Spring School on “Stochastic Control in Finance”
Roscoff, France, March 7-17, 2010
1. Equity Linked Insurance Pricing
2. The Indifference Pricing Problem
3. The UVL Insurance Problem
4. General Life Insurance Models
5. The Case of Bereaved Partner
6. Counter-Party Risk Models
7. UVL Insurance Problem Once More
An Equity-Linked Life insurance is one that
- allows a separate account with cash/investment options
- links the death benefits to the cash/investment performance

Examples of such insurance include
- "ELEPAVG" (Equity-Linked Endowment Policy with Asset Value Guarantee)
- "UVL" (Universal Variable Life)
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Equity Linked Life Insurance

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Literature:
- Brennan-Schwartz (’76), Boyle-Schwartz (’77), Delbaen (’86), Aase-Persson (’94), Nielson-Sandmann (1995), Kurz (’96), ...
- Also, Young (with Bayraktar, Jaimungal, Ludkovski, Zariphopoulou, ...), Schweizer, Frittelli, Rouge-El Karoui, ...
Basic elements involved in an UVL insurance

A Life Model

- Single life
- Multiple life
Basic elements involved in an UVL insurance

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A Market Model
- Tradable assets vs. Non-tradable assets, ...
Basic elements involved in an UVL insurance

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A Market Model
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Benefit Specifications
- Guaranteed benefit/return
- “Multiple decrements” (including death, retirement, long term disability, ...)
- ...
Basic Elements

- $T(x)$ — Future Life-time r.v., where $x$ is the current age
- $G_x(t) \triangleq P\{T(x) > t\} \triangleq t p_x$, $t \geq 0$ — survival function
- $h q_{x+t} \triangleq P\{T(x) \leq t + h | T(x) > t\} = 1 - h p_{x+t}.$
- $\lambda_x(t) = \lim_{h \to 0} \frac{h q_{x+t}}{h} = -\frac{f_x(t)}{G_x(t)}$ — force of mortality
The Single Life Case

### Basic Elements

- **$T(x)$** — Future Life-time r.v., where $x$ is the current age
- **$G_x(t)$** $\triangleq P\{T(x) > t\} \triangleq t p_x$, $t \geq 0$ — survival function
- **$hq_{x+t}$** $\triangleq P\{T(x) \leq t + h | T(x) > t\} = 1 - hp_{x+t}$.
- **$\lambda_x(t) = \lim_{h \to 0} \frac{hq_{x+t}}{h} = -\frac{f_x(t)}{G_x(t)}$** — force of mortality
- **$X_t \in \{0, 1, \ldots, m\}$** — State Process (finite state Markov, representing “multiple decreements”, e.g. short/long term disabilities, withdrawal, retirement, death, etc. $X_0 = 0$, and the state “1” is cemetery/absorbing, representing “death”.)
- **$dS_t^0 = r_t S_t^0; \quad S_0^0 = s^0$** — money market
- **$dS_t = S_t \{\mu_t dt + \sigma_t dB_t\}, \quad S_0 = s$, **— tradable
- **$dZ_t = Z_t^0 \{\mu_t^Z dt + \sigma_t^Z dB_t + \sigma_t d\tilde{B}_t\}, \quad Z_0 = z$** — non-tradable
The original form of "Principle of Equivalent Utility" states that the premium $\Pi$ of a claim $X$ should be determined by the equation

$$u(x) = E[u(x + \Pi - X)],$$

where $u$ is a utility function, and $x$ is the initial wealth.
Principle of Equivalent Utility

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$$u(x) = E[u(x + \Pi - \mathcal{X})],$$

where $u$ is a utility function, and $x$ is the initial wealth.

- If $x = 0$, then it is called Zero Utility Principle.
- If furthermore $u(x) = x$, then is often referred to as “Equivalence Principle”.
- Dynamically, assume that $X_t = x + \int_0^t c_s ds - S_t$, $t \geq s \geq 0$, and $\mathcal{X} = S_T$, then at any time $t \in [0, T]$ the premium $c_t$ can be determined by solving the equation

$$u(x) = E\{u(X_T)|X_t = x\}. $$
Principle of Equivalent Utility

If we use the risk reserve with investment, that is, the dynamic of the risk reserve $X$ follows the following SDE:

$$X_t = x + \int_0^t \left[ r_s X_s + c_s (1 + \rho_s) \right] ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle - S_t, \quad (1)$$

then we can require that the premium is determined so that the expected utility maximized. In other words, one solves

$$u(x) = \sup_{\pi \in \mathcal{A}} E \{ u(X_T^\pi) | X_t = x \} ,$$
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(\textbf{Note:} This is almost like an optimal control problem for maximizing the expected terminal utility by Merton (1969, 1971). But determining the premium process is rather difficult.)
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then we can require that the premium is determined so that the expected utility maximized. In other words, one solves

$$u(x) = \sup_{\pi \in \mathcal{A}} E \{ u(X^\pi_T) | X_t = x \},$$

(Note: This is almost like an optimal control problem for maximizing the expected terminal utility by Merton (1969, 1971). But determining the premium process is rather difficult.)

A more practical version of the “premium” is that it is paid as a lump-sum at the time of the contract. Although it is still priced “dynamically”, it is paid only once at the initial time $t$. 

(Jin Ma (USC) Finance, Insurance, and Mathematics Roscoff 3/2010 7/65)
A Stochastic Control Point of View

Assume we are in a “risk neutral world”. Rewrite (1) as

\[ X_t^\pi = X_0 + p + \int_0^t r_s X_s^\pi \, ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle - Y_t = W_t - Y_t, \]

where

- \( p \) is the (lump-sum) premium paid at \( t = 0 \),
- \( W_t^\pi \triangleq X_0 + p + \int_0^t r_s X_s \, ds + \int_0^t \langle \pi_s, \sigma_s dB_s \rangle \),
- \( Y \) is a general “Loss process” (e.g., \( Y_t = S_t \))

**Note**

If the insurer does not sell the insurance, then \( Y = 0 \), and therefore \( p = 0 \). The utility maximization problem becomes a usual stochastic control problem, and we denote its value function by

\[ V^0(x, t) \triangleq \sup_{\pi \in \mathcal{A}} E \{ u(W_T^\pi) | W_t = x \}. \quad (2) \]
The Indifference Pricing Problem

If the insurance is sold, and the liability cannot be traded after its transfer and before the expiration. Then the value function of the insurer should be

\[ U(t, x + p, y) = \sup_{\pi \in \mathcal{A}} E \{ u(W_T - Y_T) \mid W_t = x + p, Y_t = y \}. \] (3)

**Definition**

Let \( y \triangleq Y_t \). A premium \( p \geq 0 \) is said to be "\( y \)-acceptable" if

\[ V^0(t, x) \leq U(t, x + p, y), \quad \forall (t, x). \] (4)

Denote \( \mathcal{P}_y = \{ \text{all } y \text{-acceptable premium} \} \). Define the universal write price, \( p^*(t, y) \) by

\[ p^*(t, y) \triangleq \inf \{ p \geq 0 : V^0(t, x) \leq U(t, x + p, y), \forall (t, x) \} = \inf \mathcal{P}_y. \]
Existence of the Fair Price

**Theorem**

*Suppose that \( \mathcal{P}_{s, z} \neq \emptyset \), and let \( p^* \triangleq \inf \mathcal{P}_y \). Then it holds that*

\[
V^0(t, x) = U(t, x + p^*, y), \forall (t, x).
\]

**Sketch of the proof**

- By Comparison Theorem, \( W_0 \geq \tilde{W}_0 \implies W_\pi^T \geq \tilde{W}_\pi^T \implies U(t, x + p, y) \) is increasing in \( p \).
- Since \( Y_\pi \geq 0 \implies u(W_\pi^T - Y_\pi) \leq u(W_\pi^T) \implies U(t, x, y) \leq V^0(t, x) \leq U(t, x + p^*, y) \).
- If \( U(t, \cdot, y) \) is continuous, then \( \exists p^{**} \in [0, p^*] \) s.t.

\[
V^0(t, x) = U(t, x + p^{**}, y)
\]

- But \( p^{**} \in \mathcal{P}_{s, z} \implies p^* \leq p^{**} \implies p^* = p^{**} \). 

\[\blacksquare\]
First introduced by Hodges and Neuberger (1989), as a pricing principle for contingent claims in an incomplete market.

The value is within the interval of arbitrage prices

\[ \left[ \inf_Q E_Q \{ X e^{-rT} \}, \sup_Q E_Q \{ X e^{-rT} \} \right], \]

where \( Q \) runs over the set of all EMMs.
Indifference Pricing in Finance/Insurance

- First introduced by Hodges and Neuberger (1989), as a pricing principle for contingent claims in an incomplete market.
- The value is within the interval of arbitrage prices

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where \( Q \) runs over the set of all EMMs.

Existing works for similar problems

- Cvitanić et al. (’01), Delbaen et al. (’02)… (martingale, duality)
- Rouge & El Karoui (’00) (BSDEs)
- M. Davis (’00), M. Musiela & Zariphopoulou (’02); Young and Zariphopoulou (’02) (PDE solutions, power/exponential utility)
- Bielecki, Jeanblanc and Rutkowski (’05) (defaultable claims)
The *Universal Variable Life* (UVL for short) is an insurance product that offers

- a separate cash account besides a death benefit
- various investment options
- different risk/return relationships (may include money market, bond, common stocks, or even non-tradable equities.)

**Main Features**

- The changes in the policy’s cash values and death benefits will be related directly to the investment performance of its underlying assets.
- The death benefit will not fall below a minimum amount (usually the initial face amount) even if the invested assets depreciate in value by a substantial amount. Although there is no similar “floor” to protect the cash values.
Consider a term life insurance with expiration date $T > 0$ and death benefit

$$b_t = g(S^1_t, \cdots, S^d_t, Z_t) = g(S_t, Z_t),$$

(5)

where $g : \mathbb{R}^{d+1} \rightarrow (0, \infty)$ is some measurable function.
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(5)
where $g: \mathbb{R}^{d+1} \mapsto (0, \infty)$ is some measurable function.

**Example**

- $g(S_t, Z_t) = S_t^i \vee s^i$, for some $i$,
- $g(S_t, Z_t) = Z_t \vee z$.
- If $Z$ is the retirement fund, one can set $g(Z_t) = Z_t \vee e^{\bar{r}t}z$, $t \geq 0$, where $\bar{r}$ is a certain growth rate (such as the interest rate or any contractually pre-determined rate.
Consider a term life insurance with expiration date \( T > 0 \) and death benefit

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**Note:**

In this case the loss process is \( Y_t = g(S_T, Z_T) \mathbb{1}_{\{T(x) \leq t\}}, \ t \geq 0 \).
We denote

\[ \mathcal{A} = \{ \pi : E \int_0^T |\pi_t|^2 dt < \infty \} \]

\[ E_{t,w,s,z} \{ \cdot \} = E \{ \cdot | W_t = w, S_t = s, Z_t = z \}. \]
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\[ J(t, w, s, z; \pi) \triangleq E_{t,w,s,z}\{ u(W_T^\pi - Y_T) \}, \]

\[ J^0(t, w; \pi) \triangleq E_{t,w}\{ u(W_T^\pi) \}. \quad (T(x) > T, \implies Y_T = 0.) \]

\[ \hat{J}(t, w, s; \pi) \triangleq E_{t,w,s}\{ u(W_T^\pi - g(S_T)Y_T) \}. \quad (g = g(S_T)) \]
Some Optimization Problems

We denote

$$\mathcal{A} = \{ \pi : E \int_0^T |\pi_t|^2 dt < \infty \}$$

$$E_{t,w,s,z}\{ \cdot \} = E\{ \cdot | W_t = w, S_t = s, Z_t = z \}.$$ 

$$J(t, w, s, z; \pi) \triangleq E_{t,w,s,z}\{ u(W^\pi_T - Y_T) \},$$

$$J^0(t, w; \pi) \triangleq E_{t,w}\{ u(W^\pi_T) \}. \quad (T(x) > T, \implies Y_T = 0.)$$

$$\hat{J}(t, w, s; \pi) \triangleq E_{t,w,s}\{ u(W^\pi_T - g(S_T)Y_T) \}. \quad (g = g(S_T))$$

The Value Functions

$$V^0(t, w) = \sup_{\pi \in \mathcal{A}} J^0(t, w; \pi)$$

$$V(t, w, s) = \sup_{\pi \in \mathcal{A}} \hat{J}(t, w, s; \pi)$$

$$U(t, w, s, z) = \sup_{\pi \in \mathcal{A}} J(t, w, s, z; \pi).$$
Solution for \( g = g(S_T) \)

- First recall the Bellman Principle: for any \( h > 0 \),
  \[
  V(t, w, s) = \sup_{\pi \in \mathcal{A}} E_{t,w,s}\{ V(t + h, W_{t+h}^\pi, S_{t+h}) \}. \tag{6}
  \]

- Since \( g(S_T) \) involves all tradeable assets, and the benefit is paid at a fixed terminal time \( T \), one can consider \( g(S_T) \) as a contingent claim, and determine its present value by
  \[
  c(t, s) = E^Q\{ e^{-r(T-t)} g(S_T) | S_t = s \}.
  \]

- If the death occurs during \([t, t+h]\), then one can set aside the amount of \( c(t+h, S_{t+h}) \) at time \( t+h \) to hedge the potential claim lost \( g(S_T) \), and consider the remaining optimization problem on \([t+h, T]\) as if there were no insurance involved. Thus,
  \[
  E_{t,w,s}\{ V(t + h, W_{t+h}^\pi, S_{t+h}) \} = E_{t,w,s}\{ V^0(t + h, W_{t+h}^\pi - c(t+h, S_{t+h})) \}. \]
Solution for $g = g(S_T)$

Now for any $\pi$ on $[t, t+h]$,

$$V(t, w, s) \geq E_{t,w,s}\{V(t+h, W_{t+h}^\pi, S_{t+h})\} h p_{x+t} + E_{t,w,s}\{V^0(t+h, W_{t+h}^\pi - c(t+h, S_{t+h}))\} h q_{x+t}.$$ 

Assume that $c(\cdot, \cdot) \in C^{1,2}$ and satisfies the Black-Scholes PDE, we can apply Itô to both $V(W_t, t, S_t)$ and $V^0(W_t - c(t, S_t), t)$ from $t$ to $t+h$, and then take conditional expectations and rearrange terms to obtain

$$V(w, t, s) \frac{hq_{x+t}}{h} \geq V^0(w - c(t, s), t) \frac{hq_{x+t}}{h}$$

$$+ E \left\{ \frac{1}{h} \int_t^{t+h} \{V_t + \mathcal{L}[V](u, W_u, S_u) | W_t = w\} \right\} h p_{x+t}$$

$$+ E \left\{ \frac{1}{h} \int_t^{t+h} \{V^0_t + \mathcal{L}[V^0](r, W_u, S_u) | W_t = w\} \right\} h q_{x+t}. $$
Solution for $g = g(S_T)$

- Letting $h \to 0$, noting that
  \[
  \lim_{h \to 0} h q_{x+t}/h = \lambda_x(t), \quad \lim_{h \to 0} h p_{x+t} = 1, \quad \lim_{h \to 0} h q_{x+t} = 0,
  \]
  and using the fact that $c$ satisfies the Black-Scholes PDE, we obtain the HJB Equation for $V$:

  \[
  \begin{cases}
  0 = V_t + \max_{\pi} \{ (\mu - r) \pi V_w + \frac{1}{2} \sigma^2 \pi^2 V_{ww} + s \sigma^2 \pi V_{ws} \} + r w V_w \\
  + s \mu V_s + \frac{1}{2} \sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(w - c, t) - V(w, t, s)),
  \end{cases}
  \]

  \[
  V(T, w, s) = u(w).
  \]
Solution for $g = g(S_T)$

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  \]

  and using the fact that $c$ satisfies the Black-Scholes PDE, we obtain the HJB Equation for $V$:

  \[
  0 = V_t + \max_{\pi} \{ (\mu - r)\pi \nu_w + \frac{1}{2} \sigma^2 \pi^2 V_{ww} + s \sigma^2 \pi \nu_{ws} \} + r \nu_w V_w \\
  + s \mu V_s + \frac{1}{2} \sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(w - c, t) - V(w, t, s)),
  \]

  \[
  V(T, w, s) = u(w).
  \]

  **Note:** In the Black-Scholes world, the HJB equation for $V^0$ is

  \[
  \left\{ \begin{array}{l}
  V_t^0 + \max_{\pi \in \mathbb{R}_+} \left\{ \frac{1}{2} |\sigma \pi|^2 V_{ww}^0 + \langle \pi, \mu - r \rangle V_w^0 \right\} + r w V_w^0 = 0, \\
  V^0(T, w) = u(w).
  \end{array} \right.
  \]
Consider now the case of exponential utility. I.e., \( u(w) = -\frac{1}{\alpha} e^{-\alpha w} \).

- \( V^0 \) has the close form solution:
  \[
  V^0(t, w) = -\frac{1}{\alpha} \exp\{-\alpha we^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\} \quad (8)
  \]
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  \]  
  (8)

- Assume \( V(t, w, s) = V^0(t, w)\Phi(t, s) \), then
  \[
  \Phi_t + rS\Phi_s + \frac{\sigma^2 s^2 \Phi_{ss}}{2} - \frac{s^2 \sigma^2 \Phi_s^2}{2\Phi} + \lambda_x (e^{\{\alpha e^{r(T-t)}\}} - \Phi) = 0
  \]
  \[
  \Phi(T, s) = 1.
  \]
Consider now the case of exponential utility. I.e., $u(w) = \frac{-1}{\alpha} e^{-\alpha w}$.

- $V^0$ has the close form solution:
  \[
  V^0(t, w) = -\frac{1}{\alpha} \exp\left\{ -\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2} (T - t) \right\}
  \]  
  \(8\)

- Assume $V(t, w, s) = V^0(t, w) \Phi(t, s)$, then
  \[\Phi_t + rS \Phi_s + \frac{\sigma^2 s^2 \Phi_{ss}}{2} - \frac{s^2 \sigma^2 \Phi^2_s}{2\Phi} + \lambda_x(e^{c\alpha e^{r(T-t)}} - \Phi) = 0 \]
  \[\Phi(T, s) = 1.\]

- Define $h(t, s) = c(t, s) \alpha e^{r(T-t)} - \ln \Phi$. Then one shows that
  \[
  \begin{cases}
  h_t + sh_s + \frac{1}{2} \sigma^2 s^2 h_{ss} - \lambda_x(t)(e^h - 1) = 0 \\
  h(T, s) = \alpha g(s)
  \end{cases}
  \]  
  \(9\)
The Case of Exponential Utility

If we change the variable: \( \nu = \log s, \tau = T - t, \) (9) becomes:

\[
\begin{cases}
    h_\tau = (r - \frac{1}{2} \sigma^2) h_\nu + \frac{1}{2} \sigma^2 h_{\nu\nu} - \lambda_x (T - \tau) (e^h - 1) \\
    h(0, \nu) = \alpha g(e^\nu)
\end{cases}
\]  \hspace{1cm} (10)

**Note:** The reaction-diffusion PDE (10) has an exponential growth, and we must show that it does not blow-up in finite time!
The Case of Exponential Utility

- If we change the variable: \( v = \log s, \tau = T - t \), (9) becomes:

\[
\begin{cases}
  h_\tau = (r - \frac{1}{2}\sigma^2)h_v + \frac{1}{2}\sigma^2 h_{vv} - \lambda_x (T - \tau)(e^h - 1) \\
  h(0, v) = \alpha g(e^v)
\end{cases}
\] (10)

**Note:** The reaction-diffusion PDE (10) has an exponential growth, and we must show that it does not blow-up in finite time!

- Now consider the Initial-Boundary value version of (10) with

\[
h(0, x) = \alpha g(x), \quad h(t, \pm N) = \alpha g(\pm N).
\]

and denote its solution by \( h^N(t, x) \).

- Define \( \tilde{K} = |\alpha|\|g\|_\infty \), and let

\[
K \triangleq - \log(1 - (1 - e^{-\tilde{K}})e^{\int_0^T \lambda(u)du}).
\]
The Case of Exponential Utility

Consider the function

\[ \beta_K(t) \triangleq - \log \{ 1 - (1 - e^{-K}) e^{- \int_0^t \lambda(u) du} \}, \quad t \geq 0. \]

Since \( \beta_K(t) \) is decreasing in \( t \), we have

\[ \tilde{K} = \beta_K(T) \leq \beta_K(t) \leq \beta_K(0) = K, \quad \forall t \in [0, T]. \]

It can be easily checked that \( h(t, x) \triangleq \beta_K(t) \), solves (10) with the *Initial-Boundary* value:

\[ h(0, x) = K, \quad h(t, \pm N) = \beta_K(t). \quad (11) \]

Thus by Comparison Theorem of PDE \( h^N(\cdot, \cdot) \) is bounded by \( \beta_{\tilde{K}}(\cdot) \).
Similarly, denote $v^N(\tau, x) = \partial_x h^N(\tau, x)$, and apply the Comparison Theorem to $v^N$ one sees that $v^N(\cdot, \cdot)$ is bounded by the function $\tilde{v}(t, x) = K' e^{\int_t^T \lambda(t) dt}$, with $K' = |\alpha||g'||\infty$.

We can now apply the Arzela-Ascoli Theorem to obtain a uniformly bounded solution of the Cauchy problem by letting $N \to \infty$!

The indifference price of the UVL insurance is given by

$$p = c(0, s) - \frac{h(0, s)}{\alpha} e^{-rT},$$
The General Case: \[ g = g(S_T, Z_T) \]

Note:
Since \( Z \) is non-tradable, this is an “incomplete market” case and the arbitrage free price for the payoff \( g(S_T, Z_T) \) cannot be determined as in the previous case.
Note:
Since $Z$ is non-tradable, this is an “incomplete market” case and the arbitrage free price for the payoff $g(S_T, Z_T)$ cannot be determined as in the previous case.

A Dynamic Strategy
We consider the following more aggressive (or adventurous) strategy:

- Assuming that the death of the insured occurs before $t + h$
- Instead of putting aside a certain amount of money at the $t + h$ to hedge the future claim, the insurer simply continue to invest all of his current wealth freely, but knowing that he is liable to pay $g(S_T, Z_T)$ at time $T$. 
Consider an auxiliary control problem assuming death happens before $T$

$$
\tilde{J}(t, x, s, z; \pi) \triangleq E_{t,x,s,z}\{u(X_t^{\pi}) - g(S_T, Z_T)\},
$$

with the corresponding value function $\tilde{U}(t, x, s, z)$.

Then $U$ satisfies a HJB equation: (assuming $\mu = r$)

$$
0 = U_t + \max_{\pi} \left\{ \frac{1}{2} \sigma^2 \pi^2 U_{ww} + (U_{ws} S \sigma^2 + U_{wz} Z \sigma^Z \sigma) \pi \right\}
$$

$$
+ r w U_w + U_s S \mu + U_z Z \mu Z + \frac{1}{2} \sigma^2 U_{ss} S^2
$$

$$
+ \frac{1}{2} U_{zz} Z^2 (\tilde{\sigma}^2 + \sigma^Z \sigma^2) + U_{sz} S Z \sigma \sigma Z + \lambda_x(t)(\tilde{U} - U),
$$

$$
U(w, T, s, z) = u(w),
$$

where $\tilde{U}$ satisfies a similar HJB equation with $\lambda_x \equiv 0$. 

The General Case: $g = g(S_T, Z_T)$
The General Case: $g = g(S_T, Z_T)$

Using the similar techniques as before, modulo the technicalities of showing the no blow-ups, we can derive the indifference price in this case:

- The premium $p(t, s, z) = \frac{1}{\alpha} e^{-r(T-t)} h(T - t, \log s, \log z)$,
- $h$ is a bounded, classical solution to the PDE

\[
\begin{align*}
    h_T - \frac{1}{2}\tilde{\sigma}^2 h_{y_2}^2 - \frac{1}{2}\sigma^2 h_{y_1 y_1} - \frac{1}{2}(\tilde{\sigma}^2 + \sigma z^2) h_{y_2 y_2} - \sigma \sigma^z h_{y_1 y_2} \\
    - \left( r - \frac{1}{2}\sigma^2 \right) h_{y_1} - \left( \mu^z - \frac{\mu - r}{\sigma} \sigma^z - \frac{\tilde{\sigma}^2 + \sigma^z z^2}{2} \right) h_{y_2} \\
    - \lambda_x (T - \tau)(e^{\tilde{h} - h} - 1) = 0; \\
    h(0, y_1, y_2) = 0,
\end{align*}
\]

and $\tilde{h}$ is a bounded, classical solution to a similar PDE as above, with $\lambda_x \equiv 0$, and $\tilde{h}(0, y_1, y_2) = \alpha g(e^{y_1}, e^{y_2})$. 
### Main Features

- Allowing “multiple decrement”: such as short/long term disabilities, withdrawal, retirement, death, etc.
- Benefit payable at a random time, e.g., “moment of death”.
- The payments may depend on the different status as well as the transitions between them.
### Multiple-decrement Case

#### Main Features
- Allowing “multiple decrement”: such as short/long term disabilities, withdrawal, retirement, death, etc.
- Benefit payable at a random time, e.g., “moment of death”.
- The payments may depend on the different status as well as the transitions between them.

#### The State/Status Process \( \{X_t\}_{t \geq 0} \)
- A Markov chain with finite state space \( \{0, 1, \ldots, m\} \), representing the numerical code of the “status”.
- \( i = 1 \) to be the “cemetary state” (death), and \( X_0 = 0 \)
- Denote \( I_t^i = 1_{\{X_t=i\}} \) to be the “status indicator” and define the counting process
  \[
  N_t^{ij} \triangleq \#\{\text{transitions of } X \text{ from state } i \text{ to } j \text{ during } [0, t]\}.  
  \]
Multiple-decrement Case

Some Important Quantities

- for each \( t \), denote \( \tau_t = \inf\{s \geq t : X_s \neq X_t\} \); and for \( i = 0, \ldots, m \), define \( \tau_t^i = \tau_t \), if \( X_{\tau_t} = i \) and \( \infty \) otherwise.

- \( t\bar{p}_i^t \triangleq P\{\tau_s > t \mid X_s = i\} \);

- \( t\bar{q}_{ij}^s \triangleq P\{\tau_s = \tau_j \leq t \mid X_s = i\}, \ s \leq t, \ i, j \in \{0, \ldots, m\} \).

- Clearly, \( t\bar{p}_i^1 = 1; \ t\bar{q}_{i,j}^1 = 0, \) for all \( j \neq 1; \) and

\[
\sum_{j \neq i} t\bar{p}_i^j + \sum_{j \neq i} t\bar{q}_{ij}^s = 1, \quad \forall i = 0, 1, \cdots, m, \quad 0 \leq s < t. \quad (12)
\]

- “force of decrement of status \( i \) due to cause \( j \)” as

\[
\bar{\lambda}_{ij}^t \triangleq \lim_{h \to 0} \frac{t + h\bar{q}_{ij}^t}{h}, \quad i, j = 0, 1, \cdots m. \quad (13)
\]
Some Remarks

- If \( m = 1 \), then the state process \( X \) becomes the one as in the simple life model, and \( \tau_0^1 = T(x) \). In that case we should have

\[
t\bar{p}_s^0 = t - s \rho_{x+s}, \quad tq_{s}^{01} = t - s q_{x+s}.
\]

- Being a Markov chain, the process \( X \) has its transition probability and the corresponding transition intensity

\[
tq_{s}^{ij} = P\{X_t = j | X_s = i\}; \quad \lambda_{t}^{ij} \triangleq \lim_{h \downarrow 0} \frac{t + hq_{t}^{ij}}{h}, \quad i \neq j.
\]

There are natural links between \( p_{ij} \)'s and \( \bar{p}_{ij} \)'s. For example:

- \( \bar{\lambda}_t^{ij} = \lambda_t^{ij} \), for all \( t \geq 0, \ i, j = 0, 1, \ldots, m; \)

- \( t + h \bar{p}_t^i = \exp\{- \int_t^{t+h} \sum_{j \neq i} \lambda_s^{ij} ds\}; \quad t + h \bar{p}_t^{ij} = \int_t^{t+h} \bar{p}_t^i \lambda^{ij}_{\tau} d\tau, \)

\( \forall h > 0, \ i, j = 0, \ldots, m. \)

... ... ... ... ...
The Payment Process $A_t$:

- Two types of payments will be considered: “life-annuity” and “life-insurance”.
- Since the non-tradability of the asset $Z$ will not make significant difference in the optimization problem, we will not distinguish $Z$ from $S$.
- The cumulative payment process is defined by

$$A_t = \int_0^t \sum_i l_i^i a_i^i(u, S_u) du + \sum_{i \neq j} a_{ij}^j(u, S_u) dN_{ij}^j, \quad t \geq 0,$$

— an $\mathbf{F}$-adapted, càdlàg, non-decreasing process in which

- $a_i^i(t, s)$ — rate of payments of annuity at state $i$, given $S_t = s$;
- $a_{ij}^j(t, s)$ — rate of payments of insurance when transit from state $i$ to $j$, given $S_t = s$. 

Dynamics of general reserve

\[ d\hat{W}_t^\pi = [r_t \hat{W}_t^\pi + \pi_t(\mu_t - r_t)]dt + \pi_t \sigma_t dB_t - dA_t, \]

where

- \( dA_t = \sum_i l_i(t) a^i(t, S_t) dt + \sum_{i \neq j} a^{ij}(t, S_t) dN_t^{ij} \)
- \( l_t^i = 1_{\{X_t = i\}}, \ N_t^{ij} \triangleq \# \{\text{jumps of } X \text{ from } i \text{ to } j \text{ during } [0, t]\} \)
Dynamics of General Reserve

Dynamics of general reserve

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\[dA_t = \sum_i l_i(t)a^i(t, S_t)dt + \sum_{i\neq j} a^{ij}(t, S_t)dN_t^{ij}\]

\[l_t^i = 1_{\{X_t = i\}}, \quad N_t^{ij} \triangleq \#\{\text{jumps of } X \text{ from } i \text{ to } j \text{ during } [0, t]\}\]

Hamiltonian

\[
\hat{H}^k \triangleq \frac{1}{2}|\sigma_t\pi|^2\psi + \langle \pi, \mu_t - r_t1 \rangle + r_tw - a^k(t, s)\varphi + \langle \pi, \sigma_t\sigma_T trD[s]p \rangle, \quad k = 0, 1, \ldots, m,
\]

\[H^k(t, w, s, \varphi, \psi, p) \triangleq \sup_\pi \hat{H}^k(t, w, s, \varphi, \psi, p; \pi).\]
The HJB Equation

**Theorem (Yu, '07; M.-Yu, '10)**

Under suitable conditions, the value function $U = (U^0, U^1, \ldots, U^m)$ is the \textit{unique} viscosity solution to the system of PDDE’s:

$$
\begin{align*}
    U^k_t + F_k(t, w, s, DU^k, D^2 U^k) + \langle \mathcal{H}_k U \rangle &= 0, \\
    U^k(T, w, s) &= u(w), \quad k = 0, \ldots, m, \\
\end{align*}
$$

where

$$
F_k(\cdots) = \sup_{\pi \in \Pi} \left\{ \pi (\mu_t - r_t) U^k_w + \frac{1}{2} |\sigma_t \pi|^2 U^k_{ww} + \pi \sigma_t^2 s U^k_{ws} \right\}
$$

$$
+ \mu_t s U^k_s + \frac{1}{2} \sigma_t^2 s^2 U^k_{ss} + (r_t w - a^k(t, s)) U^k_w
$$

$$
(\mathcal{H}_k U) = \sum_{j \neq k} \lambda^k_{tj} \left( U^j(t, w - a^{kj}(t, s), s) - U^k(t, w, s) \right).
$$
Viscosity Solution for System of PDDEs

Main Difficulties

- **Definition** of viscosity solution for the system of PDDE.
- **Uniqueness**
  - Different from Ishii et al.’s results: *Parabolic PDDE vs. Elliptic PDEs*
  - Different from Pardoux et al.’s results: *Fully Nonlinear System vs. Semilinear System*
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Main idea:

- Taking the index vector of the value function as an additional “spatial” variable with values in a finite set: the *system* of PDDEs becomes a *single* PDDE!
- The abstract framework of viscosity solutions (e.g., Fleming & Soner book) applies!
Recall Fleming-Soner (II.3)

- $\Sigma$ — a closed subset of a Banach space
- $\mathcal{C}$ — a collection of functions on $\Sigma$
- $\mathcal{T}_{tr}, 0 \leq t \leq r \leq T$ — a family of operators on $\mathcal{C}$, s.t.,
  
  \begin{enumerate}
  \item $\mathcal{T}_{tt}\varphi = \varphi$;
  \item $\mathcal{T}_{tr}\varphi \leq \mathcal{T}_{ts}\psi$, if $\varphi \leq (\mathcal{T}_{rs}\psi)$, $\forall 0 \leq t \leq r \leq s$;
  \item $\mathcal{T}_{tr}\varphi \geq \mathcal{T}_{ts}\psi$, if $\varphi \geq (\mathcal{T}_{rs}\psi)$, $\forall 0 \leq t \leq r \leq s$.
  \end{enumerate}

\textbf{Note}

- $r = s$ in (ii) $\Longrightarrow$ \textit{monotonicity}: $\mathcal{T}_{tr}\varphi \leq \mathcal{T}_{tr}\psi$, if $\varphi \leq \psi$,
- (iiia) $\oplus$ (iib) $\Longrightarrow$ \textit{semigroup property}:

\[ \mathcal{T}_{ts}\varphi = \mathcal{T}_{tr}(\mathcal{T}_{rs}\varphi), \quad t \leq r \leq s \leq T, \quad \text{if } \mathcal{T}_{tr}\varphi \in \mathcal{C}, \forall \varphi \in \mathcal{C}. \]

Of course, the fact that $\mathcal{T}_{tr}\varphi \in \mathcal{C}$ must be verified!
Abstract Bellman (Dynamic Programming) Principle

- $\Sigma \subseteq \overline{\Theta}$, where $\Theta$ is an open set in $\mathbb{R}^n$, and $\mathcal{C} = \mathcal{M}(\Sigma)$,
- $T_{t,r;u}\psi(x) \triangleq J(t, r; u) = E_{t,x} \left\{ \int_t^r L(s, X_s, u_s) ds + \psi(X_r) \right\}$.
- $\mathcal{T}_{t,r}\psi(x) \triangleq \inf_{u \in \mathcal{U}_{ad}} T_{t,r;u}\psi(x)$ (Thus, $T_{t,T}\psi(x) = V(t, x)$).
Abstract Bellman (Dynamic Programming) Principle

- $\Sigma \subseteq \overline{O}$, where $O$ is an open set in $\mathbb{R}^n$, and $C = M(\Sigma)$,
- $T_{t,r;u}\psi(x) \triangleq J(t, r; u) = E_{t,x} \left\{ \int_t^r L(s, X_s, u_s)ds + \psi(X_r) \right\}$.
- $T_{t,r}\psi(x) \triangleq \inf_{u \in \mathcal{U}_{ad}} T_{t,r;u}\psi(x)$ (Thus, $T_{t,T}\psi(x) = V(t, x)$).

**Note**

Semigroup Property = (Abstract) Bellman Principle(!)
Abstract Bellman (Dynamic Programming) Principle

- $\Sigma \subseteq \overline{\mathcal{O}}$, where $\mathcal{O}$ is an open set in $\mathbb{R}^n$, and $\mathcal{C} = \mathcal{M}(\Sigma)$,

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Note

Semigroup Property = (Abstract) Bellman Principle(!)

- Let $\{\mathcal{G}_t\}_{t \geq 0}$ be the "infinitesimal generator" of the semigroup $\mathcal{T}$, that is, for all $\varphi \in \mathcal{D}, y \in \Sigma$,

$$\lim_{h \downarrow 0} \frac{1}{h} \left\{ (\mathcal{T}_{tt+h}\varphi(t+h, \cdot))(y) - \varphi(t, y) \right\} = \left[ \frac{\partial}{\partial t} + \mathcal{G}_t \right] \varphi(t, y),$$

where $\mathcal{D} \subset C([0, T) \times \Sigma)$ is the set of "test functions" [i.e., $\forall \varphi \in \mathcal{D}, \frac{\partial}{\partial t} \varphi(t, y)$ and $(\mathcal{G}_t\varphi(t, \cdot))(y)$ are continuous.]
Assume $V \in C^{1,2} \subset \mathcal{D}$. Then use the semigroup property one derives the HJB equation:

$$
0 = \lim_{h \downarrow 0} \frac{1}{h} \left\{ (\mathcal{T}_{tt+h} V(t+h, \cdot))(y) - V(t, y) \right\}
$$

$$
= \left[ \frac{\partial}{\partial t} + \mathcal{G}_t \right] V(t, y), \quad \forall y \in \Sigma,
$$

$$
V(T, y) = \psi(y).
$$

(16)
Assume \( V \in C^{1,2} \subset \mathcal{D} \). Then use the semigroup property one derives the HJB equation:

\[
\begin{align*}
0 &= \lim_{h \downarrow 0} \frac{1}{h} \{( \mathcal{T}_{t+h} V(t+h, \cdot))(y) - V(t, y) \} \\
&= \left[ \frac{\partial}{\partial t} + G_t \right] V(t, y), \quad \forall y \in \Sigma, \\
V(T, y) &= \psi(y).
\end{align*}
\]

(16)

**Theorem (Fleming-Soner, Theorem II.5.1)**

If the value function of a control problem \( V \in C[0, T] \times \Sigma \), then \( V \) is a **viscosity solution** to the (abstract) HJB equation (16).
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If the value function of a control problem $V \in C[0, T] \times \Sigma)$, then $V$ is a **viscosity solution** to the (abstract) HJB equation (16).

**Question:**

What are $\mathcal{G}$, $\mathcal{D}$, ..., etc. in our case?
Back to UVL Model

- $\Sigma = \{(w, s, k) : w, s \in \mathbb{R}, k \in \{0, 1, ..., m\}\}$,
- $\mathcal{C} = C(\Sigma)$.
- $(\mathcal{T}_{tr}\varphi)(w, s, k) \triangleq \sup_{\pi \in \mathcal{A}} E_{w, s, k}\{\varphi(\hat{W}_r^\pi, S_r, X_r)\}, \quad t \geq r$
- $(\mathcal{T}_{tT}u)(w, s, k) = U^k(t, w, s), \quad \forall (t, w, s)$ and $k$

Note: It is easy to check that the family $\{\mathcal{T}_{tr}\varphi\}$ satisfies (i), (ii).
Since $U^k(t, w, s)$'s are all continuous, the function $(t, w, s, k) \mapsto U^k(t, w, s)$ (on $\Sigma$) should satisfy an abstract HJB equation!
\[ \Sigma = \{(w, s, k) : w, s \in \mathbb{R}, k \in \{0, 1, \ldots, m\}\}, \]
\[ \mathcal{C} = C(\Sigma). \]
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**Note**

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- Since \(U^k(t, w, s)\)'s are all continuous, the function \((t, w, s, k) \mapsto U^k(t, w, s)\) (on \(\Sigma\)) should satisfy an abstract HJB equation!

**Problems:**

- Identify the infinitesimal generator of the semigroup \(\mathcal{T}\).
- Define the “viscosity solutions” to the corresponding abstract HJB equation (vs. the system of the HJB equations!)
Denote $U(t, w, s, k) = U^k(t, w, s)$, and recall the PDDEs (15):

$$
\begin{cases}
\frac{\partial}{\partial t} U^k(t, w, s) + F_k(t, w, s, DU^k, D^2 U^k) + (\mathcal{H}_k U)(t, w, s) = 0, \\
U^k(T, w, s) = u(w), \quad k = 0, \cdots, m.
\end{cases}
$$

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\end{cases} \tag{17}$$

**Theorem**

The viscosity solutions of the abstract HJB equation (16) with respect to the operator $\mathcal{T}$ and that of the system of PDDEs (17) are equivalent if and only if

$$(\mathcal{G}_t \varphi(t, \cdot))(w, s, k) = [F_k(\cdot, \cdots, D\varphi, D^2 \varphi) + (\mathcal{H}_k \varphi)](t, w, s). \tag{18}$$
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$$
\begin{cases}
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(18)
The Case of Bereaved Partner

Main Rationales

- The usual "Multi-Life Contingency" (e.g., pension plans) assumes independent mortality, even for married couples.
- Empirical evidence of the bereaved spouse (Hu-Goldman ('90) Mariikainen-Valkonen ('96), and Valkonen et al. ('04)) indicated the possible correlated mortality.
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\[ T_{x_1}, T_{x_2}, \ldots, T_{x_n} \] — future life time random variables,
\[ T_m = T_{x_1, \ldots, x_n} \overset{\Delta}{=} \min\{ T_{x_1}, \ldots, T_{x_n} \} \] — (Joint-life)
\[ T_M = T_{x_1, \ldots, x_n} \overset{\Delta}{=} \max\{ T_{x_1}, \ldots, T_{x_n} \} \] — (Last-survivor)
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Let $T_{x_1}, T_{x_2}, \ldots, T_{x_n}$ be future lifetime random variables, then

- $T_m = \min\{ T_{x_1}, \ldots, T_{x_n} \}$ — (Joint-life)
- $T_M = \max\{ T_{x_1}, \ldots, T_{x_n} \}$ — (Last-survivor)

If $n = 2$, one has $T_M + T_m = T_{x_1} + T_{x_2}$, $T_M T_m = T_{x_1} T_{x_2}$.

$$F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t), \quad t \geq 0 \text{ where } F_T \text{ is the distribution function of } T.$$

If $T_{x_1} \perp T_{x_2}$, then $F_M(t) = F_{T_{x_1}}(t) F_{T_{x_2}}(t)$. ...
The Case of Bereaved Partner

Assume $n = 2$, and that the individual force of mortalities take the form:

\[
\begin{align*}
\mu_{x_1}(t) &= \lambda_{x_1}(t) + \mathbf{1}_{\{T_{x_2}\leq t\}} \gamma_{x_1}(t - T_{x_2}), \\
\mu_{x_2}(t) &= \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1}\leq t\}} \gamma_{x_2}(t - T_{x_1}),
\end{align*}
\tag{19}
\]

where $\lambda_{x_i}$'s are the (marginal) force of mortality and

\[
\gamma_{x_i}(t) = \frac{n_i}{r_i e^t + 1}, \quad i = 1, 2, \quad r_1, r_2, n_1, n_2 > 0.
\]
The Case of Bereaved Partner

Assume \( n = 2 \), and that the individual force of mortalities take the form:

\[
\begin{align*}
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\mu_{x_2}(t) &= \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1} \leq t\}} \gamma_{x_2}(t - T_{x_1}),
\end{align*}
\]

where \( \lambda_{x_i} \)'s are the (marginal) force of mortality and

\[
\gamma_{x_i}(t) = \frac{n_i}{r_i e^t + 1}, \quad i = 1, 2, \quad r_1, r_2, n_1, n_2 > 0.
\]

\( t \geq 0 \), \hspace{1cm} (19)

Note:

This essentially becomes a problem of “Counter-Party Risk”, a well-know topic in “Contagion Models” of correlated default!

Existing literature include

- King-Wadhwani, Kodres-Pritsker, Collin-Dufresne, …
- Jarrow-Yu, Yu (2001, counterparty, two firms)
- ...........
Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a given filtered probability space.

- \(\mathbb{P}\) is *risk neutral* (in a default free bond market)
- \(\exists\) a factor process \(X = \{X_t : t \geq 0\}\)
- There are \(I\) firms, with default times \(\tau^i, i = 1, \cdots, I\)
Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a given filtered probability space.

- \(\mathbb{P}\) is \textit{risk neutral} (in a default free bond market)
- \(\exists\) a factor process \(X = \{X_t : t \geq 0\}\)
- There are \(I\) firms, with default times \(\tau^i, i = 1, \ldots, I\)

Denote

- \(N^i_t \triangleq 1_{\{\tau^i \leq t\}}\) — default process with respect to \(\tau^i\),
- \(\mathcal{F}_t = \mathcal{F}^X_t \vee \mathcal{F}_t^1 \vee \ldots \vee \mathcal{F}^I_t\), where \(\mathcal{F}^i_t = \sigma\{N^i_s : 0 \leq s \leq t\}\), \(\forall i\)
- \(\mathcal{H}^i_t = \mathcal{F}^X_t \vee \mathcal{F}^1_t \vee \ldots \vee \mathcal{F}^{i-1}_t \vee \mathcal{F}^{i+1}_t \vee \ldots \vee \mathcal{F}^I_t\),

\[\implies \mathcal{F}_t = \mathcal{H}^i_t \vee \mathcal{F}^i_t.\]
Define

- \( S^i_t = \mathbb{P}\{\tau^i > t | \mathcal{H}^i_t\} > 0 \) (\( \Rightarrow \) \( S^i \) is an \( \mathcal{H}^i \)-supermg)
- \( H^i_t \triangleq -\ln(S^i_t), \ t \geq 0 \) — Hazard Process
Define

- $S_t^i = \mathbb{P}\{\tau^i > t | \mathcal{H}_t^i\} > 0 \quad (\implies S^i \text{ is an } \mathcal{H}^i\text{-supermg})$
- $H_t^i \triangleq -\ln(S_t^i), \ t \geq 0$ — Hazard Process

**Note:**

- $S_t^i > 0$ implies that $\tau^i$ cannot be an $\mathcal{H}^i$-stopping time!
Basic Setup

Define

- \( S_t^i = \mathbb{P}\{\tau^i > t | \mathcal{H}_t^i\} > 0 \) (\( \implies S^i \) is an \( \mathcal{H}^i \)-supermg)
- \( H_t^i \triangleq -\ln(S_t^i), t \geq 0 \) — Hazard Process

Note:

- \( S_t^i > 0 \) implies that \( \tau^i \) cannot be an \( \mathcal{H}^i \)-stopping time!
- If \( \exists \lambda_t^i \in \mathcal{H}_t^i \), such that \( H_t^i = \int_0^t \lambda_s^i \, ds, t \geq 0 \), then

\[
S_t^i = \mathbb{P}\{\tau^i > t | \mathcal{H}_t^i\} = \exp\left\{ - \int_0^t \lambda_s^i \, ds \right\}.
\] (20)

\( \lambda^i \) is called the (conditional) intensity process of \( \tau^i \), and it holds that \( \lambda_t^i = -dS_t^i/S_t^i, t \geq 0 \).
Lemma

For any $\mathcal{F}$-measurable random variable $Z$ we have, for any $t \geq 0$,

$$1_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}} Z | \mathcal{H}_t^i\}}{\mathbb{E}\{1_{\{\tau^i > t\}} | \mathcal{H}_t^i\}} \quad (21)$$
A Useful Lemma

Lemma

For any $\mathcal{F}$-measurable random variable $Z$ we have, for any $t \geq 0$,

$$1_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}}Z | \mathcal{H}^i_t\}}{\mathbb{E}\{1_{\{\tau^i > t\}}| \mathcal{H}^i_t\}}$$ (21)

Idea: Define

$$\mathcal{F}_t^* \triangleq \{ A \in \mathcal{F} | \exists B \in \mathcal{H}^i_t, A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\} \}.$$

Then one can check that $\mathcal{F}_t = \mathcal{F}_t^*$, $t \geq 0$. 
Lemma

For any $\mathcal{F}$-measurable random variable $Z$ we have, for any $t \geq 0$,

$$
1\{\tau^i > t\} \mathbb{E}\{Z \mid \mathcal{F}_t\} = 1\{\tau^i > t\} \frac{\mathbb{E}\{1\{\tau^i > t\}Z \mid \mathcal{H}^i_t\}}{\mathbb{E}\{1\{\tau^i > t\} \mid \mathcal{H}^i_t\}} \tag{21}
$$

Idea: Define

$$
\mathcal{F}^*_t \triangleq \{A \in \mathcal{F} \mid \exists B \in \mathcal{H}_t^i, A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\}\}.
$$

Then one can check that $\mathcal{F}_t = \mathcal{F}^*_t$, $t \geq 0$.

Applying “Monotone Class”, one shows that, $\forall Z \in \mathcal{F}$, $\exists X \in \mathcal{H}^i_t$, s.t.

$$
\mathbb{E}\{1\{\tau^i > t\}Z \mid \mathcal{F}_t\} = 1\{\tau^i > t\} \mathbb{E}\{Z \mid \mathcal{F}_t\} = 1\{\tau^i > t\} X.
$$

Taking $\mathbb{E}\{\cdot \mid \mathcal{H}^i_t\}$ on both sides and solve for $X$.  

\[\blacksquare\]
The Conditional Survival Probability

Note that \( \mathbb{P}\{\tau^i > T | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \mathbb{E}\{1_{\{\tau^i > T\}} | \mathcal{F}_t\} \). Applying Lemma we have

\[
\mathbb{P}\{\tau^i > T | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \frac{\mathbb{E}[1_{\{\tau^i > T\}} | \mathcal{H}_t^i]}{\mathbb{E}\{1_{\{\tau^i > t\}} | \mathcal{H}_t^i\}}. 
\] (22)
The Conditional Survival Probability

Note that \( P\{\tau^i > T | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \mathbb{E}\{1_{\{\tau^i > T\}} | \mathcal{F}_t\} \). Applying Lemma we have

\[
P\{\tau^i > T | \mathcal{F}_t\} = 1_{\{\tau^i > t\}} \frac{\mathbb{E}\{1_{\{\tau^i > T\}} | \mathcal{H}_t^i\}}{\mathbb{E}\{1_{\{\tau^i > t\}} | \mathcal{H}_t^i\}}.
\]  

(22)

Since

- \( \mathbb{E}\{1_{\{\tau^i > T\}} | \mathcal{H}_t^i\} = \mathbb{E}\{P\{\tau^i > T | \mathcal{H}_T^i\} | \mathcal{H}_t^i\} = \mathbb{E}\left\{ e^{-\int_0^T \lambda^i_s ds} | \mathcal{H}_t^i \right\} \).

- \( \mathbb{E}\{1_{\{\tau^i > t\}} | \mathcal{H}_t^i\} = e^{-\int_0^t \lambda^i_s ds} \).
Note that $\mathbb{P}\{\tau^i > T | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} | \mathcal{F}_t\}$. Applying Lemma we have

$$\mathbb{P}\{\tau^i > T | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} | \mathcal{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathcal{H}_t^i\}}. \quad (22)$$

Since

- $\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathcal{H}_t^i\} = \mathbb{E}\{\mathbb{P}\{\tau^i > T | \mathcal{H}_T^i\} | \mathcal{H}_t^i\} = \mathbb{E}\left\{e^{-\int_0^T \lambda_s^i ds} \left| \mathcal{H}_t^i\right.\right\}$.
- $\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathcal{H}_t^i\} = e^{-\int_0^t \lambda_s^i ds}$

Consequently:

- $\mathbb{P}\{\tau^i > T | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\left\{e^{-\int_t^T \lambda_s^i ds} \left| \mathcal{H}_t^i\right.\right\}$.  


The Conditional Survival Probability

Note that \( P\{\tau^i > T \mid \mathcal{F}_t \} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} \mid \mathcal{F}_t \} \). ApplyingLemma we have

\[
P\{\tau^i > T \mid \mathcal{F}_t \} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} \mid \mathcal{H}^i_t \}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} \mid \mathcal{H}^i_t \}}. \tag{22}
\]

Since

\[
\mathbb{E}\{\mathbf{1}_{\{\tau^i > T\}} \mid \mathcal{H}^i_t \} = \mathbb{E}\{P\{\tau^i > T \mid \mathcal{H}^i_T \} \mid \mathcal{H}^i_t \} = \\
\mathbb{E}\left\{ e^{-\int_0^T \lambda^i_s ds} \right\}.
\]

\[
\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} \mid \mathcal{H}^i_t \} = e^{-\int_0^t \lambda^i_s ds}
\]

Consequently:

\[
P\{\tau^i > T \mid \mathcal{F}_t \} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\left\{ e^{-\int_t^T \lambda^i_s ds} \right\}.
\]

\[
M^i_t \triangleq N^i_t - H^i_{t \wedge \tau^i} = \mathbf{1}_{\{\tau^i \leq t\}} - \int_0^t \mathbf{1}_{\{\tau^i > s\}} \lambda^i_s ds, \ i = 1, \ldots, l, \text{ are } \{\mathcal{F}_t\}\text{-martingales.}
\]
Standing Assumptions

(\textbf{H1}) $\lambda_t^i$ satisfy the following condition:

$$\mathbb{E}\left\{ \exp \left( 2 \int_0^t \sum_{i=1}^I \lambda_s^i ds \right) \right\} < \infty, \quad \forall t < \infty.$$

(\textbf{H2}) For each $i$, $\mathbb{P}\{\tau^i > 0\} = 1$. Furthermore, there are no simultaneous defaults among the $I$ firms. In other words, it holds that $\mathbb{P}\{\tau^i \neq \tau^j\} = 1$, whenever $i \neq j$. 

\[ \text{Main Task} \] Find effective, tractable way to calculate the joint distribution (survival probability):

$$\mathbb{P}\{\tau^1 \leq t_1, \ldots, \tau^I \leq t_I\}, \quad \text{and/or} \quad \mathbb{P}\{\tau^1 > t_1, \ldots, \tau^I > t_I\},$$

given the conditional intensities.
Standing Assumptions

(H1) $\lambda^i_t$ satisfy the following condition:

$$\mathbb{E}\left\{ \exp \left( 2 \int_0^t \sum_{i=1}^l \lambda^i_s \, ds \right) \right\} < \infty, \quad \forall t < \infty.$$

(H2) For each $i$, $\mathbb{P}\{\tau^i > 0\} = 1$. Furthermore, there are no simultaneous defaults among the $l$ firms. In other words, it holds that $\mathbb{P}\{\tau^i \neq \tau^j\} = 1$, whenever $i \neq j$.

Main Task

Find effective, tractable way to calculate the joint distribution (survival probability):

$$\mathbb{P}\{\tau^1 \leq t_1, \ldots, \tau^l \leq t_l\}, \quad \text{and/or} \quad \mathbb{P}\{\tau^1 > t_1, \ldots, \tau^l > t_l\},$$
given the conditional intensities.
Define, for $i = 1, ..., I$, $\Gamma_t^i \triangleq \exp\{\int_0^t \lambda^i_s ds\}$, and

$$Z_t^i \triangleq 1_{\{\tau^i > t\}} \Gamma_t^i = 1_{\{\tau^i > t\}} \exp \left\{ \int_0^t \lambda^i_s ds \right\}.$$  \hfill (23)
Define, for $i = 1, \ldots, I$, $\Gamma_t^i \triangleq \exp\{\int_0^t \lambda_s^i \, ds\}$, and

$$Z_t^i \triangleq 1_{\{\tau^i > t\}} \Gamma_t^i = 1_{\{\tau^i > t\}} \exp\left\{ \int_0^t \lambda_s^i \, ds \right\}. \quad (23)$$

Then

- $Z_t^i \geq 0$; and $Z_0^i = 1$, $\forall i$.
- $Z^i$'s are $\mathcal{F}_t$-adapted, and $\mathbb{E}\{Z_t^i\} = 1$. 

Define, for $i = 1, \ldots, I$, $\Gamma^i_t \triangleq \exp\{\int_0^t \lambda^i_s ds\}$, and
\[
Z^i_t \triangleq 1_{\{\tau^i > t\}} \Gamma^i_t = 1_{\{\tau^i > t\}} \exp \left\{ \int_0^t \lambda^i_s ds \right\}.
\] (23)

Then
- $Z^i_t \geq 0$; and $Z^i_0 = 1$, $\forall i$.
- $Z^i$'s are $\{\mathcal{F}_t\}$-adapted, and $\mathbb{E}\{Z^i_t\} = 1$.

**Proposition**

Assume (H1) and (H2). Then, for $k = 1, \ldots, l$, the processes
\[
\prod_{i=1}^k Z^i_t \triangleq \prod_{i=1}^k 1_{\{\tau^i > t\}} \Gamma^i_t, \quad t \geq 0
\] (24)
are all $\{\mathcal{F}_t\}$-martingales.
Sketch of the proof. (i) $Z_t$'s are martingales.
[Sketch of the proof.] (i) $Z_t^i$'s are martingales.

\[
\mathbb{E}\{Z_t^i|\mathcal{F}_s\} = \mathbb{E}\{1_{\{\tau^i > t\}}\Gamma_t^i|\mathcal{F}_s\} = 1_{\{\tau^i > s\}}\mathbb{E}\{1_{\{\tau^i > t\}}\Gamma_t^i|\mathcal{F}_s\}
\]

\[
= 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}}\Gamma_t^i|\mathcal{H}_s^i\}}{\mathbb{E}\{1_{\{\tau^i > s\}}|\mathcal{H}_s^i\}}
\]

\[
= 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}}\Gamma_t^i|\mathcal{H}_s^i\}}{\left(\Gamma_t^i\right)^{-1}} = Z_s^i\mathbb{E}\{1_{\{\tau^i > t\}}\Gamma_t^i|\mathcal{H}_s^i\}
\]

\[
= Z_s^i\mathbb{E}\{\mathbb{E}\{1_{\{\tau^i > t\}}|\mathcal{H}_t^i\}\Gamma_t^i|\mathcal{H}_s^i\} = Z_s^i.
\]
[Sketch of the proof.] (i) $Z^i_t$’s are martingales.

$$
\mathbb{E}\{Z^i_t|\mathcal{F}_s\} = \mathbb{E}\{1_{\{\tau^i_t > t\}} \Gamma^i_t|\mathcal{F}_s\} = 1_{\{\tau^i_s > s\}} \mathbb{E}\{1_{\{\tau^i_s > s\}} \Gamma^i_s|\mathcal{F}_s\}
$$

$$
= 1_{\{\tau^i_t > t\}} \frac{\mathbb{E}\{1_{\{\tau^i_t > t\}} \Gamma^i_t|\mathcal{H}^i_t\}}{\mathbb{E}\{1_{\{\tau^i_s > s\}}|\mathcal{H}^i_s\}}
$$

(Lemma)

$$
= 1_{\{\tau^i_t > t\}} \frac{\mathbb{E}\{1_{\{\tau^i_t > t\}} \Gamma^i_s|\mathcal{H}^i_s\}}{(\Gamma^i_t)^{-1}} = Z^i_s \mathbb{E}\{1_{\{\tau^i_s > s\}} \Gamma^i_t|\mathcal{H}^i_s\}
$$

$$
= Z^i_s \mathbb{E}\{\mathbb{E}\{1_{\{\tau^i_t > t\}}|\mathcal{H}^i_t\} \Gamma^i_t|\mathcal{H}^i_s\} = Z^i_s.
$$

(ii) If $\tilde{Z}^k_t = \prod_{i=1}^k Z^i_t$ is an mg, then so is $\prod_{i=1}^{k+1} Z^i_t = \tilde{Z}^k_t Z^{k+1}_t$. 
[Sketch of the proof.] (i) $Z_t^i$’s are martingales.

\[
\mathbb{E}\{Z_t^i | \mathcal{F}_s\} = \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i | \mathcal{F}_s\} = 1_{\{\tau^i > s\}} \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i | \mathcal{F}_s\} = 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i | \mathcal{H}_t^i\}}{\mathbb{E}\{1_{\{\tau^i > s\}} | \mathcal{H}_s^i\}}
\]

(Lemma)

\[
= 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i | \mathcal{H}_s^i\}}{(\Gamma_t^i)^{-1}} = Z_s^i \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i | \mathcal{H}_s^i\}
\]

\[
= Z_s^i \mathbb{E}\{\mathbb{E}\{1_{\{\tau^i > t\}} | \mathcal{H}_t^i\} \Gamma_t^i | \mathcal{H}_s^i\} = Z_s^i.
\]

(ii) If $\tilde{Z}_t^k \triangleq \prod_{i=1}^{k} Z_t^i$ is an mg, then so is $\prod_{i=1}^{k+1} Z_t^i = \tilde{Z}_t^k Z_t^{k+1}$.

\[
\tilde{Z}_t^k Z_t^{k+1} = \int_0^t \tilde{Z}_s^k dZ_s^{k+1} + \int_0^t Z_s^{k+1} d\tilde{Z}_s^k + [\tilde{Z}_t^k, Z_t^{k+1}]_t.
\]
[Sketch of the proof.] (i) \( Z_t^i \)'s are martingales.

\[
\begin{align*}
\mathbb{E}\{Z_t^i \mid \mathcal{F}_s\} &= \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i \mid \mathcal{F}_s\} = 1_{\{\tau^i > s\}} \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i \mid \mathcal{F}_s\} \\
&= 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i \mid \mathcal{H}_t^i\}}{\mathbb{E}\{1_{\{\tau^i > s\}} \mid \mathcal{H}_s^i\}} \\
&= 1_{\{\tau^i > s\}} \frac{\mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i \mid \mathcal{H}_s^i\}}{(\Gamma_t^i)^{-1}} = Z_s^i \mathbb{E}\{1_{\{\tau^i > t\}} \Gamma_t^i \mid \mathcal{H}_s^i\} \\
&= Z_s^i \mathbb{E}\{\mathbb{E}\{1_{\{\tau^i > t\}} \mid \mathcal{H}_t^i\} \Gamma_t^i \mid \mathcal{H}_s^i\} = Z_s^i. 
\end{align*}
\]

(Lemma)

(ii) If \( \tilde{Z}_t^k \triangleq \prod_{i=1}^k Z_t^i \) is an mg, then so is \( \prod_{i=1}^{k+1} Z_t^i = \tilde{Z}_t^k Z_t^{k+1} \).

\[
\tilde{Z}_t^k Z_t^{k+1} = \int_0^t \tilde{Z}_s^k dZ_s^{k+1} + \int_0^t Z_s^{k+1} d\tilde{Z}_s^k + [\tilde{Z}_s^k, Z_s^{k+1}]_t.
\]

Since both \( \tilde{Z}_s^k \) and \( Z_s^{k+1} \) are FV and quadratic pure jump,

\[
[\tilde{Z}_s^k, Z_s^{k+1}]_t = \tilde{Z}_0^k Z_0^{k+1} + \sum_{0 < s \leq t} \Delta \tilde{Z}_s^k \Delta Z_s^{k+1} = \tilde{Z}_0^k Z_0^{k+1}.
\]
Define

\[
\frac{d\mathbb{P}^i}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} \triangleq Z^i_T; \quad \frac{d\mathbb{P}^{1,\ldots,k}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} \triangleq \tilde{Z}^k_T = \prod_{i=1}^k Z^i_T. \quad (25)
\]

and \( \mathbb{E}^{1,\ldots,k}\{X\} \triangleq \mathbb{E}^{\mathbb{P}^{1,\ldots,k}}\{X\} = \mathbb{E}\{Z^1_T Z^2_T \ldots Z^k_T X\} \).
Define
\[
\frac{d\mathbb{P}^i}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} \triangleq Z^i_T; \quad \frac{d\mathbb{P}^1,\ldots,k}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} \triangleq \tilde{Z}^k_T = \prod_{i=1}^k Z^i_T. \quad (25)
\]

and \( \mathbb{E}^{1,\ldots,k}\{X\} \triangleq \mathbb{E}^{\mathbb{P}^1,\ldots,k}\{X\} = \mathbb{E}\{Z^1_T Z^2_T \cdots Z^k_T X\} \).

Then, for each \( k \) and \( A \in \mathcal{F}_t \), it holds that
\[
\mathbb{E}\{1_A \tilde{Z}^k_t \mathbb{E}^{1,\ldots,k}\{X|\mathcal{F}_t\}\} = \mathbb{E}\{1_A \mathbb{E}\{\tilde{Z}^k_T|\mathcal{F}_t\} \mathbb{E}^{1,\ldots,k}\{X|\mathcal{F}_t\}\}
= \mathbb{E}^{1,\ldots,k}\{1_A \mathbb{E}^{1,\ldots,k}\{X|\mathcal{F}_t\}\}
= \mathbb{E}^{1,\ldots,k}\{1_A X\} = \mathbb{E}\{1_A \mathbb{E}\{\tilde{Z}^k_T X|\mathcal{F}_t\}\}. \quad (26)
\]
Representation of Joint Survival Probability

Define
\[
\frac{d\mathbb{P}^i}{d\mathbb{P}}|_{\mathcal{F}_T} \triangleq Z^i_T; \quad \frac{d\mathbb{P}^1,\ldots,k}{d\mathbb{P}}|_{\mathcal{F}_T} \triangleq \tilde{Z}^k_T = \prod_{i=1}^{k} Z^i_T. \tag{25}
\]

and \(\mathbb{E}^1,\ldots,k\{X\} \triangleq \mathbb{E}^{1,\ldots,k}\{X\} = \mathbb{E}\{Z^1_T Z^2_T \ldots Z^k_T X\}\).

Then, for each \(k\) and \(A \in \mathcal{F}_t\), it holds that
\[
\mathbb{E}\{1_A \tilde{Z}^k_t \mathbb{E}^1,\ldots,k\{X|\mathcal{F}_t\}\} = \mathbb{E}\{1_A \mathbb{E}\{\tilde{Z}^k_T|\mathcal{F}_t\} \mathbb{E}^1,\ldots,k\{X|\mathcal{F}_t\}\} = \mathbb{E}^1,\ldots,k\{1_A X\} = \mathbb{E}\{1_A \mathbb{E}\{\tilde{Z}^k_T X|\mathcal{F}_t\}\}.\]

This leads to
\[
\mathbb{E}\{Z^1_T Z^2_T \ldots Z^k_T X|\mathcal{F}_t\} = Z^1_t Z^2_t \ldots Z^k_t \mathbb{E}^1,\ldots,k\{X|\mathcal{F}_t\}, \quad \mathbb{P} - \text{a.s.} \tag{26}
\]
Assume $I = 2$, and $t_1 \leq t_2$. Apply (26) we get

$$
\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2\} = \mathbb{E}\left\{1_{\{\tau^1 > t_1\}} \mathbb{E}\left\{Z_{t_2}^2 (\Gamma_{t_2}^2)^{-1}\right\} \bigg| \mathcal{F}_{t_1}\right\}
$$

$$
= \mathbb{E}\left\{1_{\{\tau^1 > t_1\}} Z_{t_1}^2 \mathbb{E}^{\mathbb{P}^2}\left\{(\Gamma_{t_2}^2)^{-1}\right\} \bigg| \mathcal{F}_{t_1}\right\}
$$

$$
= \mathbb{E}\left\{Z_{t_1}^1 Z_{t_1}^2 \mathbb{E}^{\mathbb{P}^2}\left\{(\Gamma_{t_1}^1)^{-1}(\Gamma_{t_2}^2)^{-1}\right\} \bigg| \mathcal{F}_{t_1}\right\}
$$

$$
= \mathbb{E}^{1,2}\left\{\mathbb{E}^{\mathbb{P}^2}\left\{(\Gamma_{t_1}^1)^{-1}(\Gamma_{t_2}^2)^{-1}\right\} \bigg| \mathcal{F}_{t_1}\right\}\right\}.
$$

In particular, if $t_1 = t_2 = t$, then we have

$$
\mathbb{P}\{\tau^1 > t, \tau^2 > t\} = \mathbb{E}^{1,2}\left\{\exp\left\{- \int_0^t (\lambda_s^1 + \lambda_s^2)ds\right\}\right\}.
$$
Theorem

Assume (H1) and (H2). Then,
(i) For any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_I < \infty$, it holds that

$$
\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, \ldots, \tau^I > t_I\} = \mathbb{E}^{1,\ldots,I}\left\{\ldots\{\mathbb{E}^{\sigma_I} \left\{ \prod_{i=1}^{I} (\Gamma^i_{t_i})^{-1} \right\} \mid \mathcal{F}_{t_{I-1}} \} \ldots \mid \mathcal{F}_{t_1}\right\};
$$

(ii) Denote $\tau^* = \min\{\tau_1, \ldots, \tau_I\}$, then for any $0 \leq t \leq T$

$$
a) \mathbb{P}\{\tau^* > t\} = \mathbb{E}^{1,\ldots,I}\left\{\ldots\{\mathbb{E}^{\sigma_I} \left\{ e^{-\int_0^t \sum_{i=1}^{I} \lambda^i s \, ds} \right\} \mid \mathcal{F}_{t_{I-1}} \} \ldots \mid \mathcal{F}_{t_1}\right\};

b) \mathbb{P}\{\tau^* > T \mid \mathcal{F}_t\} = \prod_{i=1}^{I} \mathbb{P}\{\tau_i > t \mid \mathcal{F}_{t_{I-1}}\} = \mathbb{E}^{1,\ldots,I}\left\{\ldots\{\mathbb{E}^{\sigma_I} \left\{ e^{-\int_t^T \sum_{i=1}^{I} \lambda^i s \, ds} \right\} \mid \mathcal{F}_{t_{I-1}} \} \ldots \mid \mathcal{F}_{t_1}\right\}.
$$
Theorem

Assume (H1) and (H2). Then,

(i) For any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_I < \infty$, it holds that

$$\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, \ldots, \tau^I > t_I\} = \mathbb{E}^1, \ldots, I\left\{ \ldots \left\{ \prod_{i=1}^{I} (\Gamma_{t_i}^i)^{-1} \right\} \bigg| \mathcal{F}_{t_{I-1}} \right\} \ldots \bigg| \mathcal{F}_{t_1} \};$$

(ii) Denote $\tau^* = \min\{\tau^1, \ldots, \tau^I\}$, then for any $0 \leq t \leq T$

a) $\mathbb{P}\{\tau^* > t\} = \mathbb{E}^1, \ldots, I\left\{ e^{-\int_0^t \sum_{i=1}^I \lambda_i^i ds} \right\};$
Theorem

Assume (H1) and (H2). Then,
(i) For any $0 \leq t_1 \leq t_2 \leq ... \leq t_I < \infty$, it holds that

$$\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, ..., \tau^I > t_I\} = \mathbb{E}^{1,\cdots,I}\left\{ \prod_{i=1}^{I} \left( \int_{t_i}^{\infty} \right)^{-1} \right\} \bigg| \mathcal{F}_{t_{I-1}} \right\} \cdots \bigg| \mathcal{F}_{t_1} \right\};$$

(ii) Denote $\tau^* = \min\{\tau^1, \cdots, \tau^I\}$, then for any $0 \leq t \leq T$

a) $\mathbb{P}\{\tau^* > t\} = \mathbb{E}^{1,\cdots,I}\left\{ e^{-\int_0^t \sum_{i=1}^{I} \lambda_i s ds} \right\};$

b) $\mathbb{P}\{\tau^* > T | \mathcal{F}_t\} = \prod_{i=1}^{I} \mathbb{1}_{\{\tau^i > t\}} \mathbb{E}^{1,\cdots,I}\left\{ e^{-\int_{t}^{T} \sum_{i=1}^{I} \lambda_i s ds} \bigg| \mathcal{F}_t \right\}.$
Two firm case:

\[
\begin{aligned}
\lambda^A_t &= a_0(t) + 1_{\{\tau^B \leq t\}} a_1(t - \tau^B), \\
\lambda^B_t &= b_0(t) + 1_{\{\tau^A \leq t\}} b_1(t - \tau^A),
\end{aligned}
\]  

\text{(27)}

where \(a_0, a_1, b_0, \text{ and } b_1\) are deterministic functions.
Two firm case:

\[
\begin{align*}
\lambda_t^A &= a_0(t) + 1_{\{\tau^B \leq t\}} a_1(t - \tau^B), \\
\lambda_t^B &= b_0(t) + 1_{\{\tau^A \leq t\}} b_1(t - \tau^A),
\end{align*}
\]  

(27)

where \(a_0, a_1, b_0, \) and \(b_1\) are deterministic functions.

Jarrow-Yu (2004) — \(a_1, b_1\) constants.
Counter-Party Risk Models

**Two firm case:**

\[
\begin{align*}
\lambda_t^A &= a_0(t) + 1_{\{\tau^B \leq t\}} a_1(t - \tau^B), \\
\lambda_t^B &= b_0(t) + 1_{\{\tau^A \leq t\}} b_1(t - \tau^A),
\end{align*}
\]  

(27)

where \(a_0, a_1, b_0,\) and \(b_1\) are deterministic functions.

Jarrow-Yu (2004) — \(a_1, b_1\) constants.

**(H3)** (i) \(a_0\) and \(b_0\) are positive functions;

(ii) \(a_1\) and \(b_1\) are either positive and decreasing or negative and increasing, such that

\[
\lim_{t \to \infty} a_1(t) = 0 \quad \lim_{t \to \infty} b_1(t) = 0;
\]  

(28)

and such that both \(\lambda_t^A\) and \(\lambda_t^B\) are positive functions.
Counter-Party Risk Models

Proposition

Assume (H1)–(H3). Then the joint survival probability

\[ \mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} \]

is given by

\[
\begin{align*}
\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} &= \begin{cases} 
  c(t_1, t_2) \left( \int_{t_1}^{t_2} a_0(x) e^{-\int_x^{t_2} b_1(s-x)ds - \int_x^{t_1} a_0(s)ds} dx 
  + \int_{t_1}^\infty a_0(x) e^{-\int_t^{t_1} a_0(s)ds} dx \right) & t_1 \leq t_2; \\
  c(t_1, t_2) \left( \int_{t_1}^{t_2} b_0(x) e^{-\int_x^{t_1} a_1(s-x)ds - \int_x^{t_2} b_0(s)ds} dx 
  + \int_{t_1}^\infty b_0(x) e^{-\int_t^{t_2} b_0(s)ds} dx \right) & t_1 > t_2.
\end{cases}
\end{align*}
\]

where \( c(t_1, t_2) = \exp \{ -\int_0^{t_1} a_0(s)ds - \int_0^{t_2} b_0(s)ds \} \).
Counter-Party Risk Models

Main Observation: $\lambda^A_s = a_0(s)$, $\lambda^B_s = b_0(s)$, $\mathbb{P}^{A,B}$-a.s.

$\implies 1 - F^B_{\tau^A}(x) = \mathbb{P}^B(\tau^A > x) = \mathbb{P}^{A,B}(\Gamma^A_x - 1) = e^{-\int_0^x a_0(s) ds}.$
Main Observation: $\lambda^A_s = a_0(s), \lambda^B_s = b_0(s), \mathbb{P}^{A,B}$-a.s.

$$1 - F^B_{\tau^A}(x) = \mathbb{P}^B(\tau^A > x) = \mathbb{P}^{A,B}((\Gamma^A_x)^{-1}) = e^{-\int_0^x a_0(s) \, ds}.$$ 

Applying the change of measure, we have

$$\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} = \mathbb{E}\left[1_{\{\tau^A > t_1\}}1_{\{\tau^B > t_2\}}(\Gamma^B_{t_2}(\Gamma^B_{t_2})^{-1})\right]$$

$$= \mathbb{E}^B\left[1_{\{\tau^A > t_1\}} \exp\left(-\int_0^{t_2} (b_0(s) + 1_{\{\tau^A \leq s\}} b_1(s - \tau^A)) \, ds\right)\right]$$

$$= c(t_2)\left\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x) \, ds} F^B_{\tau^A}(dx) + \int_{t_2}^{\infty} F^B_{\tau^A}(dx)\right\}$$

$$= c(t_2)\left\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x) \, ds} f_{\tau^A}(x) \, dx + \int_{t_2}^{\infty} f_{\tau^A}(x) \, dx\right\}$$

$$= \text{RHS } (t_1 \leq t_2)$$
Multiple Firm Case

Assume that $l > 2$, and that the default intensities are given by

$$\lambda^i_t = a^i_0(t) + \sum_{j=1 \atop j \neq i}^l 1_{\{\tau^j \leq t\}} a^j_{i-1}(t - \tau^j), \quad i = 1, \ldots, l,$$

(29)

where $a^j_i$'s are deterministic functions satisfying (H3).
Assume that $I > 2$, and that the default intensities are given by

$$
\lambda^i_t = a^i_0(t) + \sum_{\substack{j=1 \\ j \neq i}} 1_{\{\tau^j \leq t\}} a^j_{i-1}(t - \tau^j), \quad i = 1, \ldots, I, \quad (29)
$$

where $a^j_i$’s are deterministic functions satisfying (H3).

- For $1 \leq m \leq I$, denote $f_m(t_1, t_2, \cdots, t_m)$ to be the joint density function of the default times $\tau^1, \tau^2, \cdots, \tau^m$.
- For example, $f_1(t_1) = f_{\tau^1}(t_1) = a_{1,0}(t_1)e^{-\int_0^{t_1} a_{1,0}(s)ds}$.

**Proposition**

For $0 = t_0 < t_1 < t_2 < \cdots < t_{m+1}$.

$$
f_{m+1}(t_1, t_2, \cdots, t_{m+1})
= \left\{ \sum_{j=0}^{m} a^{m+1}_j(t_{m+1} - t_j) \right\} e^{-\sum_j \int_{t_j}^{t_{m+1}} a^{m+1}_j(s-t_j)ds} f_m(t_1, \cdots, t_m).
$$
Let $\mathcal{P}(I)$ be all the permutations $p = p(1, \cdots, I)$, then $|\mathcal{P}(I)| = I!$. 
Multiple Firm Case (General)

- Let $\mathcal{P}(I)$ be all the permutations $p = p(1, \cdots, I)$, then $|\mathcal{P}(I)| = I!$.

- $\forall p \in \mathcal{P}(I)$, permute $(t_1, \cdots, t_I)$ to $(t_1^{(p)}, \cdots, t_I^{(p)})$, and

$$\mathcal{D}(p) \triangleq \{(t_1, \cdots, t_I) \in \mathbb{R}_+^I : t_1^{(p)} < \cdots < t_I^{(p)}\}.$$
Multiple Firm Case (General)

- Let \( \mathcal{P}(l) \) be all the permutations \( p = p(1, \cdots, l) \), then \( |\mathcal{P}(l)| = l! \).
- \( \forall p \in \mathcal{P}(l) \), permute \( (t_1, \cdots, t_l) \) to \( (t_1^{(p)}, \cdots, t_l^{(p)}) \), and
  \[
  \mathcal{D}(p) \triangleq \{(t_1, \cdots, t_l) \in \mathbb{R}^l_+ : t_1^{(p)} < \cdots < t_l^{(p)} \}.
  \]
- \( \mathbb{R}^l_+ = \bigcup_{i \in \mathcal{P}(l)} \mathcal{D}(p) ; \mathcal{D}(p) \cap \mathcal{D}(p) = \emptyset \).
Let $\mathcal{P}(I)$ be all the permutations $p = p(1, \cdots, I)$, then $|\mathcal{P}(I)| = I!$.

$\forall p \in \mathcal{P}(I)$, permute $(t_1, \cdots, t_I)$ to $(t_1^{(p)}, \cdots, t_I^{(p)})$, and

$$\mathcal{D}(p) \triangleq \{(t_1, \cdots, t_I) \in \mathbb{R}_+^I : t_1^{(p)} < \cdots < t_I^{(p)}\}.$$ 

$\mathbb{R}_+^I = \bigcup_{i \in \mathcal{P}(I)} \mathcal{D}(p); \mathcal{D}(p) \cap \mathcal{D}(p) = \emptyset$.

$\forall p \in \mathcal{P}(I)$, define $(\tau_1^{(p)}, \cdots, \tau_I^{(p)})$ accordingly, and

$$\lambda_{t_i}^{i,(p)} = a_{0,(p)}^{i}(t) + \sum_{j=1}^{j \neq i} \mathbf{1}_{\{\tau_j^{(p)} \leq t\}} b_{j-1}^{i}(t - \tau_j^{(p)}),$$

where $b_{j,0}(t) = a_{j(p),0}(t)$, $j = 1, \cdots, I$, $j^{(p)}$ is the image position of $j$ after the permutation $p \in \mathcal{P}(I)$, and $b_j^i$ are appropriately defined functions from $a_j^i$’s.
∀ \( p \in \mathcal{P}(l) \) apply the Proposition on the region \( D^{(i)} \), with 
\((\lambda_1, \cdots, \lambda_I)\) being replaced by \((\lambda_1^{(p)}, \cdots, \lambda_I^{(p)})\), to obtain the joint 
density function on \( D^{(p)} \), denoted by \( f^{(p)}_I \). We can then define

\[
g_I(t_1, \cdots, t_I) = f^{(p)}_I(t_1^{(p)}, \cdots, t_I^{(p)}), \quad (t_1, \cdots, t_I) \in D^{(p)}.\]
∀ \( p \in \mathcal{P}(I) \) apply the Proposition on the region \( D^{(i)} \), with \((\lambda_1, \cdots, \lambda_I)\) being replaced by \((\lambda_1^{(p)}, \cdots, \lambda_I^{(p)})\), to obtain the joint density function on \( D^{(p)} \), denoted by \( f^{(p)}_I \). We can then define

\[
g_I(t_1, \cdots, t_I) = f^{(p)}_I(t_1^{(p)}, \cdots, t_I^{(p)}), \quad (t_1, \cdots, t_I) \in D^{(p)}.
\]

**Theorem**

Assume (H1)–(H3). The joint distribution of \( \tau_1, \tau_2, \cdots, \tau_I \) can be expressed as

\[
P\{\tau^1 \leq t_1, \cdots, \tau^I \leq t_I\} = \int_0^{t_1} \cdots \int_0^{t_I} g_I(u_1, \cdots, u_I) \, du_1 \, du_2 \cdots \, du_I.
\]

where \( g_I \)'s are defined above.
Calculated Case:

- Force of mortality take the form:

\[
\begin{align*}
\mu_t^{x_1} &= \lambda_{x_1}(t) + 1_{\{T_{x_2} \leq t\}} \gamma_{x_1}(t - T_{x_2}) \\
\mu_t^{x_2} &= \lambda_{x_2}(t) + 1_{\{T_{x_1} \leq t\}} \gamma_{x_2}(t - T_{x_1}).
\end{align*}
\]  

(30)

- \(\gamma_{x_i}(t) = \frac{n_i}{r_i e^t + 1}\), \(i = 1, 2\), and \(r_1, r_2, n_1, n_2 > 0\).

- Denote

\[
\begin{align*}
\Delta_k^i(t) &= \int_0^t \frac{k}{g_i} e^{-\frac{h_i}{g_i} y} dy \text{ for } i = 1, 2, \\
\tilde{D}_k^i(t) &= D_k^i \left( \frac{\lambda_{x_i}(t)}{h_i} \right), \ i = 1, 2 \\
B^1 &= e^{-k(t_2 + x_1) + \frac{h_1}{g_1} e^{g_1(x_1 + t_1)}} \\
B^2 &= e^{-k(t_1 + x_2) + \frac{h_2}{g_2} e^{g_2(x_2 + t_2)}}.
\end{align*}
\]
Dependent Mortality Models

The joint survival probability can then be calculated as:

\[
\mathbb{P}\{ T_{x_1} > t_1, T_{x_2} > t_2 \} = \begin{cases} 
\frac{c(t_1, t_2)}{(r_2 + 1)^{n_2}} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{h_1}{g_1} r_2^{n_2-k} B^1 \left( \tilde{D}_k^1(t_2) - \tilde{D}_k^1(t_1) \right) 
+ c(t_2, t_2) & t_1 \leq t_2; \\
\frac{c(t_1, t_2)}{(r_1 + 1)^{n_1}} \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{h_2}{g_2} r_1^{n_1-k} B^2 \left( \tilde{D}_k^2(t_1) - \tilde{D}_k^2(t_2) \right) 
+ c(t_1, t_1) & t_1 > t_2,
\end{cases}
\]

where

\[
c(t_1, t_2) = \exp \left\{ - \frac{h_1}{g_1} [e^{g_1(x_1+t_1)} - e^{g_1x_1}] - \frac{h_2}{g_2} [e^{g_2(x_2+t_2)} - e^{g_2x_2}] \right\}.
\]
Joint-life vs. Last-survivor

Let \( T_{x_1}, T_{x_2}, \ldots, T_{x_n} \) be \( n \) future life time random variables, then their and are given by, respectively:

\[
T_m = T_{x_1, \ldots, x_n} \triangleq \min\{ T_{x_1}, T_{x_2}, \ldots, T_{x_n} \},
\]

— (Joint-life = first default)

\[
T_M = T_{x_1, \ldots, x_n} \triangleq \max\{ T_{x_1}, T_{x_2}, \ldots, T_{x_n} \},
\]

— (Last-survivor = last default)
Joint-life vs. Last-survivor

Let $T_{x_1}, T_{x_2}, \cdots, T_{x_n}$ be $n$ future life time random variables, then their and are given by, respectively:

$$T_m = T_{x_1, \cdots, x_n} \triangleq \min\{ T_{x_1}, T_{x_2}, \cdots, T_{x_n} \},$$

— (Joint-life = first default)

$$T_M = T_{\underline{x_1, \cdots, x_n}} \triangleq \max\{ T_{x_1}, T_{x_2}, \cdots, T_{x_n} \},$$

— (Last-survivor = last default)

If $n = 2$, one has

- $T_M + T_m = T_{x_1} + T_{x_2}, \quad T_M T_m = T_{x_1} T_{x_2}$.
- $\{ T_{x_1} \leq t \} \cap \{ T_{x_2} \leq t \} = \{ T_M \leq t \}$,
  $\{ T_{x_1} \leq t \} \cup \{ T_{x_2} \leq t \} = \{ T_m \leq t \}$,
- $F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t), \quad t \geq 0$ where $F_T$ is the distribution function of $T$. 

$\text{Jin Ma (USC) Finance, Insurance, and Mathematics Roscoff 3/2010 57/ 65}$
First Default in Multi-firm Case

Assume for $i = 1, \cdots, I$,

$$
\lambda_i^i = a_{0i}(t) + \sum_{k \neq i} a_{ki}(t) \mathbf{1}_{\{\tau^k \leq t\}} = a_{0i}(t) + \sum_{k \neq i} a_{ki}(t) N^i_s,
$$
First Default in Multi-firm Case

Assume for $i = 1, \cdots, I$,

$$\lambda_t^i = a_0^i(t) + \sum_{k \neq i} a_k^i(t)1_{\{\tau^k \leq t\}} = a_0^i(t) + \sum_{k \neq i} a_k^i(t)N_s^i,$$

Then

$$\mathbb{P}\{\tau_m > t\} = \mathbb{P}\{\tau^1 > t, \tau^2 > t, \cdots, \tau^I > t\} = \mathbb{E}^{1,2,\cdots,I}\left\{ e^{-\int_0^t (\lambda_1^1 + \lambda_2^2 + \cdots + \lambda_s^I) ds} \right\} = \mathbb{E}^{1,2,\cdots,I}\left\{ e^{-\int_0^t [a_0^1(s) + a_0^2(s) + \cdots + a_0^I(s)] ds} \right\}.$$

If all $a_0^i$'s are deterministic, then

$$\mathbb{P}\{\tau_m > t\} = \exp \left\{ -\int_0^t [a_0^1(s) + a_0^2(s) + \cdots + a_0^I(s)] ds \right\}.$$
Similarly one can obtain the conditional survival probability of $\tau_m$:

\[
\mathbb{P}\{\tau_m > T | \mathcal{F}_t\} = \mathbb{P}\{\tau^1 > T, \tau^2 > T, \cdots, \tau^l > T | \mathcal{F}_t\}
\]

\[
= \prod_{i=1}^{l} 1_{\{\tau^i_t > t\}} \mathbb{E}^{1,2,\cdots,l} \left\{ \exp \left\{ - \int_t^T \left[ \sum_{i=1}^{l} \lambda^i_s \right] ds \right\} | \mathcal{F}_t \right\}
\]

\[
= 1_{\{\tau^m > t\}} \mathbb{E}^{1,2,\cdots,l} \left\{ \exp \left\{ - \int_t^T \left[ \sum_{i=1}^{l} a^i_0(s) \right] ds \right\} | \mathcal{F}_t \right\}.
\]

If $a^i_0$’s are all deterministic, then

\[
\mathbb{P}\{\tau_m > T | \mathcal{F}_t\} = 1_{\{\tau^m > t\}} \exp \left\{ - \int_t^T \sum_{i=1}^{l} a^i_0(s) ds \right\}.
\]
The term “flight to quality” refers to the phenomenon that investors move their capital away from riskier investments to the safest possible investment vehicles, e.g., treasury bonds.
The term “flight to quality” refers to the phenomenon that investors move their capital away from riskier investments to the safest possible investment vehicles, e.g., treasury bonds.

One firm model (Collins-Dufresne et al. (03,04)):

\[ r_t = r_0 + J \mathbf{1}_{\{ \tau \leq t \}} \geq 0, \quad t \geq 0, \quad (31) \]
The term “flight to quality” refers to the phenomenon that investors move their capital away from riskier investments to the safest possible investment vehicles, e.g., treasury bonds.

One firm model (Collins-Dufresne et al. (03,04)):

\[ r_t = r_0 + J 1_{\{\tau \leq t\}} \geq 0, \quad t \geq 0, \quad (31) \]

We will consider multi-firm model:

\[ r_t = r_0(X_t) + J 1_{\{\tau_M \leq t\}}, \quad t \geq 0, \quad (32) \]

where \( \tau_M \overset{\triangle}{=} \max\{\tau^1, \cdots, \tau^I\} \) is the last-to-default time, \( X \) is a factor process.

Main purpose: pricing defaultable zero-coupon bonds.
Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.'s. Denote $N^i_t = 1\{T_{x_i} \leq t\}$, $i = 1, 2$, and

$$
\mathcal{F}_t = \mathcal{F}^X_t \lor \mathcal{F}^1_t \lor \mathcal{F}^2_t, \quad t \geq 0,
$$

where $\mathcal{F}^i_t = \sigma\{N^i_s, 0 \leq s \leq t\}$, $t \geq 0$, $i = 1, 2$, and $X$ is a factor process, assumed to be a diffusion process.
Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.’s. Denote $N^i_t = 1\{T_{x_i} \leq t\}$, $i = 1, 2$, and

$$\mathcal{F}_t = \mathcal{F}_t^X \lor \mathcal{F}_t^1 \lor \mathcal{F}_t^2, \quad t \geq 0,$$

where $\mathcal{F}_t^i = \sigma\{N^i_s, 0 \leq s \leq t\}$, $t \geq 0$, $i = 1, 2$, and $X$ is a factor process, assumed to be a diffusion process.

Death benefit is a lump-sum (e.g., $1$) payable at a terminal time $T$, contingent on the survivorship of a married couple.
Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.’s. Denote $N^i_t = 1\{T_{x_i} \leq t\}$, $i = 1, 2$, and

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2, \quad t \geq 0,$$

where $\mathcal{F}^i_t = \sigma\{N^i_s, 0 \leq s \leq t\}$, $t \geq 0$, $i = 1, 2$, and $X$ is a factor process, assumed to be a diffusion process.

Death benefit is a lump-sum (e.g., $1$) payable at a terminal time $T$, contingent on the survivorship of a married couple.

Let $K_t$ be a generic status process, e.g., $K$ could be one of the following:

$$JLI_t = 1\{T_{x_1x_2} \leq t\}, \quad SLI_t = 1\{T_{\overline{x_1x_2}} \leq t\}, \quad t \geq 0.$$
Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.'s. Denote $N_i^t = 1\{T_{x_i} \leq t\}$, $i = 1, 2$, and 

$$
\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2, \quad t \geq 0,
$$

where $\mathcal{F}_t^i = \sigma\{N_s^i, 0 \leq s \leq t\}$, $t \geq 0$, $i = 1, 2$, and $X$ is a factor process, assumed to be a diffusion process.

- Death benefit is a lump-sum (e.g., $1$) payable at a terminal time $T$, contingent on the survivorship of a married couple.
- Let $K_t$ be a generic status process, e.g., $K$ could be one of the following:

$$
JLI_t = 1\{T_{x_1x_2} \leq t\}, \quad SLI_t = 1\{T_{\overline{x_1x_2}} \leq t\}, \quad t \geq 0,
$$
Assume that the individual $T_{x_i}$'s follow the Gompertz's law (1825):

$$\lambda_{x_1}(t) = h_1 e^{g_1(x_1 + t)}, \quad \lambda_{x_2}(t) = h_2 e^{g_2(x_2 + t)}, \quad h_i > 0, \quad g_i > 0.$$ Then

$$\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\} = \begin{cases} 
\frac{c(t_1, t_2)}{(r_2 + 1)^{n_2}} \sum_{k=0}^{n_2} \binom{n_2}{k} h_1^{n_2-k} r_2^{n_2-k} B^1 \left( \tilde{D}_k^1(t_2) - \tilde{D}_k^1(t_1) \right) + c(t_2, t_2) & t_1 \leq t_2; \\
\frac{c(t_1, t_2)}{(r_1 + 1)^{n_1}} \sum_{k=0}^{n_1} \binom{n_1}{k} h_2^{n_1-k} r_1^{n_1-k} B^2 \left( \tilde{D}_k^2(t_1) - \tilde{D}_k^2(t_2) \right) + c(t_1, t_1) & t_1 > t_2,
\end{cases}$$

where

- $\Delta_k^i(t) = \int_0^t y^{\frac{k}{g_i}} e^{-\frac{h_i}{g_i} y} dy$, $\tilde{D}_k^i(t) = D_k^i \left( \frac{\lambda_{x_i}(t)}{h_i} \right)$, $i = 1, 2$,

- $B^1 = e^{-k(t_2 + x_1) + \frac{h_1}{g_1} e^{g_1(x_1 + t_1)}}$, $B^2 = e^{-k(t_1 + x_2) + \frac{h_2}{g_2} e^{g_2(x_2 + t_2)}}$,

- $c(t_1, t_2) = \exp \left\{ - \frac{h_1}{g_1} [e^{g_1(x_1 + t_1)} - e^{g_1 x_1}] - \frac{h_2}{g_2} [e^{g_2(x_2 + t_2)} - e^{g_2 x_2}] \right\}$. 

Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.’s and let $K_t$ be a generic status process, e.g., $K$ could be one of the following:

$$JL\mathbb{I}_t = 1\{T_{x_1} x_{2} \leq t\}, \quad SL\mathbb{I}_t = 1\{T_{x_1} x_{2} \leq t\}, \quad t \geq 0,$$

[Then the pdf of $K_T$ could be computable!]
• Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.’s and let $K_t$ be a generic status process, e.g., $K$ could be one of the following:

$$JLt = 1\{T_{x_1x_2} \leq t\}, \quad SLt = 1\{T_{x_1x_2} \leq t\}, \quad t \geq 0,$$

[Then the pdf of $K_T$ could be computable!]

• Let $u$ be an exponential utility function:

$$u(w) = -\frac{1}{\alpha} e^{-\alpha w}, \quad w \in \mathbb{R}. \quad (33)$$

• Define $J(t, w; \pi) \triangleq \mathbb{E}_{t,w}\{u(W^\pi_T - K_T)\}$, where $W$ is the wealth process with investment portfolio $\pi.$
Let $T_{x_1}$ and $T_{x_2}$ be two future life time r.v.’s and let $K_t$ be a generic status process, e.g., $K$ could be one of the following:

$$JLl_t = 1\{T_{x_1 x_2} \leq t\}, \quad SLl_t = 1\{T_{x_1 x_2} \leq t\}, \quad t \geq 0,$$

[Then the pdf of $K_T$ could be computable!]

Let $u$ be an exponential utility function:

$$u(w) = -\frac{1}{\alpha} e^{-\alpha w}, \quad w \in \mathbb{R}. \quad (33)$$

Define $J(t, w; \pi) \triangleq \mathbb{E}_{t, w}\{u(W_T^\pi - K_T)\}$, where $W$ is the wealth process with investment portfolio $\pi$.

If $K_T \equiv 0$, then denote $J^0(t, w; \pi) \triangleq \mathbb{E}_{t, w}\{u(W_T^\pi)\}, \pi \in \mathcal{A}$.

$$U(t, w) \triangleq \sup_{\pi \in \mathcal{A}} J(t, w; \pi), \quad V(t, w) \triangleq \sup_{\pi \in \mathcal{A}} J^0(t, w; \pi).$$
Recall the “separation of variable”: \( U(t, w) = V(t, w)\Phi(t, w) \), where

\[
V(t, w) = -\frac{1}{\alpha} \exp \left( -\alpha we^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2} (T - t) \right).
\]

**Question**

What is \( \Phi \)?
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**Question**

What is \( \Phi \)?

**Theorem (M.-Yun ’10)**

\[ \Phi(t, w) = \mathbb{E}_{t, w} \{ e^{\alpha K_T} \}. \]

[Note that \( J(t, w; \pi) = J^0(t, w; \pi) \mathbb{E}_{t, w} \{ e^{\alpha K_T} \} ! \)]]
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**Question**

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**Theorem (M.-Yun ’10)**

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[Note that \( J(t, w; \pi) = J^0(t, w; \pi) \mathbb{E}_{t, w}\{e^{\alpha K_T}\} \)!]

- The indifference (selling) price is

\[
p_t^* = \frac{1}{\alpha} e^{-r(T-t)} \log \Phi(t, w) = \frac{1}{\alpha} e^{-r(T-t)} \log \mathbb{E}_{t, w}[e^{\alpha K_T}].
\]

