Backward Stochastic Differential Equations with Infinite Time Horizon

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Outline

1. General setup and standard results
   - The multi-dimensional nonlinear case
   - The one-dimensional nonlinear case

2. Multi-dimensional linear case
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2 Multi-dimensional linear case
General setup

Throughout this talk, we are given

- a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), carrying a standard \(d\)-dimensional Brownian motion \((W_t)_{t\geq 0}\),
- the filtration \((\tilde{\mathcal{F}}_t)\) generated by \(W\),
- the filtration \((\mathcal{F}_t)\), which is \((\tilde{\mathcal{F}}_t)\) augmented by all \(\mathbb{P}\)-null sets.

\[ \implies (\mathcal{F}_t) \text{ satisfies the usual conditions} \]

Adapted processes are always assumed to be \((\mathcal{F}_t)\)-adapted.
General setup

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We denote by \(\mathcal{M}^{2,q}(E)\) the Hilbert space of processes \(X\) with:

- \(X\) is progressively measurable, with values in the Euclidean space \(E\),
- \(\mathbb{E} \left[ \int_0^\infty e^{qs} \|X_s\|_E^2 \, ds \right] < \infty\).
Consider the BSDE with infinite time horizon

\[-dY_t = \psi(t, Y_t, Z_t)dt - Z_t dW_t, \quad t \in [0, T], \; T \geq 0. \quad (1)\]

- \(\psi: \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n) \rightarrow \mathbb{R}^n\) is such that \(\psi(\cdot, y, z)\) is a progressively measurable process.
- A solution is a couple of progressively measurable processes \((Y, Z)\) with values in \(\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n)\), such that, for all \(t \leq T\) with \(t, T \geq 0\),

\[Y_t = Y_T + \int_t^T \psi(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.\]
Assumption (A1)

(A1) There exist $C \geq 0$, $\gamma \geq 0$ and $\mu \in \mathbb{R}$, such that

1. $\psi$ is uniformly lipschitz, i.e.
   \[ |\psi(t, y, z) - \psi(t, y', z')| \leq C|y - y'| + \gamma||z - z'||; \]

2. $\psi$ is monotone in $y$:
   \[ \langle y - y', \psi(t, y, z) - \psi(t, y', z) \rangle \leq -\mu|y - y'|^2; \]

3. There exists $\rho \in \mathbb{R}$, such that $\rho > \gamma^2 - 2\mu$, and
   \[ \mathbb{E} \left[ \int_0^\infty e^{\rho s} |\psi(s, 0, 0)|^2 \, ds \right] \leq C. \]
Set $\lambda := \frac{\gamma^2}{2} - \mu$. This implies $\varrho > 2\lambda$. Darling and Pardoux (1997) established the following result.

**Theorem**

*If (A1) holds then BSDE (1) has a unique solution $(Y, Z)$ in $\mathcal{M}^{2,2\lambda}(\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n))$. The solution actually belongs to $\mathcal{M}^{2,\varrho}(\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n))$.***

The major restriction is the structural condition in part (3) of (A1):

- We want to solve the equation for arbitrary bounded $\psi(\cdot, 0, 0)$.
- So we need $\mu > \frac{1}{2}\gamma^2$.

This condition is not natural in applications and, hence, is very unpleasant.
The one-dimensional case \((n = 1)\)

- Significant improvement due to Briand and Hu (1998).
- Solution exists for all \(\mu > 0\), if \(\psi(\cdot, 0, 0)\) is bounded, i.e. \((3')\) \(|\psi(t, 0, 0)| \leq K\).
- \(\mu > 0\) means, \(\psi\) is dissipative with respect to \(y\).

**Theorem \((n = 1)\)**

Assume parts (1) and (2) of (A1) with \(\mu > 0\), and (3'). Then BSDE (1) has a solution \((Y, Z)\) which belongs to \(\mathcal{M}^{2, -2\mu}(\mathbb{R} \times \mathbb{R}^d)\) and such that \(Y\) is a bounded process.

*This solution is unique in the class of processes \((Y, Z)\), such that \(Y\) is continuous and bounded and \(Z\) belongs to \(\mathcal{M}^{2}_{loc}(\mathbb{R}^d)\).*
Idea of the proof

1. Consider the equation with finite time horizon $[0, m]$. Call the unique solution $(Y_m, Z_m)$.

2. Establish the a priori bound

$$|Y_m(\theta)| \leq \frac{K}{\mu}, \text{ for all } \theta.$$ 

3. Use this a priori bound to show that $(Y_m, Z_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^{2,-2\mu}(\mathbb{R} \times \mathbb{R}^d)$.

The crucial part is to establish the a priori bound.
The a priori bound

- Linearise \( \psi \) to

\[
\psi(s, Y_m, Z_m) = \alpha_m(s)Y_m(s) + \beta_m(s)Z_m(s) + \psi(s, 0, 0)
\]

with \( \alpha_m(s) \leq -\mu \) and \( \beta_m \) bounded.

- \( (Y_m, Z_m) \) solves the equation

\[
Y_m(t) = \int_t^m \left[ \alpha_m(s)Y_m(s) + \beta_m(s)Z_m(s) + \psi(s, 0, 0) \right] ds \\
- \int_t^m Z_m(s) dW_s.
\]
• Introduce

\[ R_m(t) := \exp\left(\int_\theta^t \alpha_m(s) \, ds\right), \]

\[ W_m(t) := W(t) - \int_0^t \beta_m(s) \, ds. \]

• Note that

\[ R_m(s) \leq e^{-\mu(s-\theta)} \]

and

\[ \int_\theta^\infty R_m(s) \, ds \leq \frac{1}{\mu}. \]
Apply Itô’s formula to the process $R_m Y_m$:

$$Y_m(\theta) = R_m(m) Y_m(m) + \int_\theta^m R_m(s) \psi(s, 0, 0) \, ds$$

$$- \int_\theta^m R_m(s) Z_m(s) \, dW_m(s).$$

Take into account that $Y_m(m) = 0$:

$$Y_m(\theta) = \int_\theta^m R_m(s) \psi(s, 0, 0) \, ds - \int_\theta^m R_m(s) Z_m(s) \, dW_m(s).$$
Using Girsanov’s theorem, we can consider $W_m$ as a Brownian motion with respect to an equivalent measure $Q_m$ and hence, we get, $Q_m$-a.s.,

$$
|Y_m(\theta)| = E^{Q_m}[|Y_m(\theta)| | F_{\theta}]
$$

$$
\leq E^{Q_m}\left[\int_{\theta}^{\infty} |\psi(s, 0, 0)| R_m(s) \, ds \mid F_{\theta}\right]
$$

$$
\leq \frac{K}{\mu}.
$$

In the end, this estimate assures also the boundedness of the limit process $Y$. 
If $Y$ is a multi-dimensional process ($n > 1$), we cannot use this Girsanov trick, because each coordinate needs its own transformation, and these transformations are not consistent among each other.

So we are restricted to the case $\mu > \frac{1}{2} \gamma^2$, whereas the case $\mu > 0$ could have multiple interesting applications, e.g. in stochastic differential games or for homogenisation of PDEs.
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Let us now consider the following equation:

\[-dY_t = [AY_t + \sum_{j=1}^{d} \Gamma_j^j Z_t^j + f_t]dt - Z_t dW_t, \ t \in [0, T], \ T \geq 0. \ (2)\]

- \(A, \Gamma_j^j \in \mathbb{R}^{n \times n}\).
- \(Z_t^j\) denotes the \(j\)-th column vector of \(Z_t \in \mathbb{R}^{n \times d}\).
- \(f_t \in \mathbb{R}^n\) is bounded by \(K\).
- \(A\) is assumed to be dissipative, i.e. there exists \(\mu > 0\) such that
  \[\langle y - y', A(y - y') \rangle \leq -\mu |y - y'|^2.\]
- The coefficients in equation (2) are non-stochastic and, except \(f_t\), time-independent.
As in the one-dimensional non-linear case, we are interested in progressively measurable solutions \((Y, Z)\), such that \(Y\) is bounded. This can be achieved by establishing the above mentioned a priori estimate

\[
|Y_m(\theta)| \leq \frac{K}{\mu}.
\]

To this end, we consider the dual process to \(Y_m\), denoted by \(X^x\). This process satisfies

\[
\begin{aligned}
dX_t^x &= A^*X_t^x\,dt + \sum_{j=1}^{d}(\Gamma^j)^*X_t^x\,dW_t^j \\
X_\theta^x &= x \in \mathbb{R}^n.
\end{aligned}
\]
By Itô’s formula and the Markov property of $X^x$, we obtain

$$|Y_m(\theta)| \leq \sup_{|x|=1} \mathbb{E} \left[ \int_{\theta}^{m} \langle X^x_t, f_t \rangle \, dt \mid \mathcal{F}_\theta \right]$$

$$\leq K \sup_{|x|=1} \mathbb{E} \int_{\theta}^{\infty} |X^x_t| \, dt.$$ 

$\implies$ Question of $L_1$-stability of $X^x$ with $|x| = 1$. We need

$$\mathbb{E} \int_{0}^{\infty} |X^x_t| \, dt \leq M.$$ 

Task: Find appropriate assumptions on $\Gamma^j$ and $\mu$. 
Lyapunov approach

Try to find “Lyapunov” function $v \in C^2(\mathbb{R}^n)$ with

1. $v \geq 0$,
2. $v(x) \leq c|x|$, for some $c > 0$,
3. $[\mathcal{L}v](x) \leq -\delta |x|$, for some $\delta > 0$.

Here $\mathcal{L}$ is the Kolmogorov operator of $X^x$, i.e.

$$dv(X^x_t) = [\mathcal{L}v](X^x_t)dt + \text{“martingale part”}.$$

This approach was used by Ichikawa (1984) to show stability properties of strongly continuous semigroups.
Itô’s formula and the Markov property of $X^x$ give us

\[
\mathbb{E}[\nu(X_t^x) - \nu(X_\theta^x)] = \mathbb{E} \int_\theta^t [\mathcal{L}\nu](X_s^x) \, ds
\]

\[
\leq -\delta \mathbb{E} \int_\theta^t |X_s^x| \, ds.
\]

By showing $\mathbb{E}[\nu(X_t^x)] \to 0$ as $t \to \infty$, we obtain

\[
\mathbb{E} \int_\theta^\infty |X_s^x| \, ds \leq \frac{1}{\delta} \mathbb{E}[\nu(X_\theta^x)] \leq \frac{c}{\delta} |x|
\]

\[
\leq \frac{c}{\delta} =: M.
\]
How to find a Lyapunov function?

- First idea: \( v(x) = |x| \).
- Problem: \( v \) is not \( C^2 \), hence Itô’s formula inapplicable.

- Second idea: Define, for \( \varepsilon > 0 \),
  \[
  v_\varepsilon(x) = \sqrt{|x|^2 + \varepsilon}.
  \]
- \( v_\varepsilon(x) \to |x| \).
How to proceed?

- Calculate \([L v_\epsilon](x)\).
- Choose \(\mu\) large enough, such that the coefficient in front of \(|x|^4\) is negative. This choice will depend on \(\Gamma^j\).
- Find appropriate \(\kappa_\epsilon > 0, \kappa_\epsilon \to 0\) and split the integral on the RHS:

\[
\mathbb{E} v_\epsilon(X^x_t) - \mathbb{E} v_\epsilon(X^x_\theta) = \mathbb{E} \int_\theta^t [L v_\epsilon](X^x_s) \, ds
\]

\[
= \mathbb{E} \int_\theta^t [L v_\epsilon](X^x_s) \mathbb{1}\{|X^x_s| \geq \kappa_\epsilon\} \, ds + \mathbb{E} \int_\theta^t [L v_\epsilon](X^x_s) \mathbb{1}\{|X^x_s| < \kappa_\epsilon\} \, ds
\]
Obtain with $\varepsilon \to 0$

$$\mathbb{E}|X_t| - \mathbb{E}|X_\theta| \leq -\delta \mathbb{E} \int_\theta^t |X_s| \, ds.$$

Apply Gronwall’s lemma to $\Phi(t) := \mathbb{E}|X_t|$.

$$\implies \lim_{t \to \infty} \mathbb{E}|X_t| = 0$$

$$\implies \mathbb{E} \int_\theta^t |X_t| \leq \frac{1}{\delta} =: M$$

So $X^x$ is $L_1$-stable and equation (2) admits a bounded solution.
Assume $\Gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ and $\gamma := \max\{ |\gamma_1|, |\gamma_2| \}$.

- $[L v_\varepsilon](x) \leq \frac{\frac{1}{8} (\gamma_1 - \gamma_2)^2 - \mu}{(|x|^2 + \varepsilon)^\frac{3}{2}} |x|^4 + \frac{1}{2} \varepsilon \gamma^2 |x|^2$

- For $\mu > \frac{1}{8} (\gamma_1 - \gamma_2)^2$ is $X^x$ $L_1$-stable, and equation (2) has a bounded solution.

- The general result from the first part requires the much stronger assumption

  $\mu > \frac{1}{2} \|\Gamma\|^2 = \frac{1}{2} (\gamma_1^2 + \gamma_2^2)$. 
**L₂-stability is strictly stronger than L₁-stability.**

**Example**

We take \( n = d = 1 \) and consider the following equation:

\[
\begin{cases}
    dX_t = -\mu X_t \, dt + \gamma X_t \, dW_t \\
    X_0 = 1.
\end{cases}
\]

The solution is a geometric Brownian motion

\[
X_t = e^{-\mu t} e^{\gamma W_t - \frac{1}{2} \gamma^2 t}
\]

and

\[
\mathbb{E}|X_t| = e^{-\mu t}, \quad \mathbb{E}|X_t|^2 = e^{-2\mu t} e^{\gamma^2 t}.
\]

So \( X \) is \( L_1 \)-stable for each \( \mu > 0 \), but \( L_2 \)-stable only for \( \mu > \frac{1}{2} \gamma^2 \).
General setup and standard results
Multi-dimensional linear case

References


