$$0 \leq g_n \leq 1, \quad g_n = 1 \text{ on } F_n \text{ and } g_n = 0 \text{ on } \{\omega \in \Omega : d(\omega, F_n) \geq \frac{1}{n}\}.$$

We set $f_n = \wedge_{i=1}^{n} g_i$, it is clear that $f_n \in C_b(\Omega)$ and $\mathbf{1}_{F_n} \leq f_n \downarrow 0$. $\mathbb{E}[\cdot]$ is regular implies $\mathbb{E}[f_n] \downarrow 0$ and thus $c(F_n) \downarrow 0$. It follows from Lemma –Lemma1 that $\mathcal{P}$ is relatively compact.

"$\Longleftarrow$" part: For each $\{X_n\}_{n=1}^{\infty} \subset C_b(\Omega)$ such that $X_n \downarrow 0$, we have

$$\mathbb{E}[X_n] = \sup_{P \in \mathcal{P}} E_P[X_n] = \sup_{P \in \mathcal{P}} \int_0^{\infty} P(\{X_n \geq t\})dt \leq \int_0^{\infty} c(\{X_n \geq t\})dt.$$

For each fixed $t > 0$, $\{X_n \geq t\}$ is a closed subset and $\{X_n \geq t\} \downarrow \emptyset$ as $n \uparrow \infty$. By Lemma –Lemma1, $c(\{X_n \geq t\}) \downarrow 0$ and thus $\int_0^{\infty} c(\{X_n \geq t\})dt \downarrow 0$. Consequently $\mathbb{E}[X_n] \downarrow 0$. □

We set, for $p > 0$,

- $\mathcal{L}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\}$;
- $\mathcal{N}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = 0\}$;
- $\mathcal{N} := \{X \in L^0(\Omega) : X = 0,\ c\text{-q.s.}\}$.

It is seen that $\mathcal{L}^p$ and $\mathcal{N}^p$ are linear spaces and $\mathcal{N}^p = \mathcal{N}$, for each $p > 0$. We denote $\mathbb{L}^p := \mathcal{L}^p / \mathcal{N}$. As usual, we do not take care about the distinction between classes and their representatives.

Lemma

Let $X \in \mathbb{L}^p$. *Then for each* $\alpha > 0$

$$c(\{|X| > \alpha\}) \leq \frac{\mathbb{E}[|X|^p]}{\alpha^p}.$$

Proof.

Just apply Markov inequality under each $P \in \mathcal{P}$. □

Similar to the classical results, we get the following proposition and the proof is omitted which is similar to the classical arguments.

Proposition.

We have

1. For each $p \geq 1$, $\mathbb{L}^p$ is a Banach space under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.
2. For each $p < 1$, $\mathbb{L}^p$ is a complete metric space under the distance $d(X, Y) := \mathbb{E}[|X - Y|^p]$.

□

We set

$$\mathcal{L}^\infty := \{X \in L^0(\Omega) : \exists \text{ a constant } M, \text{ s.t. } |X| \leq M, \quad \text{q.s.}\};$$
$$\mathbb{L}^\infty := \mathcal{L}^\infty / \mathcal{N}.$$

Proposition.

Under the norm

$$\|X\|_\infty := \inf\left\{M \geq 0 : |X| \leq M, \quad \text{q.s.}\right\},$$

$\mathbb{L}^\infty$ is a Banach space. □

Proof.

From $\{|X| > \|X\|_\infty\} = \cup_{n=1}^\infty \left\{|X| \geq \|X\|_\infty + \frac{1}{n}\right\}$ we know that $|X| \leq \|X\|_\infty$, q.s., then it is easy to check that $\|\cdot\|_\infty$ is a norm. The proof of the completeness of $\mathbb{L}^\infty$ is similar to the classical result. □

With respect to the distance defined on $\mathbb{L}^p$, $p > 0$, we denote by

- $\mathbb{L}_b^p$ the completion of $B_b(\Omega)$.
- $\mathbb{L}_c^p$ the completion of $C_b(\Omega)$.

By Proposition –Prop3, we have

$$\mathbb{L}_c^p \subset \mathbb{L}_b^p \subset \mathbb{L}^p, \quad p > 0.$$

The following Proposition is obvious and the proof is left to the reader.

Proposition.

We have

1. Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $X \in \mathbb{L}^p$ and $Y \in \mathbb{L}^q$ implies

$$XY \in \mathbb{L}^1 \text{ and } \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}};$$

   Moreover $X \in \mathbb{L}_c^p$ and $Y \in \mathbb{L}_c^q$ implies $XY \in \mathbb{L}_c^1$;
2. $\mathbb{L}^{p_1} \subset \mathbb{L}^{p_2}$, $\mathbb{L}_b^{p_1} \subset \mathbb{L}_b^{p_2}$, $\mathbb{L}_c^{p_1} \subset \mathbb{L}_c^{p_2}$, $0 < p_2 \leq p_1 \leq \infty$;
3. $\|X\|_p \uparrow \|X\|_\infty$, for each $X \in \mathbb{L}^\infty$.

□

Proposition.

Let $p \in (0, \infty]$ and (X_n) be a sequence in $\mathbb{L}^p$ which converges to X in $\mathbb{L}^p$. Then there exists a subsequence (X_{n_k}) which converges to X quasi-surely in the sense that it converges to X outside a polar set. □

Proof.

Let us assume $p \in (0, \infty)$, the case $p = \infty$ is obvious since the convergence in $\mathbb{L}^\infty$ implies the convergence in $\mathbb{L}^p$ for all p.
One can extract a subsequence (X_{n_k}) such that

$$\mathbb{E}[|X - X_{n_k}|^p] \leq 1/k^{p+2}, \quad k \in \mathbb{N}.$$

We set for all k

$$A_k = \{|X - X_{n_k}| > 1/k\},$$

then as a consequence of the Markov property (Lemma –markov) and the Borel-Cantelli Lemma –BorelC, $c(\overline{\lim}_{k\to\infty} A_k) = 0$. As it is clear that on $(\overline{\lim}_{k\to\infty} A_k)^c$, (X_{n_k}) converges to X, the proposition is proved. □

We now give a description of $\mathbb{L}^p_b$.

Proposition.

"~~Prop5~~For each $p > 0$,

$$\mathbb{L}^p_b = \{X \in \mathbb{L}^p : \lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0\}.$$

□

Proof.

We denote $J_p = \{X \in \mathbb{L}^p : \lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0\}$. For each $X \in J_p$ let $X_n = (X \wedge n) \vee (-n) \in B_b(\Omega)$. We have

$$\mathbb{E}[|X - X_n|^p] \leq \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] \to 0, \text{ as } n \to \infty.$$

Thus $X \in \mathbb{L}_b^p$.
On the other hand, for each $X \in \mathbb{L}_b^p$, we can find a sequence $\{Y_n\}_{n=1}^\infty$ in $B_b(\Omega)$ such that $\mathbb{E}[|X - Y_n|^p] \to 0$. Let $y_n = \sup_{\omega\in\Omega} |Y_n(\omega)|$ and $X_n = (X \wedge y_n) \vee (-y_n)$. Since $|X - X_n| \leq |X - Y_n|$, we have $\mathbb{E}[|X - X_n|^p] \to 0$. This clearly implies that for any sequence (α_n) tending to ∞, $\lim_{n\to\infty} \mathbb{E}[|X - (X \wedge \alpha_n) \vee (-\alpha_n)|^p] = 0$.
Now we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] &= \mathbb{E}[(|X| - n + n)^p \mathbf{1}_{\{|X|>n\}}] \\ &\leq (1 \vee 2^{p-1}) \left(\mathbb{E}[(|X| - n)^p \mathbf{1}_{\{|X|>n\}}] + n^p c(|X| > n)\right).\end{aligned}$$

The first term of the right hand side tends to 0 since

$$\mathbb{E}[(|X| - n)^p \mathbf{1}_{\{|X|>n\}}] = \mathbb{E}[|X - (X \wedge n) \vee (-n)|^p] \to 0.$$

Proposition.

Let $X \in \mathbb{L}_b^1$. Then for each $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $A \in \mathcal{B}(\Omega)$ with $c(A) \leq \delta$, we have $\mathbb{E}[|X|\mathbf{1}_A] \leq \varepsilon$. □

Proof.

For each $\varepsilon > 0$, by Proposition ~~Prop5~~, there exists an $N > 0$ such that $\mathbb{E}[|X|\mathbf{1}_{\{|X|>N\}}] \leq \frac{\varepsilon}{2}$. Take $\delta = \frac{\varepsilon}{2N}$. Then for a subset $A \in \mathcal{B}(\Omega)$ with $c(A) \leq \delta$, we have

$$\mathbb{E}[|X|\mathbf{1}_A] \leq \mathbb{E}[|X|\mathbf{1}_A\mathbf{1}_{\{|X|>N\}}] + \mathbb{E}[|X|\mathbf{1}_A\mathbf{1}_{\{|X|\leq N\}}]$$
$$\leq \mathbb{E}[|X|\mathbf{1}_{\{|X|>N\}}] + Nc(A) \leq \varepsilon.$$

□

It is important to note that not every element in $\mathbb{L}^p$ satisfies the condition $\lim_{n\to\infty}\mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0$. We give the following two counterexamples to show that $\mathbb{L}^1$ and $\mathbb{L}_b^1$ are different spaces even under the case that $\mathcal{P}$ is weakly compact.

Example

Let $\Omega = \mathbb{N}$, $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ where $P_1(\{1\}) = 1$ and $P_n(\{1\}) = 1 - \frac{1}{n}$, $P_n(\{n\}) = \frac{1}{n}$, for $n = 2, 3, \cdots$. $\mathcal{P}$ is weakly compact. We consider a function X on $\mathbb{N}$ defined by $X(n) = n$, $n \in \mathbb{N}$. We have $\mathbb{E}[|X|] = 2$ but $\mathbb{E}[|X|\mathbf{1}_{\{|X|>n\}}] = 1 \not\to 0$. In this case, $X \in \mathbb{L}^1$ but $X \notin \mathbb{L}_b^1$.

Example

Let $\Omega = \mathbb{N}$, $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ where $P_1(\{1\}) = 1$ and $P_n(\{1\}) = 1 - \frac{1}{n^2}$, $P_n(\{kn\}) = \frac{1}{n^3}$, $k = 1, 2, \ldots, n$,for $n = 2, 3, \cdots$. $\mathcal{P}$ is weakly compact. We consider a function X on $\mathbb{N}$ defined by $X(n) = n$, $n \in \mathbb{N}$. We have $\mathbb{E}[|X|] = \frac{25}{16}$ and $n\mathbb{E}[\mathbf{1}_{\{|X| \geq n\}}] = \frac{1}{n} \longrightarrow 0$, but $\mathbb{E}[|X|\mathbf{1}_{\{|X| \geq n\}}] = \frac{1}{2} + \frac{1}{2n} \not\longrightarrow 0$. In this case, X is in $\mathbb{L}^1$, continuous and $n\mathbb{E}[\mathbf{1}_{\{|X| \geq n\}}] \longrightarrow 0$, but it is not in $\mathbb{L}_b^1$.

Definition

A mapping X on Ω with values in a topological space is said to be quasi-continuous (q.c.) if

$\forall \varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continu

Definition

We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s..

Proposition.

Let $p > 0$. Then each element in $\mathbb{L}_c^p$ has a quasi-continuous version. □

Proof.

Let (X_n) be a Cauchy sequence in $C_b(\Omega)$ for the distance on $\mathbb{L}^p$. Let us choose a subsequence $(X_{n_k})_{k\geq 1}$ such that

$$\mathbb{E}[|X_{n_{k+1}} - X_{n_k}|^p] \leq 2^{-2k}, \quad \forall k \geq 1,$$

and set for all k,

$$A_k = \bigcup_{i=k}^{\infty} \{|X_{n_{i+1}} - X_{n_i}| > 2^{-i/p}\}.$$

Thanks to the subadditivity property and the Markov inequality, we have

$$c(A_k) \leq \sum_{i=k}^{\infty} c(|X_{n_{i+1}} - X_{n_i}| > 2^{-i/p}) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}.$$

As a consequence $\lim_{k\to\infty} c(A_k) = 0$, so the Borel set $A = \bigcap_{k=1}^{\infty} A_k$ is

The following theorem gives a concrete characterization of the space $\mathbb{L}_c^p$.

Theorem

For each $p > 0$,

$$\mathbb{L}_c^p = \{X \in \mathbb{L}^p : X \textit{ has a q.-c. version, } \lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0\}.$$

Proof.

We denote

$$J_p = \{X \in \mathbb{L}^p : X \text{ has a quasi-continuous version, } \lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0$$

Let $X \in \mathbb{L}_c^p$, we know by Proposition –qc that X has a quasi-continuous version. Since $X \in \mathbb{L}_b^p$, we have by Proposition –Prop5 that $\lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0$. Thus $X \in J_p$.
On the other hand, let $X \in J_p$ be quasi-continuous. Define $Y_n = (X \wedge n) \vee (-n)$ for all $n \in \mathbb{N}$. As $\mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] \to 0$, we have $\mathbb{E}[|X - Y_n|^p] \to 0$.
Moreover, for all $n \in \mathbb{N}$, as Y_n is quasi-continuous , there exists a closed set F_n such that $c(F_n^c) < \frac{1}{n^{p+1}}$ and Y_n is continuous on F_n. It follows from Tietze's extension theorem that there exists $Z_n \in C_b(\Omega)$ such that

$$|Z_n| \leq n \text{ and } Z_n = Y_n \text{ on } F_n.$$

We then have

$$(2n)^p$$

We give the following example to show that $\mathbb{L}_c^p$ is different from $\mathbb{L}_b^p$ even under the case that $\mathcal{P}$ is weakly compact.

Example

Let $\Omega = [0, 1]$, $\mathcal{P} = \{\delta_x : x \in [0, 1]\}$ is weakly compact. It is seen that $\mathbb{L}_c^p = C_b(\Omega)$ which is different from $\mathbb{L}_b^p$.

We denote $\mathbb{L}_c^\infty := \{X \in \mathbb{L}^\infty : X \text{ has a quasi-continuous version}\}$, we have

Proposition.

$\mathbb{L}_c^\infty$ is a closed linear subspace of $\mathbb{L}^\infty$. □

Proof.

For each Cauchy sequence $\{X_n\}_{n=1}^{\infty}$ of $\mathbb{L}_c^{\infty}$ under $\|\cdot\|_{\infty}$, we can find a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ such that $\|X_{n_{i+1}} - X_{n_i}\|_{\infty} \leq 2^{-i}$. We may further assume that each X_n is quasi-continuous. Then it is easy to prove that for each $\varepsilon > 0$, there exists an open set G such that $c(G) < \varepsilon$ and $|X_{n_{i+1}} - X_{n_i}| \leq 2^{-i}$ for all $i \geq 1$ on G^c, which implies that the limit belongs to $\mathbb{L}_c^{\infty}$. □

As an application of Theorem –Thm8, we can easily get the following results.

Proposition.

Assume that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version and that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{t\to\infty} \frac{f(t)}{t^p} = \infty$ and $\mathbb{E}[f(|X|)] < \infty$. Then $X \in \mathbb{L}_c^p$. □

Proof.

For each $\varepsilon > 0$, there exists an $N > 0$ such that $\frac{f(t)}{t^p} \geq \frac{1}{\varepsilon}$, for all $t \geq N$. Thus

$$\mathbb{E}[|X|^p \mathbf{1}_{\{|X|>N\}}] \leq \varepsilon \mathbb{E}[f(|X|)\mathbf{1}_{\{|X|>N\}}] \leq \varepsilon \mathbb{E}[f(|X|)].$$

Hence $\lim_{N\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>N\}}] = 0$. From Theorem –Thm8 we infer $X \in \mathbb{L}^p_c$. □

Lemma

Let $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}$ *converge weakly to* $P \in \mathcal{P}$. *Then for each* $X \in \mathbb{L}_c^1$, *we have* $E_{P_n}[X] \to E_P[X]$.

Proof.

We may assume that X is quasi-continuous, otherwise we can consider its quasi-continuous version which does not change the value E_Q for each $Q \in \mathcal{P}$. For each $\varepsilon > 0$, there exists an $N > 0$ such that $\mathbb{E}[|X|\mathbf{1}_{\{|X|>N\}}] < \frac{\varepsilon}{2}$. Set $X_N = (X \wedge N) \vee (-N)$. We can find an open subset G such that $c(G) < \frac{\varepsilon}{4N}$ and X_N is continuous on G^c. By Tietze's extension theorem, there exists $Y \in C_b(\Omega)$ such that $|Y| \leq N$ and $Y = X_N$ on G^c. Obviously, for each $Q \in \mathcal{P}$,

$$
\begin{aligned}
|E_Q[X] - E_Q[Y]| &\leq E_Q[|X - X_N|] + E_Q[|X_N - Y|] \\
&\leq \frac{\varepsilon}{2} + 2N\frac{\varepsilon}{4N} = \varepsilon.
\end{aligned}
$$

It then follows that

$$
\limsup_{n\to\infty} E_{P_n}[X] \leq \lim_{n\to\infty} E_{P_n}[Y] + \varepsilon = E_P[Y] + \varepsilon \leq E_P[X] + 2\varepsilon,
$$

and similarly $\liminf_{n\to\infty} E_{P_n}[X] \geq E_P[X] - 2\varepsilon$. Since ε can be arbitrarily small, we then have $E_{P_n}[X] \to E_P[X]$. □

Remark.

For continuous X, the above lemma is Lemma 3.8.7 in [?]. □

Now we give an extension of Theorem –Thm2.

Theorem

Let $\mathcal{P}$ be weakly compact and let $\{X_n\}_{n=1}^{\infty} \subset \mathbb{L}_c^1$ be such that $X_n \downarrow X$, q.s.. Then $\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$.

Remark.

It is important to note that X does not necessarily belong to $\mathbb{L}^1_c$. □

Proof.

For the case $\mathbb{E}[X] > -\infty$, if there exists a $\delta > 0$ such that $\mathbb{E}[X_n] > \mathbb{E}[X] + \delta$, $n = 1, 2, \cdots$, we then can find a $P_n \in \mathcal{P}$ such that $E_{P_n}[X_n] > \mathbb{E}[X] + \delta - \frac{1}{n}$, $n = 1, 2, \cdots$. Since $\mathcal{P}$ is weakly compact, we then can find a subsequence $\{P_{n_i}\}_{i=1}^{\infty}$ that converges weakly to some $P \in \mathcal{P}$. From which it follows that

$$E_P[X_{n_i}] = \lim_{j\to\infty} E_{P_{n_j}}[X_{n_i}] \geq \limsup_{j\to\infty} E_{P_{n_j}}[X_{n_j}]$$

$$\geq \limsup_{j\to\infty}\{\mathbb{E}[X] + \delta - \frac{1}{n_j}\} = \mathbb{E}[X] + \delta, \quad i = 1, 2, \cdots.$$

Thus $E_P[X] \geq \mathbb{E}[X] + \delta$. This contradicts the definition of $\mathbb{E}[\cdot]$. The proof for the case $\mathbb{E}[X] = -\infty$ is analogous. □

We immediately have the following corollary.

Corollary

Let $\mathcal{P}$ be weakly compact and let $\{X_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{L}_c^1$ decreasingly converging to 0 q.s.. Then $\mathbb{E}[X_n] \downarrow 0$.

Definition

Let I be a set of indices, $(X_t)_{t \in I}$ and $(Y_t)_{t \in I}$ be two processes indexed by I . We say that Y is a quasi-modification of X if for all $t \in I$, $X_t = Y_t$ q.s..

Remark.

In the above definition, quasi-modification is also called modification in some papers. □

We now give a Kolmogorov criterion for a process indexed by $\mathbb{R}^d$ with $d \in \mathbb{N}$.

Theorem

Let $p > 0$ and $(X_t)_{t\in[0,1]^d}$ be a process such that for all $t \in [0,1]^d$, X_t belongs to $\mathbb{L}^p$. Assume that there exist positive constants c and ε such that

$$\mathbb{E}[|X_t - X_s|^p] \leq c|t-s|^{d+\varepsilon}.$$

Then X admits a modification $\tilde{X}$ such that

$$\mathbb{E}\left[\left(\sup_{s\neq t}\frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\right)^p\right] < \infty,$$

for every $\alpha \in [0, \varepsilon/p)$. As a consequence, paths of $\tilde{X}$ are quasi-surely Höder continuous of order α for every $\alpha < \varepsilon/p$ in the sense that there exists a Borel set N of capacity 0 such that for all $w \in N^c$, the map $t \to \tilde{X}(w)$ is Höder continuous of order α for every $\alpha < \varepsilon/p$. Moreover, if $X_t \in \mathbb{L}^p_c$ for each t, then we also have $\tilde{X}_t \in \mathbb{L}^p_c$.

Proof.

Let D be the set of dyadic points in $[0,1]^d$:

$$D = \left\{ (\frac{i_1}{2^n}, \cdots, \frac{i_d}{2^n});\ n \in \mathbb{N}, i_1, \cdots, i_d \in \{0, 1, \cdots, 2^n\} \right\}.$$

Let $\alpha \in [0, \varepsilon/p)$. We set

$$M = \sup_{s,t \in D, s \neq t} \frac{|X_t - X_s|}{|t-s|^\alpha}.$$

Thanks to the classical Kolmogorov's criterion (see Revuz-Yor [**?**]), we know that for any $P \in \mathcal{P}$, $E_P[M^p]$ is finite and uniformly bounded with respect to P so that

$$\mathbb{E}[M^p] = \sup_{P \in \mathcal{P}} E_P[M^p] < \infty.$$

As a consequence, the map $t \to X_t$ is uniformly continuous on D quasi-surely and so we can define

In the following sections of this Chapter, let $\Omega = C_0^d(\mathbb{R}^+)$ denote the space of all $\mathbb{R}^d$–valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1,\omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

and let $\bar{\Omega} = (\mathbb{R}^d)^{[0,\infty)}$ denote the space of all $\mathbb{R}^d$–valued functions $(\bar{\omega}_t)_{t\in\mathbb{R}^+}$. Let $\mathcal{B}(\Omega)$ denote the σ-algebra generated by all open sets and let $\mathcal{B}(\bar{\Omega})$ denote the σ-algebra generated by all finite dimensional cylinder sets. The corresponding canonical process is $B_t(\omega) = \omega_t$ (respectively, $\bar{B}_t(\bar{\omega}) = \bar{\omega}_t$), $t \in [0,\infty)$ for $\omega \in \Omega$ (respectively, $\bar{\omega} \in \bar{\Omega}$). The spaces of Lipschitzian cylinder functions on Ω and $\bar{\Omega}$ are denoted respectively by

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) : \forall n \geq 1, t_1, \cdots, t_n \in [0,\infty), \forall \varphi \in C_{Lip}(\mathbb{R}^{d\times$$

$$L_{ip}(\bar{\Omega}) := \{\varphi(\bar{B}_{t_1}, \bar{B}_{t_2}, \cdots, \bar{B}_{t_n}) : \forall n \geq 1, t_1, \cdots, t_n \in [0,\infty), \forall \varphi \in C_{Lip}(\mathbb{R}^{d\times$$

Let $G(\cdot) : \mathbb{S}(d) \to \mathbb{R}$ be a given continuous monotonic and sublinear function. Following Sec.2 in Chap.–ch3, we can construct the corresponding G-expectation $\hat{\mathbb{E}}$ on $(\Omega, L_{ip}(\Omega))$. Due to the natural correspondence of $L_{ip}(\bar{\Omega})$ and $L_{ip}(\Omega)$, we also construct a sublinear expectation $\bar{\mathbb{E}}$ on $(\bar{\Omega}, L_{ip}(\bar{\Omega}))$ such that $(\bar{B}_t(\bar{\omega}))_{t\geq 0}$ is a G-Brownian motion.

The main objective of this section is to find a weakly compact family of (σ-additive) probability measures on $(\Omega, \mathcal{B}(\Omega))$ to represent G-expectation $\hat{\mathbb{E}}$. The following lemmas are a variety of Lemma –l-le3 and –l-le4.

Lemma

Let $0 \leq t_1 < t_2 < \cdots < t_m < \infty$ *and* $\{\varphi_n\}_{n=1}^{\infty} \subset C_{Lip}(\mathbb{R}^{d \times m})$ *satisfy* $\varphi_n \downarrow 0$. *Then* $\bar{\mathbb{E}}[\varphi_n(\bar{B}_{t_1}, \bar{B}_{t_2}, \cdots, \bar{B}_{t_m})] \downarrow 0$.

We denote
$\mathcal{T} := \{\underline{t} = (t_1, \ldots, t_m) : \forall m \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_m < \infty\}$.

Lemma

Let E be a finitely additive linear expectation dominated by $\bar{\mathbb{E}}$ on $L_{ip}(\bar{\Omega})$. Then there exists a unique probability measure Q on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that $E[X] = E_Q[X]$ for each $X \in L_{ip}(\bar{\Omega})$.

Proof.

For each fixed $\underline{t} = (t_1, \ldots, t_m) \in \mathcal{T}$, by Lemma –le3, for each sequence $\{\varphi_n\}_{n=1}^{\infty} \subset C_{Lip}(\mathbb{R}^{d\times m})$ satisfying $\varphi_n \downarrow 0$, we have $E[\varphi_n(\bar{B}_{t_1}, \bar{B}_{t_2}, \cdots, \bar{B}_{t_m})] \downarrow 0$. By Daniell-Stone's theorem (see Appendix B), there exists a unique probability measure $Q_{\underline{t}}$ on $(\mathbb{R}^{d\times m}, \mathcal{B}(\mathbb{R}^{d\times m}))$ such that $E_{Q_{\underline{t}}}[\varphi] = E[\varphi(\bar{B}_{t_1}, \bar{B}_{t_2}, \cdots, \bar{B}_{t_m})]$ for each $\varphi \in C_{Lip}(\mathbb{R}^{d\times m})$. Thus we get a family of finite dimensional distributions $\{Q_{\underline{t}} : \underline{t} \in \mathcal{T}\}$. It is easy to check that $\{Q_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ is consistent. Then by Kolmogorov's consistent theorem, there exists a probability measure Q on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that $\{Q_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ is the finite dimensional distributions of Q. Assume that there exists another probability measure $\bar{Q}$ satisfying the condition, by Daniell-Stone's theorem, Q and $\bar{Q}$ have the same finite-dimensional distributions. Then by monotone class theorem, $Q = \bar{Q}$. The proof is complete. □

Lemma

There exists a family of probability measures $\mathcal{P}_e$ *on* $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ *such that*

$$\bar{\mathbb{E}}[X] = \max_{Q \in \mathcal{P}_e} E_Q[X], \quad \textit{for } X \in L_{ip}(\bar{\Omega}).$$

Proof.

By the representation theorem of sublinear expectation and Lemma –le4, it is easy to get the result. □

For this $\mathcal{P}_e$, we define the associated capacity:

$$\tilde{c}(A) := \sup_{Q \in \mathcal{P}_e} Q(A), \quad A \in \mathcal{B}(\bar{\Omega}),$$

and the upper expectation for each $\mathcal{B}(\bar{\Omega})$-measurable real function X which makes the following definition meaningful:

$$\tilde{\mathbb{E}}[X] := \sup_{Q \in \mathcal{P}_e} E_Q[X].$$

Theorem

For $(\bar{B})_{t \geq 0}$, there exists a continuous modification $(\tilde{B})_{t \geq 0}$ of $\bar{B}$ (i.e., $\tilde{c}(\{\tilde{B}_t \neq \bar{B}_t\}) = 0$, for each $t \geq 0$) such that $\tilde{B}_0 = 0$.

Proof.

By Lemma –le5, we know that $\bar{\mathbb{E}} = \tilde{\mathbb{E}}$ on $L_{ip}(\bar{\Omega})$. On the other hand, we have

$$\tilde{\mathbb{E}}[|\bar{B}_t - \bar{B}_s|^4] = \bar{\mathbb{E}}[|\bar{B}_t - \bar{B}_s|^4] = d|t-s|^2 \quad \text{for } s,t \in [0,\infty),$$

where d is a constant depending only on G. By Theorem –ch6t128, there exists a continuous modification $\tilde{B}$ of $\bar{B}$. Since $\tilde{c}(\{\bar{B}_0 \neq 0\}) = 0$, we can set $\tilde{B}_0 = 0$. The proof is complete. □

For each $Q \in \mathcal{P}_e$, let $Q \circ \tilde{B}^{-1}$ denote the probability measure on $(\Omega, \mathcal{B}(\Omega))$ induced by $\tilde{B}$ with respect to Q. We denote $\mathcal{P}_1 = \{Q \circ \tilde{B}^{-1} : Q \in \mathcal{P}_e\}$. By Lemma –le6, we get

$$\tilde{\mathbb{E}}[|\tilde{B}_t - \tilde{B}_s|^4] = \tilde{\mathbb{E}}[|\bar{B}_t - \bar{B}_s|^4] = d|t-s|^2, \forall s, t \in [0, \infty).$$

Applying the well-known result of moment criterion for tightness of Kolmogorov-Chentsov's type (see Appendix B), we conclude that $\mathcal{P}_1$ is tight. We denote by $\mathcal{P} = \overline{\mathcal{P}}_1$ the closure of $\mathcal{P}_1$ under the topology of weak convergence, then $\mathcal{P}$ is weakly compact.

Now, we give the representation of G-expectation.

Theorem

For each continuous monotonic and sublinear function $G : \mathbb{S}(d) \to \mathbb{R}$, let $\hat{\mathbb{E}}$ be the corresponding G-expectation on $(\Omega, L_{ip}(\Omega))$. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X] \quad \text{for } X \in L_{ip}(\Omega).$$

Proof.

By Lemma –le5 and Lemma –le6, we have

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}_1} E_P[X] \quad \text{for } X \in L_{ip}(\Omega).$$

For each $X \in L_{ip}(\Omega)$, by Lemma –le3, we get $\hat{\mathbb{E}}[|X - (X \wedge N) \vee (-N)|] \downarrow 0$ as $N \to \infty$. Noting that $\mathcal{P} = \overline{\mathcal{P}}_1$, by the definition of weak convergence, we get the result. □

Remark.

In fact, we can construct the family $\mathcal{P}$ in a more explicit way: Let $(W_t)_{t\geq 0} = (W_t^i)_{i=1,t\geq 0}^d$ be a d-dimensional Brownian motion in this space. The filtration generated by W is denoted by $\mathcal{F}_t^W$. Now let Γ be the bounded, closed and convex subset in $\mathbb{R}^{d\times d}$ such that

$$G(A) = \sup_{\gamma\in\Gamma} \mathrm{tr}[A\gamma\gamma^T], \;\; A \in \mathbb{S}(d),$$

(see (–GaChII) in Ch. II) and $\mathcal{A}_\Gamma$ the collection of all Θ-valued $(\mathcal{F}_t^W)_{t\geq 0}$-adapted process $[0, \infty)$. We denote

$$B_t^\gamma := \int_0^T \gamma_s dW_s, \; t \geq 0, \;\; \gamma \in \mathcal{A}_\Gamma.$$

and $\mathcal{P}_0$ the collection of probability measures on the canonical space $(\Omega, \mathcal{B}(\Omega))$ induced by $\{B^\gamma : \gamma \in \mathcal{A}_\Gamma\}$. Then $\mathcal{P} = \overline{\mathcal{P}}_0$ (see [?] for details). □

According to Theorem –Gt34, we obtain a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ to represent G-expectation $\hat{\mathbb{E}}[\cdot]$. For this $\mathcal{P}$, we define the associated G-capacity:

$$\hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega)$$

and upper expectation for each $X \in L^0(\Omega)$ which makes the following definition meaningful:

$$\bar{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$

By Theorem –Gt34, we know that $\bar{\mathbb{E}} = \hat{\mathbb{E}}$ on $L_{ip}(\Omega)$, thus the $\hat{\mathbb{E}}[|\cdot|]$-completion and the $\bar{\mathbb{E}}[|\cdot|]$-completion of $L_{ip}(\Omega)$ are the same. For each $T > 0$, we also denote by $\Omega_T = C_0^d([0,T])$ equipped with the distance

$$\rho(\omega^1, \omega^2) = \left\|\omega^1 - \omega^2\right\|_{C_0^d([0,T])} := \max_{0 \le t \le T} |\omega_t^1 - \omega_t^2|.$$

We now prove that $L_G^1(\Omega) = \mathbb{L}_c^1$, where $\mathbb{L}_c^1$ is defined in Sec.1. First, we need the following classical approximation lemma.

Lemma

For each $X \in C_b(\Omega)$ *and* $n = 1, 2, \cdots$, *we denote*

$$X^{(n)}(\omega) := \inf_{\omega' \in \Omega} \{X(\omega') + n \|\omega - \omega'\|_{C_0^d([0,n])}\} \quad \textit{for } \omega \in \Omega.$$

Then the sequence $\{X^{(n)}\}_{n=1}^{\infty}$ *satisfies:*

(i) $-M \le X^{(n)} \le X^{(n+1)} \le \cdots \le X$, $M = \sup_{\omega \in \Omega} |X(\omega)|$;

(ii) $|X^{(n)}(\omega_1) - X^{(n)}(\omega_2)| \le n \|\omega_1 - \omega_2\|_{C_0^d([0,n])}$ for $\omega_1, \omega_2 \in \Omega$;

(iii) $X^{(n)}(\omega) \uparrow X(\omega)$ for $\omega \in \Omega$.

Proof.

(i) is obvious.
For (ii), we have

$$\begin{aligned} & X^{(n)}(\omega_1) - X^{(n)}(\omega_2) \\ \leq & \sup_{\omega' \in \Omega}\{[X(\omega') + n\|\omega_1 - \omega'\|_{C_0^d([0,n])}] - [X(\omega') + n\|\omega_2 - \omega'\|_{C_0^d} \\ \leq & n\|\omega_1 - \omega_2\|_{C_0^d([0,n])} \end{aligned}$$

and, symmetrically, $X^{(n)}(\omega_2) - X^{(n)}(\omega_1) \leq n\|\omega_1 - \omega_2\|_{C_0^d([0,n])}$. Thus (ii) follows.
We now prove (iii). For each fixed $\omega \in \Omega$, let $\omega_n \in \Omega$ be such that

$$X(\omega_n) + n\|\omega - \omega_n\|_{C_0^d([0,n])} \leq X^{(n)}(\omega) + \frac{1}{n}.$$

It is clear that $n\|\omega - \omega_n\|_{C_0^d([0,n])} \leq 2M + 1$ or $\|\omega - \omega_n\|_{C_0^d([0,n])} \leq \frac{2M+1}{n}$. Since $X \in C_b(\Omega)$, we get $X(\omega_n) \to X(\omega)$ as $n \to \infty$. We have

Proposition.

For each $X \in C_b(\Omega)$ and $\varepsilon > 0$, there exists $Y \in L_{ip}(\Omega)$ such that $\bar{\mathbb{E}}[|Y - X|] \leq \varepsilon$. $\square$

Proof.

We denote $M = \sup_{\omega\in\Omega}|X(\omega)|$. By Theorem –Thm2 and Lemma –le10, we can find $\mu > 0$, $T > 0$ and $\bar{X} \in C_b(\Omega_T)$ such that $\bar{\mathbb{E}}[|X - \bar{X}|] < \varepsilon/3$, $\sup_{\omega\in\Omega}|\bar{X}(\omega)| \le M$ and

$$|\bar{X}(\omega) - \bar{X}(\omega')| \le \mu \left\|\omega - \omega'\right\|_{C_0^d([0,T])} \quad \text{for } \omega, \omega' \in \Omega.$$

Now for each positive integer n, we introduce a mapping $\omega^{(n)}(\omega) : \Omega \to \Omega$:

$$\omega^{(n)}(\omega)(t) = \sum_{k=0}^{n-1} \frac{\mathbf{1}_{[t_k^n, t_{k+1}^n)}(t)}{t_{k+1}^n - t_k^n}[(t_{k+1}^n - t)\omega(t_k^n) + (t - t_k^n)\omega(t_{k+1}^n)] + \mathbf{1}_{[T,\infty)}($$

where $t_k^n = \frac{kT}{n}$, $k = 0, 1, \cdots, n$. We set $\bar{X}^{(n)}(\omega) := \bar{X}(\omega^{(n)}(\omega))$, then

$$\begin{aligned}|\bar{X}^{(n)}(\omega) - \bar{X}^{(n)}(\omega')| &\le \mu \sup_{t\in[0,T]} |\omega^{(n)}(\omega)(t) - \omega^{(n)}(\omega')(t)| \\ &= \mu \sup_{k\in[0,\cdots,n]} |\omega(t_k^n) - \omega'(t_k^n)|.\end{aligned}$$

By Proposition –pr11, we can easily get $L_G^1(\Omega) = \mathbb{L}_c^1$. Furthermore, we can get $L_G^p(\Omega) = \mathbb{L}_c^p$, $\forall p > 0$.
Thus, we obtain a pathwise description of $L_G^p(\Omega)$ for each $p > 0$:

$$L_G^p(\Omega) = \{X \in L^0(\Omega) : X \text{ has a quasi-continuous version and } \lim_{n\to\infty} \bar{\mathbb{E}}[|X|^p I_{\{$$

Furthermore, $\bar{\mathbb{E}}[X] = \hat{\mathbb{E}}[X]$, for each $X \in L_G^1(\Omega)$.

Exercise.

Show that, for each $p > 0$,

$$L_G^p(\Omega_T) = \{X \in L^0(\Omega_T) : X \text{ has a quasi-continuous version and } \lim_{n\to\infty} \bar{\mathbb{E}}[|X|$$

□