Finance, Insurance, and Stochastic Control (II)

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Outline

Equity Linked Life Insurance

An Equity-Linked Life insurance is one that

- allows a separate account with cash/investment options
- links the death benefits to the cash/investment performance

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Literature:

- Brennan-Schwartz ('76), Boyle-Schwartz ('77), Delbaen ('86), Aase-Persson ('94), Nielson-Sandmann (1995), Kurz ('96), ...
- Also, Young (with Bayraktar, Jaimungal, Ludkovski, Zariphopoulou, ...), Schweizer, Frittelli, Rouge-El Karoui, ...

Basic elements involved in an UVL insurance

A Life Model

- Single life
- Multiple life

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Benefit Specifications

- Guaranteed benefit/return
- "Multiple decrements" (including death, retirement, long term disability, ...)
- ... ...

Basic Elements

- $T(x)$ — Future Life-time r.v., where x is the current age
- $G_x(t) \stackrel{\triangle}{=} P\{T(x) > t\} \stackrel{\triangle}{=} {}_tp_x$, $t \geq 0$ — survival function
- ${}_hq_{x+t} \stackrel{\triangle}{=} P\{T(x) \leq t + h | T(x) > t\} = 1 - {}_hp_{x+t}$.
- $\lambda_x(t) = \lim_{h \to 0} \frac{{}_hq_{x+t}}{h} = -\frac{f_x(t)}{G_x(t)}$ — *force of mortality*

The Single Life Case

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- $\lambda_x(t) = \lim_{h \to 0} \frac{{}_hq_{x+t}}{h} = -\frac{f_x(t)}{G_x(t)}$ — *force of mortality*
- $X_t \in \{0, 1, ..., m\}$ — State Process (finite state Markov, representing "multiple decreements", e.g. short/long term disabilities, withdrawal, retirement, death, etc. $X_0 = 0$, and the state "1" is cemetery/absorbing, representing "death".)
- $dS_t^0 = r_t S_t^0$; $S_0^0 = s^0$ —money market
- $dS_t = S_t\{\mu_t dt + \sigma_t dB_t\}$, $S_0 = s$, — tradable
- $dZ_t = Z_t^0\{\mu_t^Z dt + \sigma_t^Z dB_t + \sigma_t d\tilde{B}_t\}$, $Z_0 = z$ —non-tradable

Principle of Equivalent Utility

The original form of "*Principle of Equivalent Utility*" states that the premium Π of a claim $\mathscr{X}$ should be determined by the equation

$$u(x) = E[u(x + \Pi - \mathscr{X})],$$

where u is a utility function, and x is the initial wealth.

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- If $x = 0$, then it is called *Zero Utility Principle*.
- If furthermore $u(x) = x$, then is often referred to as "*Equivalence Principle*".)
- Dynamically, assume that $X_t = x + \int_0^t c_s ds - S_t$, $t \geq s \geq 0$, and $\mathscr{X} = S_T$, then at any time $t \in [0, T]$ the premium c_t can be determined by solving the equation

$$u(x) = E\{u(X_T) | X_t = x\}.$$

Principle of Equivalent Utility

- If we use the risk reserve with investment, that is, the dynamic of the risk reserve X follows the following SDE:

$$X_t = x + \int_0^t [r_s X_s + c_s(1+\rho_s)]ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle - S_t, \quad (1)$$

then we can require that the premium is determined so that the expected utility maximized. In other words, one solves

$$u(x) = \sup_{\pi \in \mathscr{A}} E\left\{ u(X_T^\pi) | X_t = x \right\},$$

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- (**Note:** This is almost like an optimal control problem for maximizing the expected terminal utility by Merton (1969, 1971). But determing the premium process is rather difficult.)
- A more practical version of the "premium" is that it is paid as a lump-sum at the time of the contract. Although it is still priced "*dynamically*", it is paid only once at the initial time t.

A Stochastic Control Point of View

Assume we are in a "risk neutral world". Rewrite (1) as

$$X_t^\pi = X_0 + p + \int_0^t r_s X_s^\pi ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle - Y_t = W_t - Y_t,$$

where

- p is the (lump-sum) premium paid at $t = 0$,
- $W_t^\pi \overset{\triangle}{=} X_0 + p + \int_0^t r_s X_s ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle$,
- Y is a general "*Loss process*" (e.g., $Y_t = S_t$)

Note

If the insurer does not sell the insurance, then $Y = 0$, and therefore $p = 0$. The utility maximization problem becomes a usual stochastic control problme, and we denote its value function by

$$V^0(x,t) \overset{\triangle}{=} \sup_{\pi \in \mathscr{A}} E\left\{u(W_T^\pi) | W_t = x\right\}. \qquad (2)$$

The Indifference Pricing Problem

If the insurance is sold, and the liability cannot be traded after its transfer and before the expiration. Then the value function of the insurer should be

$$U(t, x+p, y) = \sup_{\pi \in \mathscr{A}} E\left\{u(W_T - Y_T) | W_t = x+p, Y_t = y\right\}. \quad (3)$$

Definition

Let $y \stackrel{\triangle}{=} Y_t$. A premium $p \geq 0$ is said to be "y-acceptable" if

$$V^0(t,x) \leq U(t, x+p, y), \qquad \forall (t,x). \quad (4)$$

Denote $\mathscr{P}_y = \{$all y-acceptable premium$\}$. Define the universal write price, $p^*(t,y)$ by

$p^*(t,y) \stackrel{\triangle}{=} \inf\{p \geq 0 : V^0(t,x) \leq U(t, x+p, y), \forall (t,x)\} = \inf \mathscr{P}_y$.

Existence of the Fair Price

Theorem

Suppose that $\mathscr{P}_{s,z} \neq \emptyset$*, and let* $p^* \triangleq \inf \mathscr{P}_y$*. Then it holds that*

$$V^0(t,x) = U(t, x + p^*, y), \ \forall (t,x).$$

Sketch of the proof

- By Comparison Theorem, $W_0 \geq \tilde{W}_0 \implies W_T^\pi \geq \tilde{W}_T^\pi$ $\implies U(t, x+p, y)$ is increasing in p.
- Since $Y_T \geq 0 \implies u(W_T^\pi - Y_T) \leq u(W_T^\pi) \implies$

$$U(t,x,y) \leq V^0(t,x) \leq U(t, x+p^*, y).$$

- If $U(t, \cdot, y)$ is continuous, then $\exists p^{**} \in [0, p^*]$ s.t.

$$V^0(t,x) = U(t, x + p^{**}, y)$$

- But $p^{**} \in \mathscr{P}_{s,z} \implies p^* \leq p^{**} \implies p^* = p^{**}$. ■

Indifference Pricing in Finance/Insurance

- First introduced by Hodges and Neuberger (1989), as a pricing principle for contingent claims in an incomplete market.
- The value is within the interval of arbitrage prices

$$\Big[\inf_Q E_Q\{\mathscr{X} e^{-rT}\},\ \sup_Q E_Q\{\mathscr{X} e^{-rT}\}\Big],$$

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Existing works for similar problems

- Cvitanić et al.('01), Delbaen et al.('02)... (martingale, duality)
- Rouge & El Karoui('00) (BSDEs)
- M. Davis ('00), M. Musiela & Zariphopoulou('02); Young and Zariphopoulou('02) (PDE solutions, power/exponential utility)
- Bielecki, Jeanblanc and Rutkowski ('05) (defaultable claims)

A Universal Variable Life Insurance Problem

The *Universal Variable Life* (UVL for short) is an insurance product that offers

- a separate cash account besides a death benefit
- various investment options
- different risk/return relationships (may include money market, bond, common stocks, or even non-tradable equities.)

Main Features

- The changes in the policy's cash values and death benefits will be related directly to the investment performance of its underlying assets.
- The death benefit will not fall below a minimum amount (usually the initial face amount) even if the invested assets depreciate in value by a substantial amount. Although there is no similar "floor" to protect the cash values.

The Death Benefit

Consider a term life insurance with expiration date $T > 0$ and death benefit

$$b_t = g(S_t^1, \cdots, S_t^d, Z_t) = g(S_t, Z_t), \qquad (5)$$

where $g : \mathbb{R}^{d+1} \mapsto (0, \infty)$ is some measurable function.

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Example

- $g(S_t, Z_t) = S_t^i \vee s^i$, for some i,
- $g(S_t, Z_t) = Z_t \vee z$.
- If Z is the retirement fund, one can set $g(Z_t) = Z_t \vee e^{\bar{r}t} z$, $t \geq 0$, where $\bar{r}$ is a certain growth rate (such as the interest rate or any contractually pre-determined rate.

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Note:

In this case the loss process is $Y_t = g(S_T, Z_T)\mathbf{1}_{\{T(x) \leq t\}}$, $t \geq 0$.

Some Optimization Problems

We denote

- $\mathscr{A} = \{\pi : E\int_0^T |\pi_t|^2 dt < \infty\}$
- $E_{t,w,s,z}\{\cdot\} = E\{\cdot | W_t = w, S_t = s, Z_t = z\}$.

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- $J(t,w,s,z;\pi) \stackrel{\triangle}{=} E_{t,w,s,z}\{u(W_T^\pi - Y_T)\}$,
- $J^0(t,w;\pi) \stackrel{\triangle}{=} E_{t,w}\{u(W_T^\pi)\}$. $(T(x) > T, \implies Y_T = 0.)$
- $\widehat{J}(t,w,s;\pi) \stackrel{\triangle}{=} E_{t,w,s}\{u(W_T^\pi - g(S_T)Y_T)\}$. $(g = g(S_T))$

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The Value Functions

- $V^0(t,w) = \sup_{\pi\in\mathscr{A}} J^0(t,w;\pi)$
- $V(t,w,s) = \sup_{\pi\in\mathscr{A}} \widehat{J}(t,w,s;\pi)$
- $U(t,w,s,z) = \sup_{\pi\in\mathscr{A}} J(t,w,s,z;\pi)$.

Solution for $g = g(S_T)$

- First recall the Bellman Principle: for any $h > 0$,

$$V(t, w, s) = \sup_{\pi \in \mathscr{A}} E_{t,w,s}\{V(t+h, W^{\pi}_{t+h}, S_{t+h})\}. \qquad (6)$$

- Since $g(S_T)$ involves all tradeable assets, and the benefit is paid at a fixed terminal time T, one can consider $g(S_T)$ as a contingent claim, and determine its present value by

$$c(t, s) = E^Q\{e^{-r(T-t)} g(S_T) | S_t = s\}.$$

- If the death occurs during $[t, t+h]$, then one can set aside the amount of $c(t+h, S_{t+h})$ at time $t+h$ to hedge the potential claim lost $g(S_T)$, and consider the remaining optimization problem on $[t+h, T]$ as if there were no insurance involved. Thus,

$$E_{t,w,s}\{V(t+h, W^{\pi}_{t+h}, S_{t+h})\}$$
$$= E_{t,w,s}\{V^0(t+h, W^{\pi}_{t+h} - c(t+h, S_{t+h}))\}.$$

Solution for $g = g(S_T)$

- Now for any π on $[t, t+h]$,

$$
\begin{aligned}
V(t,w,s) \;\geq\; & E_{t,w,s}\{V(t+h, W^{\pi}_{t+h}, S_{t+h})\}{}_h p_{x+t} \\
& + E_{t,w,s}\{V^0(t+h, W^{\pi}_{t+h} - c(t+h, S_{t+h}))\}{}_h q_{x+t}.
\end{aligned}
$$

- Assume that $c(\cdot,\cdot) \in C^{1,2}$ and satisfies the Black-Scholes PDE, we can apply Itô to both $V(W_t, t, S_t)$ and $V^0(W_t - c(t, S_t), t)$ from t to $t+h$, and then take conditional expectations and rearrange terms to obtain

$$
\begin{aligned}
& V(w,t,s)\frac{{}_h q_{x+t}}{h} \geq V^0(w - c(t,s), t)\frac{{}_h q_{x+t}}{h} \\
& + E\left\{\frac{1}{h}\int_t^{t+h} \{V_t + \mathscr{L}[V](u, W_u, S_u)\Big| W_t = w\right\}{}_h p_{x+t} \\
& + E\left\{\frac{1}{h}\int_t^{t+h} \{V^0_t + \mathscr{L}[V^0](r, W_u, S_u)\Big| W_t = w\right\}{}_h q_{x+t}.
\end{aligned}
$$

Solution for $g = g(S_T)$

- Letting $h \to 0$, noting that

$$\lim_{h\to 0} {}_hq_{x+t}/h = \lambda_x(t), \ \lim_{h\to 0} {}_hp_{x+t} = 1, \ \lim_{h\to 0} {}_hq_{x+t} = 0,$$

and using the fact that c satisfies the Black-Scholes PDE, we obtain the HJB Equation for V:

$$\begin{cases} 0 = V_t + \max_{\pi}\{(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww} + s\sigma^2\pi V_{ws}\} + rwV_w \\ \qquad + s\mu V_s + \frac{1}{2}\sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(w - c, t) - V(w, t, s)), \\ V(T, w, s) = u(w). \end{cases}$$

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Note: In the Black-Scholes world, the HJB equation for V^0 is

$$\begin{cases} V_t^0 + \max_{\pi \in \mathbb{R}_+} \left\{ \frac{1}{2}|\sigma\pi|^2 V_{ww}^0 + \langle \pi, \mu - r \rangle V_w^0 \right\} + rwV_w^0 = 0, \\ V^0(T, w) = u(w). \end{cases} \tag{7}$$

The Case of Exponential Utility

Consider now the case of exponential utility. I.e., $u(w) = -\frac{1}{\alpha} e^{-\alpha w}$.

- V^0 has the close form solution:

$$V^0(t, w) = -\frac{1}{\alpha} \exp\{-\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\} \quad (8)$$

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- Assume $V(t, w, s) = V^0(t, w)\Phi(t, s)$, then

$$\Phi_t + rS\Phi_s + \frac{\sigma^2 s^2 \Phi_{ss}}{2} - \frac{s^2\sigma^2\Phi_s^2}{2\Phi} + \lambda_x(e^{\{c\alpha e^{r(T-t)}\}} - \Phi) = 0$$
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$$\Phi(T,s) = 1.$$

- Define $h(t,s) = c(t,s)\alpha e^{r(T-t)} - \ln\Phi$. Then one shows that

$$\begin{cases} h_t + srh_s + \frac{1}{2}\sigma^2 s^2 h_{ss} - \lambda_x(t)(e^h - 1) = 0 \\ h(T,s) = \alpha g(s) \end{cases} \qquad (9)$$

The Case of Exponential Utility

- If we change the variable: $v = \log s$, $\tau = T - t$, (9) becomes:

$$\begin{cases} h_\tau = (r - \frac{1}{2}\sigma^2)h_v + \frac{1}{2}\sigma^2 h_{vv} - \lambda_x(T - \tau)(e^h - 1) \\ h(0, v) = \alpha g(e^v) \end{cases} \quad (10)$$

Note: The reaction-diffusion PDE (10) has a exponential growth, and we must show that it does not blow-up in finite time!

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- Now consider the Initial-Boundary value version of (10) with

$$h(0,x) = \alpha g(x), \quad h(t, \pm N) = \alpha g(\pm N).$$

and denote its solution by $h^N(t,x)$.

- Define $\tilde{K} = |\alpha| \|g\|_\infty$, and let

$$K \stackrel{\triangle}{=} -\log(1 - (1 - e^{-\tilde{K}})e^{\int_0^T \lambda(u)du}).$$

The Case of Exponential Utility

- Consider the function

$$\beta_K(t) \stackrel{\triangle}{=} -\log\{1-(1-e^{-K})e^{-\int_0^t \lambda(u)du}\}, \ t \geq 0.$$

Since $\beta_K(t)$ is decreasing in t, we have

$$\tilde{K} = \beta_K(T) \leq \beta_K(t) \leq \beta_K(0) = K, \qquad \forall t \in [0, T].$$

- It can be easily checked that $h(t,x) \stackrel{\triangle}{=} \beta_K(t)$, solves (10) with the *Initial-Boundary* value:

$$h(0,x) = K, \quad h(t, \pm N) = \beta_K(t). \tag{11}$$

- Thus by Comparison Theorem of PDE $h^N(\cdot,\cdot)$ is bounded by $\beta_{\tilde{K}}(\cdot)$.

The Case of Exponential Utility

- Similarly, denote $v^N(\tau, x) = \partial_x h^N(\tau, x)$, and apply the Comparison Theorem to v^N one sees that $v^N(\cdot, \cdot)$ is bounded by the function $\tilde{v}(t, x) = K' e^{\int_t^T \lambda(t)dt}$, with $K' = |\alpha| \|g'\|_\infty$.

- We can now apply the Arzela-Ascoli Theorem to obtain a uniformly bounded solution of the Cauchy problem by letting $N \to \infty$!

- The indifference price of the UVL insurance is given by

$$p = c(0, s) - \frac{h(0, s)}{\alpha} e^{-rT},$$

The General Case: $g = g(S_T, Z_T)$

Note:

Since Z is non-tradable, this is an "incomplete market" case and the arbitrage free price for the payoff $g(S_T, Z_T)$ cannot be determined as in the previous case.

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A Dynamic Strategy

We consider the following more aggressive (or adventurous) strategy:

- Assuming that the death of the insured occurs before $t + h$
- Instead of putting aside a certain amount of money at the $t + h$ to hedge the future claim, the insurer simply continue to invest all of his current wealth freely, but knowing that he is liable to pay $g(S_T, Z_T)$ at time T.

The General Case: $g = g(S_T, Z_T)$

- Consider an auxiliary control problem assuming death happens before T

$$\tilde{J}(t,x,s,z;\pi) \stackrel{\triangle}{=} E_{t,x,s,z}\{u(X_T^\pi) - g(S_T, Z_T)\},$$

with the corresponding value function $\tilde{U}(t,x,s,z)$.

- Then U satisfies a HJB equation: (assuming $\mu = r$)

$$\begin{cases} 0 = U_t + \max_\pi \left\{ \frac{1}{2}\sigma^2\pi^2 U_{ww} + (U_{ws}S\sigma^2 + U_{wz}Z\sigma^Z\sigma)\pi \right\} \\ \qquad +rwU_w + U_sS\mu + U_zZ\mu^Z + \frac{1}{2}\sigma^2 U_{ss}S^2 \\ \qquad +\frac{1}{2}U_{zz}Z^2(\tilde{\sigma}^2 + {\sigma^Z}^2) + U_{sz}SZ\sigma\sigma^Z + \lambda_x(t)(\tilde{U} - U), \\ U(w,T,s,z) = u(w), \end{cases}$$

where $\tilde{U}$ satisfies a similar HJB equation with $\lambda_x \equiv 0$.

Using the similar techniques as before, modulo the technicalities of showing the no blow-ups, we can derive the indifference price in this case:

- The premium $p(t,s,z) = \frac{1}{\alpha} e^{-r(T-t)} h(T-t, \log s, \log z)$,
- h is a bounded, classical solution to the PDE

$$\begin{cases} h_\tau - \frac{1}{2}\tilde{\sigma}^2 h_{y_2}^2 - \frac{1}{2}\sigma^2 h_{y_1 y_1} - \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{z2}) h_{y_2 y_2} - \sigma\sigma^z h_{y_1 y_2} \\ -\left(r - \frac{1}{2}\sigma^2\right) h_{y_1} - \left(\mu^z - \frac{\mu - r}{\sigma}\sigma^z - \frac{\tilde{\sigma}^2 + \sigma^{z2}}{2}\right) h_{y_2} \\ -\lambda_x(T-\tau)(e^{\tilde{h}-h} - 1) = 0; \\ h(0, y_1, y_2) = 0, \end{cases}$$

and $\tilde{h}$ is a bounded, classical solution to a similar PDE as above, with $\lambda_x \equiv 0$, and $\tilde{h}(0, y_1, y_2) = \alpha g(e^{y_1}, e^{y_2})$.

Multiple-decrement Case

Main Features

- Allowing "multiple decrement": such as short/long term disabilities, withdrawl, retirement, death, etc.
- benefit payable at a random time, e.g., "moment of death".
- the payments may depend on the different status as well as the transitions between them.

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The State/Status Process $\{X_t\}_{t\geq 0}$

- A Markov chain with finite state space $\{0, 1, ..., m\}$, representing the numerical code of the "status".
- $i = 1$ to be the "cemetary state" (death), and $X_0 = 0$
- denote $I_t^i = \mathbf{1}_{\{X_t = i\}}$ to be the "status indicator" and define the counting process

$$N_t^{ij} \stackrel{\triangle}{=} \#\{\text{transitions of } X \text{ from state } i \text{ to } j \text{ during } [0, t]\}.$$

Multiple-decrement Case

Some Important Quantities

- for each t, denote $\tau_t = \inf\{s \geq t : X_s \neq X_t\}$; and for $i = 0, ..., m$, define $\tau_t^i = \tau_t$, if $X_{\tau_t} = i$ and ∞ otherwise.
- ${}_t\bar{p}_s^i \stackrel{\triangle}{=} P\{\tau_s > t | X_s = i\}$;
- ${}_t\bar{q}_s^{ij} \stackrel{\triangle}{=} P\{\tau_s^j = \tau_s \leq t | X_s = i\}$, $s \leq t$, $i, j \in \{0, ..., m\}$.
- Clearly, ${}_t\bar{p}_s^1 = 1$; ${}_t\bar{q}_s^{1j} = 0$, for all $j \neq 1$; and

$$ {}_t\bar{p}_s^i + \sum_{j \neq i} {}_t\bar{q}_s^{ij} = 1, \qquad \forall i = 0, 1, \cdots, m, \quad 0 \leq s < t. \quad (12) $$

- "*force of decrement of status i due to cause j*" as

$$ \bar{\lambda}_t^{ij} \stackrel{\triangle}{=} \lim_{h \to 0} \frac{{}_{t+h}\bar{q}_t^{ij}}{h}, \qquad i, j = 0, 1, \cdots m. \quad (13) $$

Some Remarks

- If $m = 1$, then the state process X becomes the one as in the simple life model, and $\tau_0^1 = T(x)$. In that case we should have

$$_t\bar{p}_s^0 = {}_{t-s}p_{x+s}, \quad {}_tq_s^{01} = {}_{t-s}q_{x+s}.$$

- Being a Markov chain, the process X has its transition probability and the corresponding transition intensity

$$_tq_s^{ij} = P\{X_t = j | X_s = i\}; \qquad \lambda_t^{ij} \triangleq \lim_{h \downarrow 0} \frac{_{t+h}q_t^{ij}}{h}, \qquad i \neq j.$$

There are natural links between p^{ij}'s and $\bar{p}^{ij}$'s. For example:

- $\bar{\lambda}_t^{ij} = \lambda_t^{ij}$, for all $t \geq 0$, $i, j = 0, 1, \cdots, m$;
- $_{t+h}\bar{p}_t^i = \exp\{-\int_t^{t+h} \sum_{j \neq i} \lambda_s^{ij} ds\}$; $_{t+h}p_t^{ij} = \int_t^{t+h} {}_\tau\bar{p}_t^i \lambda_\tau^{ij} d\tau$, $\forall h > 0$, $i, j = 0, \cdots, m$.

... ...

The Payment Process A_t:

- Two types of payments will be considered: "life-annuity" and "life-insurance".
- Since the non-tradability of the asset Z will not make significant difference in the optimization problem, we will not distinguish Z from S.
- The cumulative payment process is defined by

$$A_t = \int_0^t \sum_i I_u^i a^i(u, S_u)du + \sum_{i \neq j} a^{ij}(u, S_u)dN_u^{ij}, \qquad t \geq 0, \tag{14}$$

— an $\mathbf{F}$-adapted, càdlàg , non-decreasing process in which

- $a^i(t,s)$ — rate of payments of annuity at state i, given $S_t = s$;
- $a^{ij}(t,s)$ — rate of payments of insurance when transit from state i to j, given $S_t = s$.

Dynamics of General Reserve

Dynamics of general reserve

$$d\hat{W}_t^{\pi} = [r_t \hat{W}_t^{\pi} + \pi_t(\mu_t - r_t)]dt + \pi_t \sigma_t dB_t - dA_t,$$

where

- $dA_t = \sum_i I_i(t) a^i(t, S_t) dt + \sum_{i \neq j} a^{ij}(t, S_t) dN_t^{ij}$
- $I_t^i = \mathbf{1}_{\{X_t = i\}}$, $N_t^{ij} \stackrel{\triangle}{=} \#\{\text{jumps of } X \text{ from } i \text{ to } j \text{ during } [0, t]\}$

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Hamiltonian

$$\begin{cases} \mathscr{H}^k \triangleq \frac{1}{2}|\sigma_t \pi|^2 \psi + [\langle \pi, \mu_t - r_t \mathbf{1} \rangle + r_t w - a^k(t, s)]\varphi \\ \qquad\qquad + \langle \pi, \sigma_t \sigma_t^T \mathrm{tr} D[s] p \rangle, \quad k = 0, 1, \cdots, m, \\ H^k(t, w, s, \varphi, \psi, p) \triangleq \sup_\pi \mathscr{H}^k(t, w, s, \varphi, \psi, p; \pi). \end{cases}$$

The HJB Equation

Theorem (Yu, '07; M.-Yu, '10)

Under suitable conditions, the value function $U = (U^0, U^1, ..., U^m)$ is the *unique* viscosity solution to the system of PDDE's:

$$\begin{cases} U_t^k + F_k(t, w, s, DU^k, D^2U^k) + (\mathscr{H}_k U) = 0, \\ U^k(T, w, s) = u(w), \qquad k = 0, \cdots, m, \end{cases} \tag{15}$$

where

$$\begin{aligned} F_k(\cdots) &= \sup_{\pi \in \Pi} \Big\{ \pi(\mu_t - r_t) U_w^k + \frac{1}{2} |\sigma_t \pi|^2 U_{ww}^k + \pi \sigma_t^2 s U_{ws}^k \Big\} \\ &\quad + \mu_t s U_s^k + \frac{1}{2} \sigma_t^2 s^2 U_{ss}^k + (r_t w - a^k(t, s)) U_w^k \\ (\mathscr{H}_k U) &= \sum_{j \neq k} \lambda_t^{kj} (U^j(t, w - a^{kj}(t, s), s) - U^k(t, w, s)). \end{aligned}$$

Viscosity Solution for System of PDDEs

Main Difficulties

- Definition of viscosity solution for the system of PDDE.
- Uniqueness
  - Different from Ishii et al.'s results: *Parabolic PDDE vs. Elliptic PDEs*
  - Different from Pardoux et al.'s results: *Fully Nonlinear System vs. Semilinear System*

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Main idea:

- Taking the index vector of the value function as an additional "spatial" variable with values in a finite set: the *system* of PDDEs becomes a *single* PDDE!
- The abstract framework of viscosity solutions (e.g., Fleming & Soner book) applies!

Abstract Dynamic Programming Principle Revisited

Recall Fleming-Soner (II.3)

- Σ — a closed subset of a Banach space
- $\mathscr{C}$ — a collection of functions on Σ
- $\mathscr{T}_{tr}$, $0 \le t \le r \le T$ — a family of operators on $\mathscr{C}$, s.t.,

(i) $\mathscr{T}_{tt}\varphi = \varphi$;

(iia) $\mathscr{T}_{tr}\varphi \le \mathscr{T}_{ts}\psi$, if $\varphi \le (\mathscr{T}_{rs}\psi)$, $\forall\ 0 \le t \le r \le s$;

(iib) $\mathscr{T}_{tr}\varphi \ge \mathscr{T}_{ts}\psi$, if $\varphi \ge (\mathscr{T}_{rs}\psi)$, $\forall\ 0 \le t \le r \le s$.

Note

- $r = s$ in (ii) $\Longrightarrow$ *monotonicity*: $\mathscr{T}_{tr}\varphi \le \mathscr{T}_{tr}\psi$, if $\varphi \le \psi$,
- (iia) $\oplus$ (iib) $\Longrightarrow$ *semigroup property*:

$$\mathscr{T}_{ts}\varphi = \mathscr{T}_{tr}(\mathscr{T}_{rs}\varphi),\ t \le r \le s \le T, \quad \text{if } \mathscr{T}_{tr}\varphi \in \mathscr{C}, \forall \varphi \in \mathscr{C}.$$

Of course, the fact that $T_{tr}\varphi \in \mathscr{C}$ must be verified!

Abstract Bellman (Dynamic Programming) Principle

- $\Sigma \subseteq \overline{\mathscr{O}}$, where $\mathscr{O}$ is an open set in $\mathbb{R}^n$, and $\mathscr{C} = \mathscr{M}(\Sigma)$,
- $T_{t,r;u}\psi(x) \stackrel{\triangle}{=} J(t,r;u) = E_{t,x}\left\{\int_t^r L(s,X_s,u_s)ds + \psi(X_r)\right\}$.
- $\mathscr{T}_{t,r}\psi(x) \stackrel{\triangle}{=} \inf_{u\in\mathscr{U}_{ad}} T_{t,r;u}\psi(x)$ (Thus, $T_{t,T}\psi(x) = V(t,x)!$).

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Note

Semigroup Property = (Abstract) Bellman Principle(!)

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Note

Semigroup Property = (Abstract) Bellman Principle(!)

- Let $\{\mathscr{G}_t\}_{t\geq 0}$ be the "*infinitesimal generator*" of the semigroup $\mathscr{T}$, that is, for all $\varphi \in \mathscr{D}$, $y \in \Sigma$,

$$\lim_{h\downarrow 0}\frac{1}{h}\{(\mathscr{T}_{tt+h}\varphi(t+h,\cdot))(y) - \varphi(t,y)\} = [\frac{\partial}{\partial t} + \mathscr{G}_t]\varphi(t,y),$$

- where $\mathscr{D} \subset C([0,T)\times\Sigma)$ is the set of "*test functions*" [i.e., $\forall \varphi \in \mathscr{D}$, $\frac{\partial}{\partial t}\varphi(t,y)$ and $(\mathscr{G}_t\varphi(t,\cdot))(y)$ are continuous.]

Abstract form of HJB Equation

Assume $V \in C^{1,2} \subset \mathscr{D}$. Then use the semigroup property one derives the HJB equation:

$$\begin{cases} 0 = \lim_{h \downarrow 0} \frac{1}{h}\{(\mathscr{T}_{tt+h}V(t+h,\cdot))(y) - V(t,y)\} \\ \quad = [\frac{\partial}{\partial t} + \mathscr{G}_t]V(t,y), \quad \forall y \in \Sigma, \\ V(T,y) = \psi(y). \end{cases} \tag{16}$$

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Theorem (Fleming-Soner, Theorem II.5.1)

If the value function of a control problem $V \in C[0,T] \times \Sigma)$, then V is a viscosity solution to the (abstract) HJB equation (16).

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Question:

What are $\mathscr{G}$, $\mathscr{D}$,..., etc. in our case?

Back to UVL Model

- $\Sigma = \{(w, s, k) : w, s \in \mathbb{R}, k \in \{0, 1, ..., m\}\}$,
- $\mathscr{C} = C(\Sigma)$.
- $(\mathscr{T}_{tr}\varphi)(w, s, k) \stackrel{\triangle}{=} \sup_{\pi \in \mathscr{A}} E_{w,s,k}\{\varphi(\hat{W}_r^\pi, S_r, X_r)\}, \quad t \geq r$
- $(\mathscr{T}_{tT} u)(w, s, k) = U^k(t, w, s), \quad \forall (t, w, s)$ and k

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Note

- It is easy to check that the family $\{\mathscr{T}_{tr}\}$ satisfies (i), (ii).
- Since $U^k(t, w, s)$'s are all continuous, the function $(t, w, s, k) \mapsto U^k(t, w, s)$ (on Σ) should satisfy an abstract HJB equation!

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Problems:

- Identify the infinitesimal generator of the semigroup $\mathscr{T}$.
- Define the "viscosity solutions" to the corresponding abstract HJB equation (vs. the system of the HJB equations!)

Abstract HJB Equation vs. System of PDDEs

Denote $U(t, w, s, k) = U^k(t, w, s)$, and recall the PDDEs (15):

$$\begin{cases} \dfrac{\partial}{\partial t} U^k + F_k(t, w, s, DU^k, D^2 U^k) + (\mathscr{H}_k U)(t, w, s) = 0, \\ U^k(T, w, s) = u(w), \qquad\qquad k = 0, \cdots, m. \end{cases} \tag{17}$$

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Theorem

The viscosity solutions of the abstract HJB equation (16) with respect to the operator $\mathscr{T}$ and that of the system of PDDEs (17) are equivalent if and only if

$$(\mathscr{G}_t \varphi(t, \cdot))(w, s, k) = [F_k(\cdot, \cdot, \cdot, D\varphi, D^2\varphi) + (\mathscr{H}_k \varphi)](t, w, s). \tag{18}$$

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The Case of Bereaved Partner

Main Rationales

- The usual "Multi-Life Contingency" (e.g., pension plans) assumes independent mortality, even for married couples
- Empirical evidence of the bereaved spouse (Hu-Goldman ('90) Mariikainen-Valkonen ('96), and Valkonen et al. ('04)) indicated the possible correlated mortality.

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- $T_{x_1}, T_{x_2}, \cdots, T_{x_n}$ — future life time random variables,
- $T_m = T_{x_1,\cdots,x_n} \stackrel{\triangle}{=} \min\{T_{x_1}, \cdots, T_{x_n}\}$ — (Joint-life)
- $T_M = T_{\overline{x_1,\cdots,x_n}} \stackrel{\triangle}{=} \max\{T_{x_1}, \cdots, T_{x_n}\}$ — (Last-survivor)

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- $T_M = T_{\overline{x_1,\cdots,x_n}} \stackrel{\triangle}{=} \max\{T_{x_1}, \cdots, T_{x_n}\}$ — (Last-survivor)
- If $n = 2$, one has $T_M + T_m = T_{x_1} + T_{x_2}$, $T_M T_m = T_{x_1} T_{x_2}$.
- $F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t)$, $t \geq 0$ where F_T is the distribution function of T.
- If $T_{x_1} \perp T_{x_2}$, then $F_M(t) = F_{T_{x_1}}(t) F_{T_{x_2}}(t)$...

The Case of Bereaved Partner

Assume $n = 2$, and that the individual force of mortalities take the form:

$$\begin{cases} \mu_{x_1}(t) = \lambda_{x_1}(t) + \mathbf{1}_{\{T_{x_2} \le t\}} \gamma_{x_1}(t - T_{x_2}) \\ \mu_{x_2}(t) = \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1} \le t\}} \gamma_{x_2}(t - T_{x_1}), \end{cases} \qquad t \ge 0, \qquad (19)$$

where λ_{x_i}'s are the (marginal) force of mortality and

$$\gamma_{x_i}(t) = \frac{n_i}{r_i e^t + 1}, \quad i = 1, 2, \quad r_1, r_2, n_1, n_2 > 0.$$

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Note:

This essentially becomes a problem of "*Counter-Party Risk*", a well-know topic in "Contagion Models" of correlated default! Existing literature include

- King-Wadhwani, Kodres-Pritsker, Collin-Dufresne, ...
- Jarrow-Yu, Yu (2001, counterparty, two firms)
- ....

Basic Setup

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ be a given filtered probability space.

- $\mathbb{P}$ is *risk neutral* (in a default free bond market)
- $\exists$ a factor process $X = \{X_t : t \geq 0\}$
- There are I firms, with default times τ^i, $i = 1, \cdots, I$

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Denote

- $N_t^i \stackrel{\triangle}{=} \mathbf{1}_{\{\tau^i \leq t\}}$ — *default process* with respect to τ^i,
- $\mathscr{F}_t \stackrel{\triangle}{=} \mathscr{F}_t^X \vee \mathscr{F}_t^1 \vee \ldots \vee \mathscr{F}_t^I$, where $\mathscr{F}_t^i = \sigma\{N_s^i : 0 \leq s \leq t\}$, $\forall i$
- $\mathscr{H}_t^i = \mathscr{F}_t^X \vee \mathscr{F}_t^1 \vee \ldots \vee \mathscr{F}_t^{i-1} \vee \mathscr{F}_t^{i+1} \vee \ldots \vee \mathscr{F}_t^I$,
  $\Longrightarrow \ \mathscr{F}_t = \mathscr{H}_t^i \vee \mathscr{F}_t^i$.

Basic Setup

Define

- $S_t^i = \mathbb{P}\{\tau^i > t | \mathscr{H}_t^i\} > 0$ ($\Longrightarrow S^i$ is an $\mathscr{H}^i$-supermg)
- $H_t^i \overset{\triangle}{=} -\ln(S_t^i)$, $t \geq 0$ — Hazard Process

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Note:

- $S_t^i > 0$ implies that τ^i cannot be an $\mathscr{H}^i$-stopping time!

Basic Setup

Define

- $S_t^i = \mathbb{P}\{\tau^i > t | \mathscr{H}_t^i\} > 0$ ($\Longrightarrow S^i$ is an $\mathscr{H}^i$-supermg)
- $H_t^i \stackrel{\triangle}{=} -\ln(S_t^i)$, $t \geq 0$ — Hazard Process

Note:

- $S_t^i > 0$ implies that τ^i cannot be an $\mathscr{H}^i$-stopping time!
- If $\exists \lambda_t^i \in \mathscr{H}_t^i$, such that $H_t^i = \int_0^t \lambda_s^i \, ds$, $t \geq 0$, then

$$S_t^i = \mathbb{P}\{\tau^i > t | \mathscr{H}_t^i\} = \exp\Big\{ -\int_0^t \lambda_s^i \, ds \Big\}. \tag{20}$$

— λ^i is called the *(conditional) intensity process* of τ^i, and it holds that $\lambda_t^i = -dS_t^i / S_t^i$, $t \geq 0$.

A Useful Lemma

Lemma

For any $\mathscr{F}$-measurable random variable Z we have, for any $t \geq 0$,

$$\mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} Z | \mathscr{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathscr{H}_t^i\}} \qquad (21)$$

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Idea: Define

$$\mathscr{F}_t^* \triangleq \{A \in \mathscr{F} | \exists B \in \mathscr{H}_t^i,\ A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\}\}.$$

Then one can check that $\mathscr{F}_t = \mathscr{F}_t^*$, $t \geq 0$.

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$$\mathscr{F}_t^* \triangleq \{A \in \mathscr{F} | \exists B \in \mathscr{H}_t^i, \; A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\}\}.$$

Then one can check that $\mathscr{F}_t = \mathscr{F}_t^*$, $t \geq 0$.
Applying "Monotone Class", one shows that, $\forall Z \in \mathscr{F}$, $\exists X \in \mathscr{H}_t^i$, s.t.

$$\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} Z | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} X.$$

Taking $\mathbb{E}\{\cdot \, | \mathscr{H}_t^i\}$ on both sides and solve for X. ■

The Conditional Survival Probability

Note that $\mathbb{P}\{\tau^i > T | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} | \mathscr{F}_t\}$. Applying Lemma we have

$$\mathbb{P}\{\tau^i > T | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}[\mathbf{1}_{\{\tau^i > T\}} | \mathscr{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathscr{H}_t^i\}}. \tag{22}$$

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Since

- $\mathbb{E}\{\mathbf{1}_{\{\tau^i>T\}}|\mathscr{H}_t^i\} = \mathbb{E}\{\mathbb{P}\{\tau^i > T|\mathscr{H}_T^i\}|\mathscr{H}_t^i\} = \mathbb{E}\left\{e^{-\int_0^T \lambda_s^i ds}\middle|\mathscr{H}_t^i\right\}$.
- $\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathscr{H}_t^i\} = e^{-\int_0^t \lambda_s^i ds}$

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Consequently:

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Note that $\mathbb{P}\{\tau^i > T|\mathscr{F}_t\} = \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>T\}}|\mathscr{F}_t\}$. Applying Lemma we have

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Consequently:

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- $M_t^i \stackrel{\triangle}{=} N_t^i - H_{t\wedge\tau^i}^i = \mathbf{1}_{\{\tau^i\le t\}} - \int_0^t \mathbf{1}_{\{\tau^i>s\}}\lambda_s^i ds$, $i = 1, \dots, I$, are $\{\mathscr{F}_t\}$-martingales.

Standing Assumptions

(H1) λ_t^i satisfy the following condition:

$$\mathbb{E}\Big\{\exp\Big(2\int_0^t \sum_{i=1}^{I} \lambda_s^i ds\Big)\Big\} < \infty, \quad \forall t < \infty.$$

(H2) For each i, $\mathbb{P}\{\tau^i > 0\} = 1$. Furthermore, there are no simultaneous defaults among the I firms. In other words, it holds that $\mathbb{P}\{\tau^i \neq \tau^j\} = 1$, whenever $i \neq j$.

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Main Task

Find effective, tractable way to calculate the joint distribution (survival probability):

$$\mathbb{P}\{\tau^1 \le t_1, \cdots, \tau^I \le t_I\}, \quad \text{and/or} \quad \mathbb{P}\{\tau^1 > t_1, \cdots, \tau^I > t_I\},$$

given the conditional intensities.

Representation of Joint Survival Probability

Define, for $i = 1, ..., I$, $\Gamma_t^i \triangleq \exp\{\int_0^t \lambda_s^i ds\}$, and

$$Z_t^i \triangleq \mathbf{1}_{\{\tau^i > t\}} \Gamma_t^i = \mathbf{1}_{\{\tau^i > t\}} \exp\Big\{ \int_0^t \lambda_s^i ds \Big\}. \tag{23}$$

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Then

- $Z_t^i \geq 0$; and $Z_0^i = 1$, $\forall i$.
- Z^i's are $\{\mathscr{F}_t\}$-adapted, and $\mathbb{E}\{Z_t^i\} = 1$.

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Proposition

Assume (H1) and (H2). Then, for $k = 1, ..., I$, the processes

$$\prod_{i=1}^k Z_t^i \triangleq \prod_{i=1}^k \mathbf{1}_{\{\tau^i > t\}} \Gamma_t^i, \quad t \geq 0 \tag{24}$$

are all $\{\mathscr{F}_t\}$-martingales.

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$$\begin{aligned}
\mathbb{E}\{Z_t^i|\mathscr{F}_s\} &= \mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{F}_s\} = \mathbf{1}_{\{\tau^i>s\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{F}_s\} \\
&= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>s\}}|\mathscr{H}_s^i\}} \qquad \text{(Lemma)} \\
&= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_s^i\}}{(\Gamma_t^i)^{-1}} = Z_s^i\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_s^i\} \\
&= Z_s^i\mathbb{E}\{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathscr{H}_t^i\}\Gamma_t^i|\mathscr{H}_s^i\} = Z_s^i.
\end{aligned}$$

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[Sketch of the proof.] (i) Z_t^i's are martingales.

$$\begin{aligned}\mathbb{E}\{Z_t^i|\mathscr{F}_s\} &= \mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{F}_s\} = \mathbf{1}_{\{\tau^i>s\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{F}_s\} \\ &= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>s\}}|\mathscr{H}_s^i\}} \qquad \text{(Lemma)} \\ &= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_s^i\}}{(\Gamma_t^i)^{-1}} = Z_s^i\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma_t^i|\mathscr{H}_s^i\} \\ &= Z_s^i\mathbb{E}\{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathscr{H}_t^i\}\Gamma_t^i|\mathscr{H}_s^i\} = Z_s^i.\end{aligned}$$

(ii) If $\tilde{Z}_t^k \stackrel{\triangle}{=} \prod_{i=1}^k Z_t^i$ is an mg, then so is $\prod_{i=1}^{k+1} Z_t^i = \tilde{Z}_t^k Z_t^{k+1}$.

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&= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathscr{H}^i_t\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>s\}}|\mathscr{H}^i_s\}} \qquad \text{(Lemma)} \\
&= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathscr{H}^i_s\}}{(\Gamma^i_t)^{-1}} = Z^i_s\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathscr{H}^i_s\} \\
&= Z^i_s\mathbb{E}\{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathscr{H}^i_t\}\Gamma^i_t|\mathscr{H}^i_s\} = Z^i_s.
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$$
\tilde{Z}^k_t Z^{k+1}_t = \int_{0^+}^t \tilde{Z}^k_{s-}dZ^{k+1}_s + \int_{0^+}^t Z^{k+1}_{s-}d\tilde{Z}^k_s + [\tilde{Z}^k, Z^{k+1}]_t.
$$

[Sketch of the proof.] (i) Z_t^i's are martingales.

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(ii) If $\tilde{Z}_t^k \triangleq \prod_{i=1}^k Z_t^i$ is an mg, then so is $\prod_{i=1}^{k+1} Z_t^i = \tilde{Z}_t^k Z_t^{k+1}$.

$$\tilde{Z}_t^k Z_t^{k+1} = \int_{0^+}^t \tilde{Z}_{s-}^k dZ_s^{k+1} + \int_{0^+}^t Z_{s-}^{k+1} d\tilde{Z}_s^k + [\tilde{Z}^k, Z^{k+1}]_t.$$

Since both $\tilde{Z}^k$ and Z^{k+1} are FV and quadratic pure jump,

$$[\tilde{Z}^k, Z^{k+1}]_t = \tilde{Z}_0^k Z_0^{k+1} + \sum_{0<s\le t} \Delta\tilde{Z}_s^k \Delta Z_s^{k+1} = \tilde{Z}_0^k Z_0^{k+1}.$$

■

Representation of Joint Survival Probability

Define

$$\left.\frac{d\mathbb{P}^i}{d\mathbb{P}}\right|_{\mathscr{F}_T} \triangleq Z_T^i; \qquad \left.\frac{d\mathbb{P}^{1,\cdots,k}}{d\mathbb{P}}\right|_{\mathscr{F}_T} \triangleq \tilde{Z}_T^k = \prod_{i=1}^{k} Z_T^i. \tag{25}$$

and $\mathbb{E}^{1,\cdots,k}\{X\} \triangleq \mathbb{E}^{\mathbb{P}^{1,\cdots,k}}\{X\} = \mathbb{E}\{Z_T^1 Z_T^2 ... Z_T^k X\}$.

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and $\mathbb{E}^{1,\cdots,k}\{X\} \triangleq \mathbb{E}^{\mathbb{P}^{1,\cdots,k}}\{X\} = \mathbb{E}\{Z_T^1 Z_T^2 ... Z_T^k X\}$.
Then,for each k and $A \in \mathscr{F}_t$, it holds that

$$\begin{aligned}
\mathbb{E}\{\mathbf{1}_A \tilde{Z}_t^k \mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}\} &= \mathbb{E}\{\mathbf{1}_A \mathbb{E}\{\tilde{Z}_T^k|\mathscr{F}_t\}\mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}\} \\
&= \mathbb{E}^{1,\cdots,k}\{\mathbf{1}_A \mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}\} \\
&= \mathbb{E}^{1,\cdots,k}\{\mathbf{1}_A X\} = \mathbb{E}\{\mathbf{1}_A \mathbb{E}\{\tilde{Z}_T^k X|\mathscr{F}_t\}\}.
\end{aligned}$$

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\end{aligned}$$

This leads to

$$\mathbb{E}\{Z_T^1 Z_T^2 \ldots Z_T^k X|\mathscr{F}_t\} = Z_t^1 Z_t^2 \ldots Z_t^k \mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}, \quad \mathbb{P}-a.s. \tag{26}$$

Representation of Joint Survival Probability

Assume $I = 2$, and $t_1 \le t_2$. Apply (26) we get

$$\begin{aligned}
\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2\} &= \mathbb{E}\Big\{\mathbf{1}_{\{\tau^1 > t_1\}}\mathbb{E}\Big\{Z^2_{t_2}(\Gamma^2_{t_2})^{-1}\Big\}\Big|\mathscr{F}_{t_1}\Big\}\Big\} \\
&= \mathbb{E}\Big\{\mathbf{1}_{\{\tau^1 > t_1\}}Z^2_{t_1}\mathbb{E}^{\mathbb{P}^2}\Big\{(\Gamma^2_{t_2})^{-1}\Big\}\Big|\mathscr{F}_{t_1}\Big\}\Big\} \\
&= \mathbb{E}\Big\{Z^1_{t_1}Z^2_{t_1}\mathbb{E}^{\mathbb{P}^2}\Big\{(\Gamma^1_{t_1})^{-1}(\Gamma^2_{t_2})^{-1}\Big\}\Big|\mathscr{F}_{t_1}\Big\}\Big\} \\
&= \mathbb{E}^{1,2}\Big\{\mathbb{E}^{\mathbb{P}^2}\Big\{(\Gamma^1_{t_1})^{-1}(\Gamma^2_{t_2})^{-1}\Big\}\Big|\mathscr{F}_{t_1}\Big\}\Big\}.
\end{aligned}$$

In particular, if $t_1 = t_2 = t$, then we have

$$\mathbb{P}\{\tau^1 > t, \tau^2 > t\} = \mathbb{E}^{1,2}\Big\{\exp\Big\{-\int_0^t(\lambda^1_s + \lambda^2_s)ds\Big\}\Big\}.$$

Theorem

Assume (H1) and (H2). Then,
(i) For any $0 \le t_1 \le t_2 \le ... \le t_I < \infty$, it holds that

$$\begin{aligned} & \mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, ..., \tau^I > t_I\} \\ = \; & \mathbb{E}^{1,\cdots,I}\Big\{ \cdots \Big\{ \mathbb{E}^{\mathbb{P}^I}\Big\{ \prod_{i=1}^{I} (\Gamma^i_{t_i})^{-1} \Big\} \Big| \mathscr{F}_{t_{I-1}} \Big\} \cdots \Big| \mathscr{F}_{t_1} \Big\}; \end{aligned}$$

Representation of Joint Survival Probability

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(ii) Denote $\tau^* = \min\{\tau^1, \cdots, \tau^I\}$, then for any $0 \le t \le T$

a) $\mathbb{P}\{\tau^* > t\} = \mathbb{E}^{1,\cdots,I}\Big\{ e^{-\int_0^t \sum_{i=1}^I \lambda^i_s ds} \Big\};$

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a) $\mathbb{P}\{\tau^* > t\} = \mathbb{E}^{1,\cdots,I}\Big\{e^{-\int_0^t \sum_{i=1}^I \lambda_s^i ds}\Big\}$;

b) $\mathbb{P}\{\tau^* > T|\mathscr{F}_t\} = \prod_{i=1}^{I} \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}^{1,\cdots,I}\Big\{e^{-\int_t^T \sum_{i=1}^I \lambda_s^i ds}\Big|\mathscr{F}_t\Big\}$.

Two firm case:

$$\begin{cases} \lambda_t^A = a_0(t) + \mathbf{1}_{\{\tau^B \le t\}} a_1(t - \tau^B), \\ \lambda_t^B = b_0(t) + \mathbf{1}_{\{\tau^A \le t\}} b_1(t - \tau^A), \end{cases} \tag{27}$$

where a_0, a_1, b_0, and b_1 are deterministic functions.

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Jarrow-Yu (2004) — a_1, b_1 constants.

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Jarrow-Yu (2004) — a_1, b_1 constants.

(H3) (i) a_0 and b_0 are positive functions;

(ii) a_1 and b_1 are either positive and decreasing or negative and increasing, such that

$$\lim_{t\to\infty} a_1(t) = 0 \qquad \lim_{t\to\infty} b_1(t) = 0; \tag{28}$$

and such that both λ_t^A and λ_t^B are positive functions.

Proposition

Assume (H1)–(H3). Then the joint survival probability $\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\}$ is given by

$$\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\}$$
$$= \begin{cases} c(t_1,t_2)\Big(\int_{t_1}^{t_2} a_0(x)\ e^{-\int_x^{t_2} b_1(s-x)ds - \int_{t_1}^x a_0(s)ds} dx \\ \qquad + \int_{t_2}^{\infty} a_0(x) e^{-\int_{t_1}^x a_0(s)ds}\ dx \Big) & t_1 \le t_2; \\ c(t_1,t_2)\Big(\int_{t_2}^{t_1} b_0(x)\ e^{-\int_x^{t_1} a_1(s-x)ds - \int_{t_2}^x b_0(s)ds} dx \\ \qquad + \int_{t_1}^{\infty} b_0(x) e^{-\int_{t_2}^x b_0(s)ds}\ dx \Big) & t_1 > t_2. \end{cases}$$

where $c(t_1,t_2) = \exp\big\{ -\int_0^{t_1} a_0(s)ds - \int_0^{t_2} b_0(s)ds \big\}$.

Counter-Party Risk Models

Main Observation: $\lambda_s^A = a_0(s)$, $\lambda_s^B = b_0(s)$, $\mathbb{P}^{A,B}$-a.s.

$$\Longrightarrow \quad 1 - F^B_{\tau^A}(x) = \mathbb{P}^B(\tau^A > x) = \mathbb{P}^{A,B}((\Gamma^A_x)^{-1}) = e^{-\int_0^x a_0(s)ds}.$$

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Applying the change of measure, we have

$$\begin{aligned}
& \mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} = \mathbb{E}\Big[\mathbf{1}_{\{\tau^A > t_1\}}\mathbf{1}_{\{\tau^B > t_2\}}\Gamma^B_{t_2}(\Gamma^B_{t_2})^{-1}\Big\}\Big] \\
= \quad & \mathbb{E}^B\Big[\mathbf{1}_{\{\tau^A > t_1\}} \exp\Big(-\int_0^{t_2}(b_0(s) + \mathbf{1}_{\{\tau^A \le s\}} b_1(s - \tau^A))\,ds\Big)\Big] \\
= \quad & c(t_2)\Big\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x)ds} F^B_{\tau^A}(dx) + \int_{t_2}^{\infty} F^B_{\tau^A}(dx)\Big\} \\
= \quad & c(t_2)\Big\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x)ds} f_{\tau^A}(x)dx + \int_{t_2}^{\infty} f_{\tau^A}(x)dx\Big\} \\
= \quad & \text{RHS} \quad (t_1 \le t_2)
\end{aligned}$$

Multiple Firm Case

Assume that $I > 2$, and that the default intensities are given by

$$\lambda_t^i = a_0^i(t) + \sum_{\substack{j=1 \\ j \neq i}} \mathbf{1}_{\{\tau^j \le t\}} a_{j-1}^i(t - \tau^j), \quad i = 1, \cdots, I, \tag{29}$$

where a_j^i's are deterministic functions satisfying (H3).

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where a_j^i's are deterministic functions satisfying (H3).

- For $1 \leq m \leq I$, denote $f_m(t_1, t_2, \cdots, t_m)$ to be the joint density function of the default times $\tau^1, \tau^2, \cdots, \tau^m$.
- For example, $f_1(t_1) = f_{\tau^1}(t_1) = a_{1,0}(t_1) e^{-\int_0^{t_1} a_{1,0}(s) ds}$.

Proposition

For $0 = t_0 < t_1 < t_2 < \ldots < t_{m+1}$.

$$f_{m+1}(t_1, t_2, \cdots, t_{m+1}) = \Big\{ \sum_{j=0}^{m} a_j^{m+1}(t_{m+1} - t_j) \Big\} e^{-\sum_j \int_{t_j}^{t_{m+1}} a_j^{m+1}(s - t_j) ds} f_m(t_1, \cdots, t_m).$$

Multiple Firm Case (General)

- Let $\mathscr{P}(I)$ be all the permutations $p = p(1, \cdots, I)$, then $|\mathscr{P}(I)| = I!$.

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$$\mathscr{D}^{(p)} \triangleq \{(t_1, \cdots, t_I) \in \mathbb{R}_+^I : t_1^{(p)} < \cdots < t_I^{(p)}\}.$$

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- $\forall p \in \mathscr{P}(I)$, define $(\tau_1^{(p)}, \cdots, \tau_I^{(p)})$ accordingly, and

$$\lambda_t^{i,(p)} = a_0^{i,(p)}(t) + \sum_{\substack{j=1 \\ j \neq i}} \mathbf{1}_{\{\tau_j^{(p)} \leq t\}} b_{j-1}^i(t - \tau_j^{(p)}),$$

where $b_{j,0}(t) = a_{j^{(p)},0}(t)$, $j = 1, \cdots, I$, $j^{(p)}$ is the image position of j after the permutation $p \in \mathscr{P}(I)$, and b_j^i are appropriately defined functions from a_j^i's.

Multiple Firm Case (General)

$\forall p \in \mathscr{P}(I)$ apply the Proposition on the region $D^{(i)}$, with $(\lambda_1, \cdots, \lambda_I)$ being replaced by $(\lambda_1^{(p)}, \cdots \lambda_I^{(p)})$, to obtain the joint density function on $D^{(p)}$, denoted by $f_I^{(p)}$. We can then define

$$g_I(t_1, \cdots, t_I) = f_I^{(p)}(t_1^{(p)}, \cdots, t_I^{(p)}), \; (t_1, \cdots, t_I) \in D^{(p)}.$$

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Theorem

Assume (H1)–(H3). The joint distribution of $\tau_1, \tau_2, \cdots, \tau_I$ can be expressed as

$$\mathbb{P}\{\tau^1 \leq t_1, \cdots, \tau^I \leq t_I\} = \int_0^{t_1} \ldots \int_0^{t_I} g_I(u_1, \cdots, u_I) du_1 du_2 \cdots du_I.$$

where g_I's are defined above. ■

Dependent Mortality Models

Calculated Case:

- Force of mortality take the form:

$$\begin{cases} \mu_t^{x_1} = \lambda_{x_1}(t) + \mathbf{1}_{\{T_{x_2} \le t\}} \gamma_{x_1}(t - T_{x_2}) \\ \mu_t^{x_2} = \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1} \le t\}} \gamma_{x_2}(t - T_{x_1}). \end{cases} \tag{30}$$

- $\gamma_{x_i}(t) = \dfrac{n_i}{r_i e^t + 1},$, $i = 1, 2$, and $r_1,\ r_2,\ n_1,\ n_2 > 0$.
- Denote
  - $\Delta_k^i(t) = \int_0^t y^{\frac{k}{g_i}} \ e^{-\frac{h_i}{g_i} y} dy$ for $i = 1, 2$,
  - $\tilde{\mathbb{D}}_k^i(t) = \mathbb{D}_k^i\left(\frac{\lambda_{x_i}(t)}{h_i}\right)$, $i = 1, 2$
  - $B^1 = e^{-k(t_2+x_1)+\frac{h_1}{g_1} e^{g_1(x_1+t_1)}}$,
  - $B^2 = e^{-k(t_1+x_2)+\frac{h_2}{g_2} e^{g_2(x_2+t_2)}}$,

The joint survival probability can then be calculated as:

$$\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\}$$
$$= \begin{cases} \dfrac{c(t_1,t_2)}{(r_2+1)^{n_2}} \displaystyle\sum_{k=0}^{n_2} \binom{n_2}{k} \dfrac{h_1}{g_1} r_2^{n_2-k} B^1\Big(\tilde{\mathbb{D}}_k^1(t_2) - \tilde{\mathbb{D}}_k^1(t_1)\Big) & \\ \quad +c(t_2,t_2) & t_1 \le t_2; \\ \dfrac{c(t_1,t_2)}{(r_1+1)^{n_1}} \displaystyle\sum_{k=0}^{n_1} \binom{n_1}{k} \dfrac{h_2}{g_2} r_1^{n_1-k} B^2\Big(\tilde{\mathbb{D}}_k^2(t_1)\Big) - \tilde{\mathbb{D}}_k^2(t_2)\Big) & \\ \quad +c(t_1,t_1) & t_1 > t_2, \end{cases}$$

where
$$c(t_1,t_2) = \exp\Big\{ -\tfrac{h_1}{g_1}[e^{g_1(x_1+t_1)} - e^{g_1x_1}] - \tfrac{h_2}{g_2}[e^{g_2(x_2+t_2)} - e^{g_2x_2}]\Big\}.$$

Let T_{x_1}, $T_{x_2}, \cdots, T_{x_n}$ be n future life time random variables, then their and are given by, respectively:

$$T_m = T_{x_1,\cdots,x_n} \stackrel{\triangle}{=} \min\{T_{x_1}, T_{x_2}, \cdots, T_{x_n}\},$$

— (Joint-life = first default)

$$T_M = T_{\overline{x_1,\cdots,x_n}} \stackrel{\triangle}{=} \max\{T_{x_1}, T_{x_2}, \cdots, T_{x_n}\},$$

— (Last-survivor = last default)

Joint-life vs. Last-survivor

Let $T_{x_1}, T_{x_2}, \cdots, T_{x_n}$ be n future life time random variables, then their and are given by, respectively:

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— (Last-survivor = last default)

If $n = 2$, one has

- $T_M + T_m = T_{x_1} + T_{x_2}$, $T_M T_m = T_{x_1} T_{x_2}$.
- $\{T_{x_1} \leq t\} \cap \{T_{x_2} \leq t\} = \{T_M \leq t\}$,
  $\{T_{x_1} \leq t\} \cup \{T_{x_2} \leq t\} = \{T_m \leq t\}$,
- $F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t)$, $t \geq 0$ where F_T is the distribution function of T.

First Default in Multi-firm Case

Assume for $i = 1, \cdots, I$,

$$\lambda_t^i = a_0^i(t) + \sum_{k \neq i} a_k^i(t) \mathbf{1}_{\{\tau^k \leq t\}} = a_0^i(t) + \sum_{k \neq i} a_k^i(t) N_s^i,$$

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Then

$$\begin{aligned} \mathbb{P}\{\tau_m > t\} &= \mathbb{P}\{\tau^1 > t, \tau^2 > t, \cdots, \tau^I > t\} \\ &= \mathbb{E}^{1,2,\cdots,I}\Big\{ e^{-\int_0^t (\lambda_s^1 + \lambda_s^2 \ + \ldots + \lambda_s^I) ds} \Big\} \\ &= \mathbb{E}^{1,2,\cdots,I}\Big\{ e^{-\int_0^t [a_0^1(s) + a_0^2(s) + \ldots + a_0^I(s)] ds} \Big\}. \end{aligned}$$

If all a_0^i's are deterministic, then

$$\mathbb{P}\{\tau_m > t\} = \exp\Big\{ -\int_0^t [a_0^1(s) + a_0^2(s) + \ldots + a_0^I(s)] ds \Big\}.$$

First Default in Multi-firm Case

Similarly one can obtain the *conditional* survival probability of τ_m:

$$\begin{aligned}
&\mathbb{P}\{\tau_m > T|\mathscr{F}_t\} = \mathbb{P}\{\tau^1 > T, \tau^2 > T, \cdots, \tau^I > T|\mathscr{F}_t\} \\
= \;& \prod_{i=1}^{I} \mathbf{1}_{\{\tau_t^i > t\}} \mathbb{E}^{1,2,\cdots,I}\Big\{ \exp\Big\{ -\int_t^T [\sum_{i=1}^{I} \lambda_s^i] ds\Big\}\Big|\mathscr{F}_t\Big\} \\
= \;& \mathbf{1}_{\{\tau_m > t\}} \mathbb{E}^{1,2,\cdots,I}\Big\{ \exp\Big\{ -\int_t^T [\sum_{i=1}^{I} a_0^i(s)] ds\Big\}\Big|\mathscr{F}_t\Big\}.
\end{aligned}$$

If a_0^i's are all deterministic, then

$$\mathbb{P}\{\tau_m > T|\mathscr{F}_t\} = \mathbf{1}_{\{\tau_m > t\}} \exp\Big\{ -\int_t^T \sum a_0^i(s) ds\Big\}.$$

Flight to Quality

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- We will consider multi-firm model:

$$r_t = r_0(X_t) + J\, \mathbf{1}_{\{\tau_M \le t\}}, \qquad t \ge 0, \tag{32}$$

where $\tau_M \stackrel{\triangle}{=} \max\{\tau^1, \cdots, \tau^I\}$ is the last-to-default time, X is a factor process.
- Main purpose: pricing defaultable zero-coupon bonds.

- Let T_{x_1} and T_{x_2} be two future life time r.v.'s. Denote $N_t^i = \mathbf{1}_{\{T_{x_i} \le t\}}$, $i = 1, 2$, and

$$\mathscr{F}_t = \mathscr{F}_t^X \vee \mathscr{F}_t^1 \vee \mathscr{F}_t^2, \qquad t \ge 0,$$

where $\mathscr{F}_t^i = \sigma\{N_s^i, 0 \le s \le t\}$, $t \ge 0$, $i = 1, 2$, and X is a factor process, assumed to be a diffusion process

Pricing of UVL Insurance Involving Married Couples

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- Death benefit is a lump-sum (e.g., \$1) payable at a terminal time T, contingent on the survivorship of a married couple.

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- Let K_t be a generic status process, e.g., K could be one of the following:

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Assume that the individual T_{x_i}'s follow the Gompertz's law (1825): $\lambda_{x_1}(t) = h_1 e^{g_1(x_1+t)}$, $\lambda_{x_2}(t) = h_2 e^{g_2(x_2+t)}$, $h_i > 0$, $g_i > 0$. Then

$$\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\}$$
$$= \begin{cases} \frac{c(t_1,t_2)}{(r_2+1)^{n_2}} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{h_1}{g_1} r_2^{n_2-k} B^1 \Big(\tilde{\mathbb{D}}_k^1(t_2) - \tilde{\mathbb{D}}_k^1(t_1)\Big) & \\ \quad +c(t_2,t_2) & t_1 \le t_2; \\ \frac{c(t_1,t_2)}{(r_1+1)^{n_1}} \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{h_2}{g_2} r_1^{n_1-k} B^2 \Big(\tilde{\mathbb{D}}_k^2(t_1)\Big) - \tilde{\mathbb{D}}_k^2(t_2)\Big) & \\ \quad +c(t_1,t_1) & t_1 > t_2, \end{cases}$$

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- $c(t_1,t_2) = \exp\Big\{-\frac{h_1}{g_1}[e^{g_1(x_1+t_1)} - e^{g_1x_1}] - \frac{h_2}{g_2}[e^{g_2(x_2+t_2)} - e^{g_2x_2}]\Big\}$.

Back to UVL Insurance Pricing

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- Let u be an exponential utility function:

$$u(w) = -\frac{1}{\alpha}e^{-\alpha w}, \qquad w \in \mathbb{R}. \tag{33}$$

- Define $J(t, w; \pi) \triangleq \mathbb{E}_{t,w}\{u(W_T^\pi - K_T)\}$, where W is the wealth process with investment portfolio π.

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- Define $J(t,w;\pi) \triangleq \mathbb{E}_{t,w}\{u(W_T^\pi - K_T)\}$, where W is the wealth process with investment portfolio π.
- If $K_T \equiv 0$, then denote $J^0(t,w;\pi) \stackrel{\triangle}{=} \mathbb{E}_{t,w}\{u(W_T^\pi)\}, \pi \in \mathscr{A}$.
- $U(t,w) \stackrel{\triangle}{=} \sup_{\pi\in\mathscr{A}} J(t,w;\pi)$, $V(t,w) \stackrel{\triangle}{=} \sup_{\pi\in\mathscr{A}} J^0(t,w;\pi)$.

Back to UVL Insurance Pricing

Recall the "separation of variable": $U(t,w) = V(t,w)\Phi(t,w)$, where

$$V(t,w) = -\frac{1}{\alpha}\exp\Big(-\alpha w e^{r(T-t)} - \frac{(\mu-r)^2}{2\sigma^2}(T-t)\Big).$$

Question

What is Φ?

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Theorem (M.-Yun '10)

- $\Phi(t, w) = \mathbb{E}_{t,w}\{e^{\alpha K_T}\}$.

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Back to UVL Insurance Pricing

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- The indifference (selling) price is

$$p_t^* = \frac{1}{\alpha}e^{-r(T-t)}\log\Phi(t,w) = \frac{1}{\alpha}e^{-r(T-t)}\log\mathbb{E}_{t,w}[e^{\alpha K_T}].$$

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