# Finance, Insurance, and Stochastic Control (III)

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# Outline

1. Reinsurance and Stochastic Control Problems
2. Proportional Reinsurance with Diffusion Models
3. General Reinsurance Problems
4. Admissibility of Strategies
5. Existence of Admissible Strategies
6. Utility Optimization

### Basic Idea

An insurance company may choose to "cede" some of its risk to a reinsurer by paying a premium. Thus the reserve may look like

$$X_t = x + \int_0^t c_s^h(1+\rho_s)ds - \int_0^t \int_{\mathbb{R}_+} h(s,x)\mu(dxds),$$

where $h$ is the "retention function"

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Common types of retention functions:

- $h(x) = \alpha x$, $0 \le \alpha \le 1$ — Proportional Reinsurance
- $h(x) = \alpha \wedge x$, $\alpha > 0$ — Stop-loss Reinsurance

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**Purpose**

Determine the "reasonable" reinsurance premium, find the "best" reinsurance policy,..., etc.

# Generalized Cramér-Lundberg model

- $(\Omega, \mathscr{F}, P)$ — a complete probability space
- $W = \{W_t\}_{t\geq 0}$— a $d$-dimensional Brownian Motion
- $p = \{p_t\}_{t\geq 0}$ — stationary Poisson point process, $\perp\!\!\!\perp W$
- $N_p(dtdz)$ — counting measure of $p$ on $(0, \infty) \times \mathbb{R}_+$
- $\hat{N}_p(dtdz) = E(N_p(dtdz)) = \nu(dz)dt$
- $\mathbf{F} = \mathbf{F}^W \otimes \mathbf{F}^p$,
- $F_p^q \stackrel{\triangle}{=} \{\varphi : \mathbf{F}^p\text{-predi'ble}, \ E \int_0^T \int_{\mathbb{R}_+} |\varphi|^q d\nu ds < \infty, \ q \geq 1\}$

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## Claim Process

$$S_t = \int_0^{t+}\int_{\mathbb{R}_+} f(s,x,\omega)N_p(dsdx), \quad t\geq 0, \quad f\in F_p. \qquad (1)$$

Compound Poisson Case: $f(t,z)\equiv z$, $\nu(\mathbb{R}_+) = \lambda$.

# Profit Margin Principle

## A "Counting Principle" for Reinsurance Premiums

- $\rho$— original safety loading of the cedent company
- $\rho^r$— safety loading of the reinsurance company
- $\rho^\alpha$— modified safety loading of the cedent company (after reinsurance)

If the claim size is $U$, then the "*profit margin principle*" states

$$\underbrace{(1+\rho)E[U]}_{\text{original premium}} = \underbrace{(1+\rho^r)E[U-h(U)]}_{\text{premium to the reinsurance company}} + \underbrace{(1+\rho^\alpha)E[h(U)]}_{\text{modified premium}}. \qquad (2)$$

$\rho^r = \rho^\alpha = \rho$ — "Cheap" Reinsurance

$\rho^r \neq \rho^\alpha$ — "Non-cheap" Reinsurance

# Existing Literature

- Stop-Loss Reinsurance (e.g., Sondermann (1991), Mnif-Sulem (2001), Azcue-Muler (2005), ...)

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  - General reserve models (Liu-M. 2009, ...)

# Proportional Reinsurance with Diffusion Models

The following case study is based on Hojgaard-Taksar (1997).

Consider the reserve with "proportional reinsurance" :

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Replacing this by the following "Diffusion Model":

$$X_t = x + \int_0^t \mu\alpha_t dt + \int_0^t \sigma\alpha_t dW_t, \qquad t \geq 0, \tag{3}$$

where $\mu > 0$, $\sigma > 0$, and $\alpha_t \in [0,1]$ is a stochastic process representing the fraction of the incoming claim that the insurance company retains to itself. We call it a "*admissible reinsurance policy*" if it is $\mathbf{F}^W$-adapted.

# Proportional Reinsurance with Diffusion Models

- "*Return Function*":

$$J(x;\alpha) \triangleq E\int_0^\tau e^{-ct}X_t^{x,\alpha}dt,$$

where $\tau = \tau^{x,\alpha} = \inf\{t \geq 0 : X_t^{x,\alpha} = 0\}$ is the ruin time and $c > 0$ is the "*discount factor*".

- "*Value Function*":

$$V(x) = \sup_{\alpha\in\mathscr{A}} J(x;\alpha)$$

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**Note**

For any $\alpha \in \mathscr{A}$ and $x > 0$, define $\hat{\alpha}_t = \alpha_t \mathbf{1}_{\{t \leq \tau^{x,\alpha}\}}$. Then $\tau^{x,\hat{\alpha}} = \tau^{x,\alpha} \implies J(x,\hat{\alpha}) = J(x,\alpha)$. we can work on $\mathscr{A}'(x) \triangleq \{\alpha \in \mathscr{A} : \alpha_t = 0 \text{ for all } t > \tau^{x,\alpha}\}$ and

$$J'(x;\alpha) \triangleq E\int_0^{\infty} e^{-ct}X_t^{x,\alpha}dt, \qquad \alpha \in \mathscr{A}'(x).$$

**1. The Concavity of $V$.**

- For any $x^1, x^2 > 0$ and $\lambda \in (0,1)$, let $\alpha^i \in \mathscr{A}(x_i)$, $i = 1,2$. Define $\xi \triangleq \lambda x^1 + (1-\lambda)x^2$, $\alpha \triangleq \lambda\alpha^1 + (1-\lambda)\alpha^2$.

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- Denote $X^i = X^{x^i,\alpha^i}$ and $\tau^i = \tau^{x^i,\alpha^i}$, $i = 1, 2$. Then by the linearity of the reserve equation (3) one has

$$X_t \triangleq X_t^{\xi,\alpha} = \lambda X_t^1 + (1-\lambda)X_t^2, \quad \text{and} \quad \tau \triangleq \tau^{\xi,\alpha} = \tau^1 \vee \tau^2.$$

$$\Longrightarrow J(\xi,\alpha) = \lambda J(x^1,\alpha^1) + (1-\lambda)J(x^2,\alpha^2).$$

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$$\Longrightarrow J(\xi,\alpha) = \lambda J(x^1,\alpha^1) + (1-\lambda)J(x^2,\alpha^2).$$

- $\forall \varepsilon > 0$, choose $\alpha^i$, s.t. $J(x^i,\alpha^i) \geq V(x^i) - \varepsilon/2$, $i = 1,2$.

$$\begin{aligned}\Longrightarrow J(\xi,\alpha) &= \lambda J(x^1,\alpha^1) + (1-\lambda)J(x^2,\alpha^2)\\ &\geq \lambda V(x^1) + (1-\lambda)V(x^2) - \varepsilon\end{aligned}$$

$$\Longrightarrow V(\xi) \geq \lambda V(x^1) + (1-\lambda)V(x^2) - \varepsilon \quad \Longrightarrow \quad \text{Done!} \quad \blacksquare$$

## 2. The HJB Equation.

- Let $\tau$ be any $\mathbf{F}$-stopping time. By "*Bellman Principle*"

$$V(x) = \sup_{\alpha \in \mathscr{A}(x)} E\left\{ \int_0^{\tau^\alpha \wedge \tau} e^{-ct} X_t^{x,\hat{\alpha}} dt + e^{-c(\tau^\alpha \wedge \tau)} V(X^{x,\hat{\alpha}}_{\tau^\alpha \wedge \tau}) \right\}.$$

- $\forall \alpha \in \mathscr{A}$ and $h > 0$ let $\tau^h = \tau^h_\alpha \stackrel{\triangle}{=} h \wedge \inf\{t : |X^\alpha_t - x| > h\}$. Then $\tau^h < \infty$, a.s. and $\tau^h \to 0$, as $h \to 0$, a.s.
- Assume $V \in C^2$. For any $a \in [0, 1]$, define $\alpha \equiv a \in \mathscr{A}$. Then for any $h < x$, we have $\tau^h < \tau^\alpha$. Letting $\tau = \tau^h$ in (4) and applying Itô (to $F(t, x) = e^{-ct} V(x)$) we deduce

$$0 \ge E\left\{ \int_0^{\tau^h} e^{-ct} X_t^{x,\alpha} dt + e^{-ct} [\mathscr{L}^a V](X_t^{x,\alpha}) dt \right\},$$

where $[\mathscr{L}^a g](x) \stackrel{\triangle}{=} \frac{\sigma^2 a^2}{2} g''(x) + \mu a g'(x) - c g(x)$.

# The HJB Equation

- Letting $h \to 0$, one has

$$0 \geq x + [\mathscr{L}^a V](x).$$

$\Longrightarrow \quad 0 \geq x + \max_{a \in [0,1]} [\mathscr{L}^a V](x)$, since $a$ is arbitrary.

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- On the other hand, $\forall \delta > 0$, we choose $\alpha^* \in \mathscr{A}(x)$ s.t.

$$V(x) \leq E\left\{\int_0^{\tau^h_{\alpha^*}} e^{-ct} X_t^{x,\alpha^*} dt + e^{-c\tau^h_{\alpha^*}} V(X^{x,\alpha^*}_{\tau^h_{\alpha^*}})\right\} + \delta.$$

Letting $\delta = E[\tau^h_{\alpha^*}]^2$ and applying Itô again we have

$$0 \leq \frac{1}{E[\tau^h_{\alpha^*}]} E\left\{\int_0^{\tau^h_{\alpha^*}} e^{-ct}\{X_t^\alpha + \max_a [\mathscr{L}^a V](X_t^{x,\alpha})\} dt + \delta\right\}$$

$$\longrightarrow \quad x + \max_{a\in[0,1]}[\mathscr{L}^a V](x), \text{ as } h \to 0.$$

# The HJB Equation

We obtain the HJB equation:

$$\begin{cases} \max_{\alpha\in[0,1]} \left\{ \frac{\sigma^2\alpha^2}{2} V''(x) + \mu\alpha V'(x) - cV(x) + x \right\} = 0, \\ V(0) = 0. \end{cases} \tag{4}$$

We shall construct a solution to the HJB equation (4) that is concave and $C^2$ by using the technique of "*Principle of Smooth fit*" that we used before.

- First we note that if $\alpha(x) \in \text{argmax}_{\alpha\in[0,1]} \left\{ -\frac{\sigma^2\alpha^2}{2} V'' + \mu\alpha V' - cV + x \right\}$, then the first order condition tells us that

$$\alpha(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)}. \tag{5}$$

# Principle of Smooth Fit

- Plugging this into the HJB equation (4) we get

$$-\frac{\mu^2[V'(x)]^2}{2\sigma^2 V''(x)} - cV(x) + x = 0, \qquad x \in [0, \infty). \qquad (6)$$

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**Main Trick:**

Find a $C^2$ function $X : \mathbb{R} \mapsto [0, \infty)$, such that $V'(X(z)) = e^{-z}$!
(Note: Since $V$ is concave, one could argue that the Implicit Function Thm applies to equation: $F(X, z) = V'(X) - e^{-z} = 0$.)

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- Since $V'(X(z)) = e^{-z}$ and $V''(X(z)) = -\frac{e^{-z}}{X'(z)}$, replacing $x$ by $X(z)$ in (6) we obtain

$$\frac{\mu^2}{2\sigma^2} X'(z)e^{-z} - cV(X(z)) + X(z) = 0. \qquad (7)$$

# Principle of Smooth Fit

- Differentiating (7) w.r.t. $z$ and eliminating $V$:

$$\frac{\mu^2}{2\sigma^2}X''(z)e^{-z} - \left(\frac{\mu^2}{2\sigma^2} + c\right) e^{-z}X'(z) + X'(z) = 0.$$

Therefore, denoting $\gamma \stackrel{\triangle}{=} 2\sigma^2/\mu^2$, the equation becomes

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- Solving (8) explicitly we have $X'(z) = k_1 e^{(1+c\gamma)z-\gamma e^z}$ or

$$\begin{aligned} X(z) &= k_1 \int_{-\infty}^{z} e^{(1+c\gamma)y-\gamma e^y} dy + k_2 \\ &= k_1 \int_0^{e^z} y^{c\gamma} e^{-\gamma y} dy + k_2, \quad (y \mapsto e^y = y') \end{aligned}$$

—This is a $\Gamma$-integral!

# Principle of Smooth Fit

- Let $G$ be the c.d.f. of a Gamma distribution with parameter $(c\gamma+1, 1/\gamma)$. Then

$$X(z) = k_1 \frac{\Gamma(c\gamma+1)}{\gamma^{c\gamma+1}} G(e^z) + k_2 = k_1 G(e^z) + k_2.$$

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- Clearly $k_2 = X(-\infty) \geq 0$. By definition of $X$ we see that

$$-\ln(V'(x)) = \ln\left(G^{-1}\left(\frac{x-k_2}{k_1}\right)\right) \quad \text{or} \quad V'(x) = \frac{1}{G^{-1}\left(\frac{x-k_2}{k_1}\right)}.$$

$$\Longrightarrow \alpha(x) = \frac{\mu}{\sigma^2} k_1 G^{-1}\left(\frac{x-k_2}{k_1}\right) g\left(G^{-1}\left(\frac{x-k_2}{k_1}\right)\right), \; x \geq k_2,$$

where $g$ is the density function of $G$.

# Principle of Smooth Fit

- Change variable: $y = G^{-1}((x - k_2)/k_1)$, we have

$$\alpha(x) = \hat{\alpha}(y) = \frac{\mu k_1}{\sigma^2} y g(y), \qquad y \geq 0.$$

- Since $\hat{\alpha}(0) = 0$ and $\hat{\alpha}(\infty) = \infty$, we can find a $y_1 \in (0, \infty)$ such that $\hat{\alpha}(y_1) = 1$. Also, since

$$\hat{\alpha}'(y) = K y^{c\gamma} e^{-\gamma y}(c\gamma + 1 - \gamma y) > 0,$$

$$k_2 < x < k_1 G(y_1) + k_2 \stackrel{\triangle}{=} x_1,$$

$\hat{\alpha}$ is strictly increasing on $(k_2, x_1)$, and $\hat{\alpha}(y_1) = \alpha(x_1) = 1$.

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**Claim: $k_2 = 0$!**

For otherwise extending $G^{-1} \equiv 0$ on $(-\infty, 0]$ we have $\alpha(x) = 0$ for $x \leq k_2$. Then HJB equation implies $V(x) = -x/c$, for $x \leq k_2$. But for such $V$ the maximizer of (7) cannot be zero, whenever $\mu > 0$, a contradiction.

# Principle of Smooth Fit

- Thus

$$V(x) = \int_0^x \frac{1}{G^{-1}\left(\frac{u}{k_1}\right)} du + k_3, \qquad 0 \le x < x_1. \qquad (9)$$

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- Solving the non-homogeneous ODE we get

$$V(x) = \frac{x}{c} + \frac{\mu}{c^2} + K_4 e^{r_- x} + k_5 e^{r_+ x}.$$

where $r_\pm = \frac{-\frac{\mu}{\sigma} \pm \sqrt{\frac{\mu^2}{\sigma^2} + 2c}}{\sigma}$.

- Note that by concavity of $V$ we have $V'(x) = \mathscr{O}(1)$ or $V(x) = \mathscr{O}(x)$, as $x \to \infty$ Thus $k_5 = 0$. Renaming the constants we have

$$V(x) = \begin{cases} \displaystyle\int_0^x \frac{1}{G^{-1}\left(\frac{z}{k_1}\right)} dz, & 0 \leq x < x_1 \\ \displaystyle\frac{x}{c} + \frac{\mu}{c^2} + k_2 e^{r_- x} & x > x_1. \end{cases} \tag{10}$$

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**Principle of Smooth Fit**

Find $k_1$ and $k_2$ so that $V$ is $C^2$ at $x = x_1$.

- First note that

$$V'(x_1+) = \frac{1}{c} + k_2 r_- e^{r_- x_1}, \quad V''(x_1+) = k_2 r_- e^{r_- x_1}.$$

# Principle of Smooth Fit

- Denoting $\beta = K_2 e^{r - x_1}$ and noting that $V'(x_1) = 1/y_1$, we derive from the HJB equation that $V''(x_1) = -\mu/\sigma^2 V'(x_1)$.
  $\Longrightarrow \dfrac{1}{y_1} = \dfrac{1}{c} + \beta r_-; \quad -\dfrac{\mu}{\sigma^2}\dfrac{1}{y_1} = \beta r_-^2.$

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- Solving for $(y_1, \beta)$ we obtain

$$(y_1, \beta) = \left( c\left(1 + \frac{\mu}{\sigma^2 r_-}\right), \ \frac{-\mu}{c(\sigma^2 r_-^2 + \mu r_-)} \right).$$

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- by definition of $r_-$ we see that $(y_1, \beta) \in (0, c) \times (-\infty, 0)$. Recall that $y_1 = G^{-1}(x_1/k_1)$ we have

$$\frac{x_1}{k_1} = G(y_1), \qquad \frac{\mu}{\sigma^2} k_1 y_1 g(y_1) = 1.$$

$$\Longrightarrow \quad (k_1, x_1) = \left( \frac{\sigma^2}{\mu y_1 g(y_1)}, \; \frac{\sigma^2 G(y_1)}{\mu y_1 g(y_1)} \right).$$

## Theorem

*The function*

$$V(x) = \begin{cases} \displaystyle\int_0^x \frac{1}{G^{-1}\left(\frac{z}{k_1}\right)} dz, & 0 \le x < x_1 \\ \displaystyle\frac{x}{c} + \frac{\mu}{c^2} + \beta e^{r_- x} & x > x_1, \end{cases} \tag{11}$$

*where* $\beta = \frac{-\mu}{c(\sigma^2 r_-^2 + \mu r_-)}$, $x_1 = \frac{\sigma^2 G(y_1)}{\mu y_1 g(y_1)}$, $k_1 = \frac{\sigma^2}{\mu y_1 g(y_1)}$,
$y_1 = c\left(1 + \frac{\mu}{\sigma^2 r_-}\right)$ *is a concave solution to the HJB equation (4).*

**Proof.** Plug in and check! ∎

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**Proof.** Plug in and check! ∎

## Warning:

This theorem does not give the solution to the optimization problem. In other words: the function $V$ may not be the value function!

# A Verification Theorem

In order to check that the $C^2$ function $V$ that we worked so hard to get is indeed the value function, and the function $a(x)$ we have obtained is the optimal policy.

**Theorem**

*Let $V$ be the function given by (11), and define a process $a_t^* \stackrel{\triangle}{=} a(X_t^*)$, where*

$$a(x) = \begin{cases} \dfrac{G^{-1}\left(\frac{x}{k_1}\right) g\left(G^{-1}\left(\frac{x}{k_1}\right)\right)}{y_1 g(y_1)} & x < x_1 \\ 1 & x > x_1, \end{cases}$$

*Then $V(x)$ is the value function and $\alpha^*$ is an optimal strategy.*

# General Reinsurance Problems

We now consider the following more general dynamics of a risk reserve:

$$X_t = x + \int_0^t (1 + \rho_s^\alpha) c^\alpha(s) ds - \int_0^t \int_{\mathbb{R}_+} \alpha(s, z) f(s, z) N_p(dsdz),$$

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where $c^\alpha$ is the adjusted premium rate after reinsurance.

**Question**

What is the general form of the reinsurance policy and the reasonable form of $c^\alpha$?

**Definition**

A (proportional) reinsurance policy is a random field $\alpha : [0,\infty) \times \mathbb{R}_+ \times \Omega \mapsto [0,1]$ such that for each fixed $z \in \mathbb{R}_+$, the process $\alpha(\cdot, z, \cdot)$ is predictable.

# Remarks

- The dependence of a reinsurance policy $\alpha$ on the variable $z$ amounts to saying that the proportion can depend on the sizes of the claims.
- One can define a reinsurance policy as a predictable process $\alpha_t$, but in general one may not be able to find an optimal strategy, unless $S_t$ has fixed size jumps. The similar issue also occurs in utility optimization problems in finance involving jump-diffusion models (See, e.g, X. X. Xue (1992).)
- Given a reinsurance policy $\alpha$, during time period $[t, t+\Delta t]$ the insurance company retains to itself

$$[\alpha * S]_t^{t+\Delta t} \triangleq \int_t^{t+\Delta t} \int_{\mathbb{R}_+} \alpha(s,z) f(s,z) N_p(dzds)$$

and cedes to the reinsurer

$$[(1-\alpha) * S]_t^{t+\Delta t} \triangleq \int_t^{t+\Delta t} \int_{\mathbb{R}_+} [1-\alpha(s,z)] f(s,z) N_p(dzds).$$

# Dynamics for the Reserve with Reinsurance

- By "*Profit Margin Principle*", one has:

$$\underbrace{(1+\rho_t)E_t^p\{[1*S]_t^{t+\Delta t}\}}_{\text{original premium}}$$

$$= \underbrace{(1+\rho_t^r)E_t^p\{[(1-\alpha)*S]_t^{t+\Delta t}\}}_{\text{premium to the reinsurance company}} + \underbrace{(1+\rho_t^\alpha)E_t^p\{[\alpha*S]_t^{t+\Delta t}\}}_{\text{modified premium}}$$

- $\Delta t \to 0 \quad \Longrightarrow$

$$\begin{aligned}(1+\rho_t)c_t &= (1+\rho_t^r)\int_{\mathbb{R}_+}(1-\alpha(t,z))f(t,z)\nu(dz)\\ &\quad +(1+\rho_t^\alpha)\int_{\mathbb{R}_+}\alpha(t,z)f(t,z)\nu(dz).\end{aligned}$$

- Denote $S_t^\alpha = \int_0^t\int_{\mathbb{R}_+}\alpha(s,z)f(s,z)N_p(dzds)$, and

$$m(t,\alpha) = \int_{\mathbb{R}_+}\alpha(t,z)f(t,z)\nu(dz),$$

# Dynamics for the Reserve with Reinsurance

We see that a general dynamics of risk reserve

$$X_t = x + \int_0^t (1+\rho_s^\alpha) m(s,\alpha) ds - \int_0^t \int_{\mathbb{R}_+} \alpha(s,z) f(s,z) N_p(dsdz).$$

# Dynamics for the Reserve with Reinsurance

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**Note**

- Whether a reinsurance is cheap or non-cheap does not change the form of the reserve equation. We will not distinguish them in the future.
- If the reinsurance policy $\alpha$ is independent of claim size $z$, then

$$S_t^\alpha = \int_0^t \alpha(s) \int_{\mathbb{R}_+} f(s,z) N_p(dzds) = \int_0^t \alpha(s) dS_s$$

  and $m(t,\alpha) = \alpha(s) c_s$, as we often see in the standard reinsurance framework.

- The Market:

$$\begin{cases} dP_t^0 = r_t P_t^0 dt; & \text{(money market)} \\ dP_t^i = P_t^i[\mu_t^i dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^j], \quad i = 1, \cdots, n. & \text{(stocks)} \end{cases}$$

- Portfolio Process:
  - $\pi_t(\cdot) = \left(\pi_t^1, \cdots, \pi_t^k\right)$ — $\pi_t^i$ is the fraction of its reserve $X_t$ allocated to the $i^{th}$ stock.
  - $X_t - \sum_{i=1}^k \pi_t^i X_t = (1 - \sum_{i=1}^k \pi_t^i) X_t$ — money market account.
- Consumption (Rate) Process:
  $D = \{D_t : t \geq 0\}$ — $\mathscr{F}$-predictable nonnegative process satisfying $D \in L^1_{\mathscr{F}}([0, T] \times \mathbb{R}_+)$ (may include dividend/bonus, etc.).

# Dynamics of Reserve with Reinsurance and Investment

$$
\begin{aligned}
dX_t &= \Big\{ X_t \Big[ r_t + \langle \pi_t, \mu_t - r_t \mathbf{1} \rangle \Big] + (1+\rho_t) m(t,\alpha) - D_t \Big\} dt \\
&\quad + X_t \langle \pi_t, \sigma_t dW_t \rangle - \int_{\mathbb{R}_+} \alpha(t,z) f(t,z,\cdot) N_p(dtdz),
\end{aligned}
$$

where $\mathbf{1} = (1, \cdots, 1)^T$. We call the pair $(\pi, \alpha)$ *D-financing*".

# Dynamics of Reserve with Reinsurance and Investment

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$$+X_t \langle \pi_t, \sigma_t dW_t \rangle - \int_{\mathbb{R}_+} \alpha(t,z) f(t,z,\cdot) N_p(dtdz),$$

where $\mathbf{1} = (1, \cdots, 1)^T$. We call the pair $(\pi, \alpha)$ *D-financing*".

**Example**

- Classical Model:

  — $r = 0$, $\rho = 0$, $\pi = 0$, $f(t,x,\cdot) = x$, $\nu(dx) = \lambda F(dx)$.
- Discounted Risk Reserve:

  — $\rho = 0$, $\pi = 0$, $f(t,x,\cdot) = x$, $\nu(dx) = \lambda F(dx)$, but $r > 0$ is deterministic
- Perturbed Risk Reserve:

  — $r = 0$, $\rho = 0$, $\pi = \varepsilon$, $f(t,x,\cdot) = x$, $\nu(dx) = \lambda F(dx)$.

# Standing Assumptions

.

**(H1)** $f \in F_p$, continuous in $t$, and piecewise continuous in $z$. Furthermore, $\exists 0 < d < L$ such that

$$d \leq f(s, z, \omega) \leq L, \qquad \forall (s, z) \in [0, \infty) \times \mathbb{R}_+, \quad P\text{-a.s.}$$

**Remark**

The bounds $d$ and $L$ in (H1) could be understood as the *deductible* and *benefit limit*. They can be relaxed to certain integrability assumptions on both $f$ and $f^{-1}$.

**(H2)** The safety loading $\rho$ and the premium $c$ are both bounded, non-negative $\mathbf{F}^p$-adapted processes,

**(H3)** The processes $r$, $\mu$, and $\sigma$ are $\mathbf{F}^W$-adapted and bounded. Furthermore, $\exists \delta > 0$, such that $\sigma_t \sigma_t^* \geq \delta I$, $\forall t \in [0, T]$, $P$-a.s.

# Admissibility of Strategies

## Main Features

- $\alpha \in [0,1]$ is intrinsic, cannot be relaxed.
- $\alpha$ CANNOT be assumed *a priori* to be proportional to the reserve $X_t$
- by nature of a reinsurance problem, (or by regulation) we require that the reserve be aloft. That is, at any time $t \geq 0$, $X_t^{x,\pi,\alpha,D} \geq C$ for some constant $C > 0$. We will set $C = 0$.

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## Definition (Admissible strategies)

For any $x \geq 0$, a portfolio/reinsurance/consumption (PRC for short) triplet $(\pi, \alpha, D)$ is called "admissible at $x$", if

$$X_t^{x,\pi,\alpha,D} \geq 0, \qquad \forall t \in [0, T], \quad P\text{-a.s.}$$

We denote the totality of all strategies admissible at $x$ by $\mathscr{A}(x)$.

# A Necessary Condition

Define

- $\theta_t \stackrel{\triangle}{=} \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$ — (*risk premium*)
- $\gamma_t \stackrel{\triangle}{=} \exp\{-\int_0^t r_s ds\}$, $t \geq 0$ — (*discount factor*)
- $W_t^0 \stackrel{\triangle}{=} W_t + \int_0^t \theta_s ds$
- $Z_t \stackrel{\triangle}{=} \exp\Big\{ -\int_0^t \langle \theta_s, dW_s \rangle - \frac{1}{2}\int_0^t \|\theta_s\|^2 ds \Big\}$
- $Y_t \stackrel{\triangle}{=} \exp\Big\{ \int_0^t \int_{\mathbb{R}_+} \ln(1+\rho_s) N_p(dsdz) - \nu(\mathbb{R}^+) \int_0^t \rho_s ds \Big\}$
- $H_t \stackrel{\triangle}{=} \gamma_t Y_t Z_t$ — *state-price-density*

**Girsanov-Meyer Transformations**

$$dQ_Z = Z_T dP;\ dQ_Y = Y_T dP;\ dQ = Y_T dQ_Z = Y_T Z_T dP.$$

# A Necessary Condition

The following facts are easy to check:

- The process $Y$ is a square-integrable $P$-martingale;
- The process $Z$ is a square-integrable $Q_Y$-martingale;
- For any reinsurance policy $\alpha$, the process

$$N_t^\alpha \triangleq \int_0^t (1+\rho_s) m(s,\alpha) ds - \int_0^{t+} \int_{\mathbb{R}_+} \alpha(s,z) f(s,z) N_p(dsdz)$$

  is a $Q_Y$-local martingale.
- The process $ZN^\alpha$ is a $Q_Y$-local martingale.

**In the "$Q$"-world:**

- the process $W^0$ is also a $Q$-Brownian motion,
- $N^\alpha$ is a $Q$-local martingale.
- $N^\alpha W^0$ is a $Q$-local martingale.

# A Necessary Condition (Budget Constraint)

Under the probability $Q$ the reserve process reads

$$\tilde{X}_t + \int_0^t \gamma_s D_s ds = x + \int_0^t \tilde{X}_s \langle \pi, \sigma_s dW_s^0 \rangle - \int_0^t \gamma_s dN_s^{\alpha}.$$

The admissibility of $(\pi, \alpha, D)$ implies that the right hand side is a positive local martingale, whence a supermartingale under $Q$!

# A Necessary Condition (Budget Constraint)

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The admissibility of $(\pi, \alpha, D)$ implies that the right hand side is a positive local martingale, whence a supermartingale under $Q$!

**Theorem**

Assume (H2) and (H3). Then for any PRC triplet $(\pi, \alpha, D) \in \mathscr{A}(x)$, the following (*"budget constraint"*) holds

$$E\Big\{ \int_0^T H_s D_s ds + H_T X_T^{x,\alpha,\pi,D} \Big\} \le x,$$

where $H_t = \gamma_t Y_t Z_t$, and $\gamma_t = \exp\{-\int_0^t r_s ds\}$.

# Wider-sense Strategies

**Definition** (*wider-sense strategies*)

A triplet of **F**-adapted processes $(\pi, \alpha, D)$ is called a *wider-sense strategy* if $\pi$ and $D$ are admissible, but $\alpha \in F_p^2$. Denote all wider-sense strategies by $\mathscr{A}^w(x)$. We call the process $\alpha$ in a wider-sense strategy a *pseudo-reinsurance policy*.

**Lemma** (Existence of wider-sense strategies)

Assume (H1)– (H3). For any consumption process $D$ and any $B \in \mathscr{F}_T$ such that $E(B) > 0$ and

$$E\Big\{ \int_0^T H_s D_s ds + H_T B \Big\} = x, \tag{12}$$

$\exists(\pi, \alpha)$ such that $(D, \pi, \alpha) \in \mathscr{A}^w(x)$, and that

$$X_t^{x,\pi,\alpha,D} > 0,\ \forall 0 \le t \le T; \quad \text{and} \quad X_T^{x,\pi,\alpha,D} = B,\ P\text{-a.s.}$$

**Sketch of the Proof.**

- Given a consumption rate process $D$, consider the BSDE:

$$\begin{aligned} X_t &= B - \int_t^T \Big\{ r_s X_s + \langle \varphi_s, \theta_s \rangle - D_s + \rho_s \int_{\mathbb{R}_+} \psi(s,z)\nu(dz) \Big\} ds \\ &\quad - \int_t^T \langle \varphi_s, dW_s \rangle + \int_t^T \int_{\mathbb{R}_+} \psi(s,z) \tilde{N}_p(dsdz). \qquad (13) \end{aligned}$$

— (Tang-Li (1994), Situ (2000))

- Define $\alpha(t,z) \stackrel{\triangle}{=} \frac{\psi(t,z)}{f(t,z)}$ — a pseudo-reinsurance policy $\Longrightarrow$

$$\begin{aligned} & -\int_t^T \Big\{ \rho_s \int_{\mathbb{R}_+} \psi(s,z)\nu(dz)ds + \int_{\mathbb{R}_+} \psi(s,z)\tilde{N}_p(dsdz) \Big\} \\ = & \; -\int_t^T \Big\{ (1+\rho_s) m(s,\alpha) ds + \int_{\mathbb{R}_+} \alpha(s,z) f(s,z) N_p(dsdz) \Big\}, \end{aligned}$$

# Wider-sense Strategies

- The BSDE (13) becomes

$$dX_t = \{r_t X_t - D_t\}dt + \langle \varphi_t, dW_t^0 \rangle - dN_t^\alpha, \qquad (14)$$

where $W^0$ is a $Q$-B.M. and $N^\alpha$ is a $Q$-local martingale.

- "Localizing" $\oplus$ "Monotone Convergence" $\oplus$ "Exponentiating" $\oplus$ $E(B) > 0$ and $D$ is non-negative:

$$\gamma_t X_t = E^Q\left\{\gamma_T B + \int_t^T \gamma_s D_s ds \middle| \mathscr{F}_t\right\} \geq E^Q\{\gamma_T B | \mathscr{F}_t\} > 0.$$

$$\Longrightarrow P\{X_t > 0, \ \forall t \geq 0; X_T = B\} = 1.$$

- Define $\pi_t \stackrel{\triangle}{=} [\sigma_t^*]^{-1}\varphi_t / X_t$ and note that

$$X_0 = E^Q\left\{\gamma_T X_T + \int_0^T \gamma_s D_s ds\right\} = E\left\{H_T X_T + \int_0^T H_s D_s ds\right\} = x$$

$$\Longrightarrow (\pi, \alpha, D) \in \mathscr{A}^w(x)! \qquad \blacksquare$$

**Question**

When will $(\pi, \alpha, D) \in \mathscr{A}^W(x)$?

# A Duality Method

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When will $(\pi, \alpha, D) \in \mathscr{A}^W(x)$?

Following the idea of "*Duality Method*" (Cvitanic-Karatzas (1993)), we begin by recalling the *support function* of $[0, 1]$

$$\delta(x) \stackrel{\triangle}{=} \delta(x|[0,1]) \stackrel{\triangle}{=} \begin{cases} 0, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

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Define a subspace of $F_p^2$:

$$\mathscr{D} \triangleq \{v \in F_p^2 : \sup_{t \in [0,R]} \int_{\mathbb{R}^+} |v(t,z)|\nu(dz) < C_R, \ \forall R > 0\}.$$

For each $v \in \mathscr{D}$, recall that

$$m(t, \delta(v)) = \int_{\mathbb{R}+} \delta(v(t,z)) f(t,z) \nu(dz), \ t \geq 0.$$

# An Auxiliary (Fictitious) Market

## The Fictitious Market

For $v \in \mathscr{D}$, consider a market in which the interest rate and appreciation rate are perturbed:

$$\begin{cases} dP_t^{v,0} = P_t^{v,0}\{r_t + m(t,\delta(v))\}dt, \\ dP_t^{v,i} = P_t^{v,i}\{(\mu_t^i + m(t,\delta(v))dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j\}, \ i = 1, \cdots, k. \end{cases}$$

Consider also a (fictitious) expense loading and interest rate

$$\rho^v(s,z,x) \stackrel{\triangle}{=} \rho_s + v(s,z)x, \quad r_t^{\alpha,v} = r_t + m(t, \alpha v + \delta(v)).$$

Under the fictitious market, the reserve equation becomes

$$X_t^v = x + \int_0^t X_s^v r_s^{\alpha,v} ds + \int_0^t X_s^v \langle \pi_s, \sigma_s dW_s^0 \rangle + N_t^\alpha - \int_0^t D_s ds.$$

- for $\alpha \in F_p^2$,

$$\begin{cases} \alpha v + \delta(v) = |v|\{\alpha \mathbf{1}_{\{v \geq 0\}} + (1-\alpha)\mathbf{1}_{\{v<0\}}\} \\ r^{\alpha,v} = r \iff m(t, \alpha v + \delta(v)) = 0. \end{cases} \tag{15}$$

- If $\alpha$ is a (true) reinsurance policy (hence $0 \leq \alpha \leq 1$), then

$$0 \leq \alpha(t,z)v(t,z) + \delta(v(t,z)) \leq |v(t,z)|, \quad \forall (t,z), \text{ -a.s.}$$

# Some Remarks

- for $\alpha \in F_p^2$,

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$$0 \leq \alpha(t,z)v(t,z) + \delta(v(t,z)) \leq |v(t,z)|, \quad \forall (t,z), \text{ -a.s.}$$

**Definition**

For $v \in \mathscr{D}$, a wider-sense strategy $(\alpha, \pi, D) \in \mathscr{A}^W(x)$ is called "$v$-admissible" if

(i) $\int_0^T |m(t, av + \delta(v))| dt < \infty$, $P$-a.s.

(ii) $X^v \stackrel{\triangle}{=} X^{v,x,\pi,\alpha,D} \geq 0$, for all $0 \leq t \leq T$, $P$-a.s.

$\mathscr{A}^v(x) \stackrel{\triangle}{=} \{$ all wider-sense $v$-admissible strategies$\}$

# Some Remarks

**Note:**

If $v \in \mathscr{D}$ and $(\alpha, \pi, D) \in \mathscr{A}^v(x)$ such that

$$\begin{cases} 0 \le \alpha(t,z) \le 1; \\ \delta(v(t,z)) + \alpha(t,z)v(t,z) = 0, \end{cases} \qquad dt \times \nu(dz)\text{-a.e. }, \ P\text{-a.s.}$$

then $(\alpha, \pi, D) \in \mathscr{A}(x)(!)$ and and $r_t^{\alpha,v} = r_t, \ t \ge 0$.

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If $v \in \mathscr{D}$ and $(\alpha, \pi, D) \in \mathscr{A}^v(x)$ such that

$$\begin{cases} 0 \le \alpha(t,z) \le 1; \\ \delta(v(t,z)) + \alpha(t,z)v(t,z) = 0, \end{cases} \qquad dt \times \nu(dz)\text{-a.e. }, \ P\text{-a.s.}$$

then $(\alpha, \pi, D) \in \mathscr{A}(x)(!)$ and and $r_t^{\alpha,v} = r_t, \ t \ge 0$.

For any $v \in \mathscr{D}$ and $(\pi, \alpha, D) \in \mathscr{A}^v(x)$, define

$$\begin{aligned} \gamma_t^{\alpha,v} &\triangleq \exp\Big\{ -\int_0^t r_s^{\alpha,v} ds \Big\}; \\ H_t^{\alpha,v} &\triangleq \gamma_t^{\alpha,v} Y_t Z_t, \qquad \psi(t,z) = \alpha(t,z) f(t,z), \\ \overline{\psi}_t^v &\triangleq \int_{\mathbb{R}_+} \psi(t,z) v(t,z) \nu(dz) \triangleq m^v(t,\psi). \end{aligned}$$

# Some Remarks

**Proposition**

*Assume (H1)—(H3). Then,*

(i) *for any* $v \in \mathscr{D}$, *and* $(\pi, \alpha, D) \in \mathscr{A}^v(x)$, *the following budget constraint still holds*

$$E\Big\{ \int_0^T H_s^{\alpha,v} D_s ds + H_T^{\alpha,v} X_T^v \Big\} \leq x; \tag{16}$$

(ii) *if* $(\pi, \alpha, D) \in \mathscr{A}(x)$, *then for any* $v \in \mathscr{D}$ *it holds that*

$$X^{v,x,\alpha,\pi,D}(t) \geq X^{x,\alpha,\pi,D}(t) \geq 0, \quad 0 \leq t \leq T, \quad \text{-a.s.} \tag{17}$$

*In other words,* $\mathscr{A}(x) \subseteq \mathscr{A}^v(x)$, $\forall v \in \mathscr{D}$.

# A BSDE with Super Linear Growth

In light of the BSDE argument before, we need to consider a BSDE based on the "fictitious" reserve. But note that

$$
\begin{aligned}
dX_t^v &= \Big\{[r_t + m(t, \alpha v + \delta(v))]X_t^v - D_t + \rho_t m(t,\alpha)\Big\}dt \\
&\quad + X_t^v \langle \pi_t, \sigma_s dW_t^0 \rangle - \int_{\mathbb{R}_+} \alpha(t,z) f(t,z) \tilde{N}_p(dtdz) \\
&= \Big\{[r_t + m(t,\delta(v))]X_t^v + \overline{\psi}_t^v X_t^v - D_t\Big\}dt \\
&\quad + \langle \varphi_t^v, dW_t^0 \rangle - \int_{\mathbb{R}_+} \psi(t,z)\tilde{N}_p^0(dtdz).
\end{aligned}
$$

where $\tilde{N}_p^0(dtdz) = \tilde{N}_p(dtdz) - \rho_t \nu(dz)dt$, $\varphi_t^v = X_t^v \sigma_t^T \pi_t$.

**Recall**

- $W^0$ is a $Q$-B.M. and $\tilde{N}^0$ is a $Q$-Poisson martingale measure.
- $m(t,\eta) = m^f(t,\eta)$, $m^1(t,\eta) = \overline{\eta}_t$.

# A BSDE with Super Linear Growth

The corresponding BSDEs is therefore: for $B \in L^2(\Omega; \mathscr{F}_T)$, $v \in F_p^2$, $r_t^v = r_t + m(t, \delta(v))$:

$$y_t = B - \int_t^T \Big\{ r_s^v y_s + \overline{\psi}_s^v y_s - D_s \Big\} ds - \int_t^T \langle \varphi_s, dW_s^0 \rangle + \int_t^T \int_{\mathbb{R}_+} \psi_s \tilde{N}_p^0(dsdz). \tag{18}$$

**Note**

The BSDE (18) is "superlinear" in both $Y$ and $Z$!
$(|ab| \le C(|a|^p + |b|^q), \ p, q > 1)$

- Continuous case:

  Lepeltier-San Martin (1998), Bahlali-Essaky-Labed (2003), Kobylanski-Lepeltier-Quenez-Torres (2003) ...

- Jump case: Liu (2006), Liu-M. (2009)

# Main Result

**Theorem**

*Assume (H1)–(H3). Assume further that processes $r$ and $D$ are all uniformly bounded. Then for any $v \in \mathscr{D}$ and $B \in L^\infty(\Omega; \mathscr{F}_T)$, the BSDE (18) has a unique adapted solution $(y^v, \varphi^v, \psi^v)$.*

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Define a "*portfolio/pseudo-reinsurance*" pair:

$$\pi_t^v = [\sigma_t^T]^{-1} \frac{\varphi_t^v}{y_t^v}; \quad \alpha^v(t,z) = \frac{\psi^v(t,z)}{f(t,z)}.$$

We call $(\pi^v, \alpha^v)$ the *portfolio/pseudo-reinsurance pair associated to $v$*.

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**Question:**

When will $(\pi^v, \alpha^v, D) \in \mathscr{A}(x)$?

# A Sufficient Condition

**Theorem**

*Assume (H1)–(H3). Let $D$ be a bounded consumption process, and $B$ be any nonnegative, bounded $\mathscr{F}_T$-measurable random variable such that $E(B) > 0$. Suppose that for some $u^* \in \mathscr{D}$ whose associated portfolio/pseudo-reinsurance pair, denoted by $(\pi^*, \alpha^*)$, satisfies that*

$$u^* \in \text{argmax}_v E\Big\{H_T^{\alpha^*,v} B + \int_0^T H_s^{\alpha^*,v} D_s ds\Big\},$$

*where for any $v \in \mathscr{D}$,*

$$H_t^{\alpha^*,v} \triangleq \gamma_t^{\alpha^*,v} Y_t Z_t, \quad \gamma_t^{\alpha^*,v} \triangleq \exp\Big\{-\int_0^t [r_s^v + m(s, \alpha^* v))]ds\Big\}.$$

*Then the triplet $(\pi^*, \alpha^*, D) \in \mathscr{A}(x)$. Further, the corresponding reserve $X^*$ satisfies $X_T^* = B$, $P$-a.s.*

# An Utility Optimization Problem

Recall that $U : [0, \infty) \mapsto [-\infty, \infty]$ is a "utility function" if it is increasing and concave. Assume that $U \in C^1$, and $U'(\infty) \stackrel{\triangle}{=} \lim_{x\to\infty} U'(x) = 0$. Define

- $\text{dom}(U) \stackrel{\triangle}{=} \{x \in [0, \infty); U(x) > -\infty\}$
- $\bar{x} \stackrel{\triangle}{=} \inf\{x \geq 0 : U(x) > -\infty\}$
- $I \stackrel{\triangle}{=} [U']^{-1}$ ($I$ is continuous and decreasing on $(0, U'(\bar{x}+))$, extendable to $(0, \infty]$ by setting $I(y) = \bar{x}$ for $y \geq U'(\bar{x}+)$)

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**"Truncated" Utility Function**

- for some $K > 0$, $U$ is a utility function on $[0, K]$ but $U(x) = U(K)$ for all $x \geq K$. (The interval $[0, K]$ is called the *"effective domain"* of $U$.)
- A truncated utility function is *"good"* if $U'(\bar{x}+) < \infty$.

# An Utility Optimization Problem

Given any UF $U$, we can define for each $n$,

$$U_n(x) = U(\underline{x}^n) - \frac{1}{2}(\underline{x}^n)^2 - n\underline{x}^n + \int_0^{x \wedge \bar{x}^n} \xi^n(y)dy,$$

where $U'(\underline{x}^n) = n$ and $U'(\bar{x}^n) = \frac{1}{n}$, and

$$\xi^n(x) = \begin{cases} \underline{x}^n - x + n & 0 \le x \le \underline{x}^n \\ U'(x) & \underline{x}^n \le x \le \bar{x}^n \\ \frac{1}{n} & x > \bar{x}^n, \end{cases} \tag{19}$$

Then $U_n$'s are good TUF's with $\bar{x} = 0$, $K = \bar{x}^n$,

$$U_n(0+) = U_n(0) = U(\underline{x}^n) - \frac{1}{2}(\underline{x}^n)^2 - n\underline{x}^n,$$

and $U_n(x) \to U(x)$ as $n \to \infty$.

# An Utility Optimization Problem

Now let $U$ be good TUF (WLOG: $\bar{x} = 0$, and $U'(0) < \infty$). Thus

- $U' : [0, K] \mapsto [U'(K), U'(0)]$
- $I(y) = [U']^{-1} : [U'(K), U'(0)] \mapsto [0, K]$ is continuous and strictly decreasing (extendable to $[0, \infty)$ by defining $I(y) = 0$ for $y \geq U'(0)$ and $I(y) = K$ for $y \in [0, U'(K)]$)
- In particular, $I$ is bounded on $[0, \infty)$(!).

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**Note**

If $U$ is a good TUF with effective domain $[0, K]$, and

$$\tilde{U}(y) \stackrel{\triangle}{=} \max_{0<x\leq K} \{U(x) - xy\}, \quad 0 < y < \infty.$$

is the Legendre-Fenchel transform of $U$. Then it holds that

$$\tilde{U}(y) = U(I(y)) - yI(y), \qquad \forall y > 0.$$

# Modified Preference Structure

We now modify the so-called "preference structure" (see Karatzas-Shreve's book) to the good TUF's:

**Definition**

*A pair of functions* $U_1 : [0, T] \times (0, \infty) \mapsto [-\infty, \infty)$ *and* $U_2 : [0, , \infty) \mapsto [-\infty, \infty)$ *is called a "modified (von Neumann-Morgenstern) preference structure" if*

(i) *for fixed* $t$, $U^1(t, \cdot)$ *is a UF with (subsistence consumption)* $\bar{x}_1(t) \stackrel{\triangle}{=} \inf\{x \in \mathbb{R}; U^1(t, x) > -\infty\}$ *being continuous on* $[0, T]$, *and* $U_1$ *and* $U_1'$ *being continuous on* $\mathscr{D}(U_1) \stackrel{\triangle}{=} \{(t, x) : x > \bar{x}^1(t), \; t \in [0, T]\}$;

(ii) $U_2$ *is a good TUF with (subsistence terminal wealth)* $\bar{x}_2 = \inf\{x : U_2'(x) > -\infty\}$.

# Utility Optimization Problem

Assume that $(U_1, U_2)$ is a modified preference structure, with effective domain of $U_2$ being $[0, K]$. For $(\pi, \alpha, D) \in \mathscr{A}(x)$, define

- Cost functional:

$$J(x; \pi, \alpha, D) \stackrel{\triangle}{=} E\Big\{ \int_0^T U_1(t, D_t)dt + U_2\Big(X_T^{x,\alpha,\pi,D}\Big)\Big\}.$$

- Value function:

$$V(x) \stackrel{\triangle}{=} \sup_{(\pi,\alpha,D)\in\mathscr{A}(x)} J(x; \pi, \alpha, D).$$

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**Duality Method**

First find a (wider-sense) optimal strategy via fictitious market, then verify that it is actually a true strategy using the sufficient condition.

Fix $v \in \mathscr{D}$.

- $\forall(\pi, \alpha, D) \in \mathscr{A}^v(x)$, denote the "fictitious" reserve by $X^v$.
- The *"fictitious" budget constraint*:

$$x \geq E^Q\Big\{\gamma_T^{\alpha,v} X_T^v + \int_0^T \gamma_s^{\alpha,v} D_s ds\Big\} = E\Big\{H_T^{\alpha,v} X_T^v + \int_0^T H_s^{\alpha,v} D_s ds\Big\}$$

- Define $I_1(t,\cdot) = [U_1'(t,\cdot)]^{-1}$ and $I_2 = [U_2']^{-1}$,

$$\mathscr{X}_v^\alpha(y) \stackrel{\triangle}{=} E\Big\{H_T^{\alpha,v} I_2(yH_T^{\alpha,v}) + \int_0^T H_t^{\alpha,v} I_1(t, yH_t^{\alpha,v}) dt\Big\}, \ y > 0.$$

( $\Longrightarrow \mathscr{X}_v^\alpha(\cdot)$ is continuous, decreasing, and $\mathscr{X}_v^\alpha(0+) = \infty$.)

- Define $\mathscr{Y}_v^\alpha(x) = \inf\{y : \mathscr{X}_v^\alpha(y) < x\} \stackrel{\triangle}{=} [\mathscr{X}_v^\alpha]^{-1}(x) \in (0, y_0)$, where $y_0 \stackrel{\triangle}{=} \sup\{y > 0; \mathscr{X}_v(y) > \mathscr{X}_v(\infty)\}$

- The "fictitious" budget constraint implies that $V(x) = -\infty$ whenever $x < \mathscr{X}_v^\alpha(\infty)$. Thus may assuem $x > \mathscr{X}_v^\alpha(\infty)$.
- consider the problem of maximizing

$$\begin{cases} \tilde{J}(D,B) \triangleq E\Big\{ \int_0^T U_1(t, D(t))dt + U_2(B)\Big\} \\ s.t. \quad E\Big\{ \int_0^T H_t^{\alpha,v} D_t dt + H_T^{\alpha,v} B\Big\} \le x. \end{cases}$$

where $D$ is a consumption process and $B \in L^\infty_{\mathscr{F}_T}(\Omega)$. s.t.,

- "Lagrange multiplier": define

$$\begin{aligned} J_v^\alpha(D,B;x,y) &\triangleq xy + E\int_0^T [U_1(t,D(t)) - yH_t^{\alpha,v}D_t]dt \\ &\quad + E[U_2(B) - yH_T^{\alpha,v}B] \\ &\le xy + E\Big\{ \int_0^T \tilde{U}_1(t, yH_t^{\alpha,v})dt + \tilde{U}_2(yH_T^{\alpha,v})\Big\}. \end{aligned}$$

# The procedure

**Note**

The equality holds $\Longleftrightarrow D_t^{\alpha,v} = I_1(t, yH_t^{\alpha,v})$ and $B^{\alpha,v} = I_2(yH_T^{\alpha,v})$, $0 \le t \le T$, $P$-a.s.

This leads to the following special "Forward-Backward SDE":

$$\begin{cases} H_t = 1 + \int_0^t H_s[r_s + m(s, \delta(v) + \alpha v))]ds - \int_0^t H_s \langle \theta_s, dW_s \rangle \\ \qquad + \int_0^t \int_{\mathbb{R}_+} H_{s-}\rho_s \tilde{N}_p(dsdz); \\ X_t = I_2(yH_T) - \int_t^T \Big\{ X_s[r_s + m(s, \delta(v) + \alpha v) + \langle \pi_s, \sigma_s \theta_s \rangle] \\ \qquad + (1 + \rho_s)m(s, \alpha) \Big\} ds - \int_t^T X_s \langle \pi_s, \sigma_s dW_s \rangle \\ \qquad + \int_t^T \int_{\mathbb{R}^+} \alpha(s, z) f(s, z) N_p(dsdz) + \int_t^T I_1(s, yH_s) ds, \end{cases}$$

# Main Result

**Theorem**

*Assume (H1)–(H3). Let $(U_1, U_2)$ be a modified preference structure. The following two statements are equivalent:*

*(i) For any $x \in \mathbb{R}$, the pair $B^* \triangleq I_2(\mathscr{Y}(x)H_T)$ and $D_t^* \triangleq I_1(t, \mathscr{Y}(x)H_t)$, satisfy*

$$
\begin{aligned}
V(x) &= E\Big\{ \int_0^T U_1(t, D_t^*)dt + U_2(B^*) \Big\} \\
&= \sup_{(\pi,\alpha,D)\in\mathscr{A}(x)} J(x; \pi, \alpha, D),
\end{aligned}
$$

*where $\mathscr{Y}(x)$ is such that*

$$
x = E\Big\{ \int_0^T I_1(t, \mathscr{Y}(x)H_t)dt + I_2(\mathscr{Y}(x)H_T) \Big\};
$$

# Main Result

(ii) There exists a $u^* \in \mathscr{D}$, such that the FBSDE (20) has an adapted solution $(H^*, X^*, \pi^*, \alpha^*)$, with $y$ satisfying

$$x = E\Big\{ \int_0^T I_1(t, yH_t^*)dt + I_2(yH_T^*) \Big\}. \qquad (20)$$

In particular, if (i) or (ii) holds, then $(\pi^*, \alpha^*, D^*) \in \mathscr{A}(x)$ is an optimal strategy for the utility maximization insurance/investment problem.

# Sketch of proof

"(i) $\Longrightarrow$ (ii)":

Assume $(\pi^*, \alpha^*, D^*) \in \mathscr{A}(x)$ is s.t. $X_T^{\pi^*,\alpha^*,D^*} = B^*$, and that

$$J(x; \pi^*, \alpha^*, D^*) = V(x) = E\Big\{ \int_0^T U_1(t, D_t^*)dt + U_2(B^*) \Big\}.$$

Define $u^*(t,z) = \mathbf{1}_{\{\alpha^*(t,z)=0\}} - \mathbf{1}_{\{\alpha^*(t,z)=1\}}$. Then $|u^*| \le 1$ and

$$\delta(u^*) + \alpha^* u^* = |u^*|\{\alpha^* \mathbf{1}_{\{u^* \ge 0\}} + (1-\alpha^*)\mathbf{1}_{\{u^*<0\}}\} \equiv 0.$$

$\Longrightarrow m(\cdot, \delta(u^*) + \alpha^* u^*) = 0$, $\gamma^{\alpha^*,u^*} = \gamma$, and $H^{\alpha^*,u^*} = H$. (since $X_T^* = B^* = I_2(\mathscr{Y}(x)H_T)$)

$\Longrightarrow (H, X^*, \pi^*, \alpha^*)$ solves FBSDE (20) with $y = \mathscr{Y}(x)$ and $v = u^*$.

# Sketch of proof

"(ii) $\Longrightarrow$ (i)":
Assume that for some $u^* \in \mathscr{D}$, FBSDE (20) has an adapted solution $(H^*, X^*, \pi^*, \alpha^*)$ with $y = \mathscr{Y}_{u^*}^{\alpha^*}(x) \stackrel{\triangle}{=} \mathscr{Y}^*(x)$. Define

$$D_t^* = I_1(t, \mathscr{Y}^*(x)H_t^*), \quad t \geq 0, \qquad B^* \stackrel{\triangle}{=} I_2(\mathscr{Y}^*(x)H_T^*).$$

Since $(D^*, B^*) \in \text{argmax} J_{u^*}^{\alpha^*}(x, \mathscr{Y}^*(x); D, B)$ (the Lagrange-Multiplier Problem), we must have

$$x = E\Big\{ \int_0^T H_t^* D_t^* dt + H^* B^* \Big\},$$

and

$$V^*(x) = \sup_{(D,B)} J_{u^*}^{\alpha^*}(\cdots) = E\Big\{ \int_0^T U_1(t, D_t^*)dt + U_2(B^*) \Big\}.$$

# Sketch of Proof

**Note**: $I_2$ is bounded(!) $\Longrightarrow |B^*| \le K$, and by the budget constraint, for any other $v \in \mathscr{D}$

$$E\Big\{\int_0^T H_t^{\alpha^*,v} D_t^* dt + H_T^{\alpha^*,v} B^*\Big\} \le x = E\Big\{\int_0^T H_t^* D_t^* dt + H_T^* B^*\Big\}.$$

$\Longrightarrow (\alpha^*, \pi^*, D^*) \in \mathscr{A}(x)$ (Sufficient Condition),
$\Longrightarrow 0 \le \alpha^*(t,z) \le 1$, $m(t, \alpha^* u^* + \delta(u^*)) = 0$, and $X_0^* = x$.
$\Longrightarrow H^* = H$, $\mathscr{Y}^*(x) = \mathscr{Y}(x)$, and $X^* = X^{x,\pi^*,\alpha^*,D^*}$.
$\Longrightarrow (D^*, B^*)$ become the same as that defined in (i), and

$$V^*(x) = V(x) = E\{\int_0^T U_1(t, D_t^*)dt + U_2(B^*)\}.$$

■

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