Finance, Insurance, and Stochastic Control (I)

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Lecture Plan

Part I. Ruin Problems (vs. Credit Risks)

Part II. Equity-Linked Insurance Problems

Part III. Reinsurance Problems

Part IV. A New Stochastic Control Problem

Outline

Introduction

Definition (Credit Default Swap (CDS))

A CDS is a contract where

- the "protection buyer" "A" pays rates "R" at times $T_{a+1}, \ldots, T_b$ (the "premium leg") in exchange for a single protection payment L_{GD} (Loss Given Default, the "protection leg").
- The buyer receives the protection leg by the protection seller "B" at the default time τ of a reference entity "C", provided that $T_a < \tau < T_b$.
- The rates R paid by "A" stop in case of default.

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- The rates R paid by "*A*" stop in case of default.

In terms of "Term Life Insurance":

- Time of death (default) — τ (of the insured "*C*")
- Death benefit — L_{GD}, payable at the moment of death
- Premium — an annuity (e.g. monthly) at (leveled) rate R
- Coverage period (term) — $[T_a, T_b]$, where $a < b$ are two ages.

Credit Risk vs. Actuarial Problems

	Credit Risk	Actuarial Science
τ	Default time	Ruin time, Future life time ($\tau = T(x)$)
$P\{\tau > t\}$	Survival Proba.	Survival Probability (${}_tp_x = P\{T(x) > t\}$)
$\Lambda(t) = -\ln {}_tp_x$	Hazard Process	Hazard Process
$\lambda(t) = \Lambda'(t)$	Default Intensity	"Force of Mortality" ($\mu(x+t) = -({}_tp_x)'/{}_tp_x$)
	Structure	Ruin Problems
	Reduced form	Life Contingencies

Basel II Accord

From Wikipedia, the free encyclopedia

Basel II (Bank for International Settlements Basel Accord)

Basel II is the second of the Basel Accords, which are recommendations on banking laws and regulations issued by the Basel Committee on Banking Supervision (Basel, Switzerland). The purpose of Basel II, which was initially published in tclblueJune 2004, is to create an international standard that banking regulators can use when creating regulations about how much capital banks need to put aside to guard against the types of financial and operational risks banks face. ......

In practice, Basel II attempts to accomplish this by setting up rigorous risk and capital management requirements designed to ensure that a bank holds capital reserves appropriate to the risk the bank exposes itself to through its lending and investment practices......

An Example in Risk Management

- Recall that the definition of "Value at Risk" of a r.v. Z:

$$\mathrm{VaR}_\alpha(Z) \stackrel{\triangle}{=} \inf\{x : \mathbb{P}\{x + Z < 0\} \le \alpha\}.$$

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- Consider the value process $V_t^\pi = x + Q_t^\pi$ ($Q_0^\pi = 0$) for an investment strategy π. Then one can assess the "risk" associated to this strategy by looking at $\text{VaR}_\alpha(\inf_{t\in[0,T]} Q_t^\pi)$.

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- Define

$$\psi(x, T) = \mathbb{P}\{V_t^\pi < 0 : \ \exists\ t \in [0, T]\}. \tag{1}$$

Then

$$\mathrm{VaR}_\alpha(\inf_{t\geq 0} Q_t^\pi) = \inf\{x : \psi(x, T) \leq \alpha\}.$$

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$$\text{VaR}_\alpha(\inf_{t\ge 0} Q_t^\pi) = \inf\{x : \psi(x, T) \le \alpha\}.$$

- Assume now that $\psi(x, T) \sim e^{-r^* x}$ for some $r^* \in \mathbb{R}$, then

$$\text{VaR}_\alpha(\inf_{t\ge 0} Q_t^\pi) \sim -\frac{\log \alpha}{r^*}!$$

Some Remarks

Note

- In Actuarial Sciences, the quantity $\psi(x, T)$ (or $\psi(x) = \mathbb{P}\{V_t^h < 0 : \ \exists\ t > 0\}$) is called "*Ruin Probability*". The estimate $\psi(x, T) \sim e^{-r^* x}$ is called the *Lundberg bound*, with *Lundberg exponent* r^*.

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- Define the "Average VaR" by

$$\rho(Z) \triangleq AVaR_\alpha(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha VaR_u(Z)du.$$

Then ρ is a "*Coherent Risk Measure*" (Cheridito-Delbaen-Kupper, '04).

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  Then ρ is a "*Coherent Risk Measure*" (Cheridito-Delbaen-Kupper, '04).
- The Lundberg bound also implies that

$$\rho(\inf_{t\geq 0} Q_t) \sim (1 - \log \alpha)/r^*.$$

  (The equality can hold if the Lundberg bound is sharp!)

Wiener-Poisson Space

- $(\Omega, \mathscr{F}, P)$ — a complete probability space
- $W = \{W_t\}_{t\geq 0}$— a d-dimensional Brownian motion
- $\mu(dtdz)$ — a Poisson random measure on $(0,\infty)\times\mathbb{R}_+$, with Lévy measure $\nu(dz)$.
- $\mathbf{F}^W = \{\mathscr{F}_t^W : t\geq 0\}$, $\mathbf{F}^\mu \stackrel{\triangle}{=} \{\mathscr{F}_t^\mu : t\geq 0\}$, $\mathbf{F} = \overline{\mathbf{F}^W\otimes\mathbf{F}^\mu}^P$,

Basic Insurance Models

Wiener-Poisson Space

- $(\Omega, \mathscr{F}, P)$ — a complete probability space
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Main Elements

- Claim Process
- Premium Process
- Reserve Process (= Premium - Claim)

Claim and Premium Processes

- **Claim Process**: $S_t = \int_0^t \int_{\mathbb{R}_+} f(s,z,\cdot)\mu(dsdz),\ t \geq 0$

  (may assume $d \leq f(s,z,\omega) \leq L$, where d and L are the *deductible* and *benefit limit*, respectively)

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Compound Poisson Case:

- $f(t,z) \equiv z$
- $S_t = \sum_{k=1}^{N_t} \Delta S_{T_k}$, where N_t is standard Poisson.
- $\nu(dz) = \lambda F_{U_1}(dz)$, and $E[S_t] = \int_0^t \int_{\mathbb{R}_+} z\nu(dz)ds = \lambda E[U_1]t$.
- $c_t = E\{\Delta S_t | \mathscr{F}_t^\mu\} = \int_{\mathbb{R}_+} z\nu(dz) = \lambda E[U_1],\ t \geq 0,$

Risk Reserve Process

Example

- Cramér-Lundberg Model: $X_t = x + \int_0^t c_s ds - S_t$

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- Add expense loading: $X_t = x + \int_0^t c_s(1 + \rho_s)ds - S_t$
- Add interest income: $X_t = x + \int_0^t [r_s X_s + c_s(1 + \rho_s)]ds - S_t$

Risk Reserve Process

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- Add interest income: $X_t = x + \int_0^t [r_s X_s + c_s(1 + \rho_s)]ds - S_t$
- Reserve with Investment

$$X_t = x + \int_0^t \Big\{ X_s[r_s + \langle \pi_s, \mu_s - r_s \mathbf{1} \rangle] + c_s(1 + \rho_s) \Big\} ds + \int_0^t X_s \langle \pi_s, \sigma_s dW_s \rangle - \int_0^t \int_{\mathbb{R}_+} f(s,z)\mu(dsdz), \quad (2)$$

General

Ruin Problems

Consider the simplest Cramér-Lundberg model:

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Ruin Problem

Find/estimate the "ruin probabilities":

$$\begin{aligned} \psi(x,T) &= P\{X_t < 0 : \ \exists\ t \in (0,T]\}; && \text{(Finite horizon)} \\ \psi(x) &= P\{X_t < 0 : \ \exists\ t > 0\}. && \text{(Infinite horizon)}. \end{aligned}$$

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Thinking finance?

Default probability? Structure model? ...

Existing ways/methods of studying ruin probabilities

- **Direct Calculation**: (e.g, vi IDE)
  — Lundberg ('26), Cramér ('35), Segerdahi ('42)...

- **Bounds**:
  — Lundberg ('26, 32, 34), Cremér ('55), Gerber ('76), Feller ('71) ...

- **Asymptotics**: (e.g., $\lim_{u\to\infty} \psi(u)e^{\gamma u} = ?$ $\lim_{u\to\infty} \psi(u,T)e^{\gamma u} = ?$)
  — Teugels-Veraverbeke ('73), Djehiche ('93), Asmussen-klüppelberg ('96)...

- **Approximations** (of claim size dist.):
  — De Vylder ('78), Daley Rolski ('84)...

Existing ways/methods of studying ruin probabilities

One of most notable discovery in ruin theory is that the ruin probablity satisfies a differential or integro-differential equation.

Main Result (Feller (1971), Gerber (1990))

Assume classical Cramér-Lundberg model. L $\psi(x)$ be the infinite horizon ruin probability with initial capital x, and $\varphi(x) = 1 - \psi(x)$ be the corresponding non-ruin probability. Then

$$\varphi(x) = \varphi(0) + \frac{\lambda}{c(1+\rho)} \int_0^x \varphi(x-z)\bar{F}_Z(z)dz, \tag{4}$$

where F is the jump size distribution and $\bar{F} = 1 - F$, and λ is the intensity of jump frequency.

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More general model— Reinhard (1984), Asmusson (1989) (Hidden Markovian), Asmusson-Petersen (1988) (reserve dependent premium) ...

Ruin Probility via Differential Equations

Assume that the risk reserve satisfies the following SDE:

$$X_t = x + \int_0^t b(s, X_s)ds - \int_0^t \int_{\mathbb{R}_+} f(s,z)N_p(dzds), \quad (5)$$

where $b : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$ is some (deterministic!) measurable function (could be Lipschitz..., if you wish). Then X is (strong) Markov.

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Then, $\forall 0 < t < T$,

$$\mathbf{1}_{\{\tau<T\}} = \mathbf{1}_{\{\tau<t\}} + \mathbf{1}_{\{t\leq\tau\}}\mathbf{1}_{\{\inf_{t\leq s<T} X_s<0\}}. \qquad (6)$$

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Define $M_t \stackrel{\triangle}{=} P\{\tau < T|\mathscr{F}_t^X\} = E\{\mathbf{1}_{\{\tau<T\}}|\mathscr{F}_t^X\}$; and

$$\Psi(t,r) \stackrel{\triangle}{=} P\left\{\inf_{t\leq s<T} X_t < 0 \middle| X_t = r\right\}. \qquad (7)$$

Ruin Probility via Differential Equations

Taking conditional expectations $E\{\cdot\,|\mathscr{F}_t^X\}$ on both sides of (6) and using the Markovian Property of X:

$$
\begin{aligned}
M_t &= \mathbf{1}_{\{\tau\le t\}} + \mathbf{1}_{\{\tau>t\}} P\left\{\inf_{t\le s<T} X_t < 0 \middle| X_t\right\} \\
&= \mathbf{1}_{\{\tau<t\}} + \mathbf{1}_{\{\tau\ge t\}} \Psi(t, X_t).
\end{aligned} \tag{8}
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Setting $t = t\wedge\tau$ in (8), we obtain that

$$M_{t\wedge\tau} = \Psi(t\wedge\tau, X_{t\wedge\tau}). \tag{9}$$

Thus by Optional Sampling $t \mapsto \Psi(t\wedge\tau, X_{t\wedge\tau})$ is an (UI) $\mathbf{F}^X$-mg!

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Now denote $\Phi(t,r) = 1 - \Psi(t,r)$ (*non-ruin probability*), and assume that $\Phi(\cdot,\cdot) \in C^{1,1}$.

Applying Itô (BV version) to get

$$
\begin{aligned}
&\Phi(t\wedge\tau, X_{t\wedge\tau}) - \Phi(0,x)\\
= \;& \int_0^{t\wedge\tau} \partial_t\Phi(s,X_s)ds + \int_0^{t\wedge\tau} \partial_r\Phi(s,X_s)b(s,X_s)ds\\
&+ \int_0^{t\wedge\tau}\int_{\mathbb{R}_+} [\Phi(s,X_{s-} - f(s,z)) - \Phi(s,X_{s-})]N_p(dzds)\\
= \;& \int_0^{t\wedge\tau} \partial_t\Phi(s,X_s)ds + \int_0^{t\wedge\tau} \partial_r\Phi(s,X_s)b(s,X_s)ds\\
&+ \int_0^{t\wedge\tau}\int_{\mathbb{R}_+} [\Phi(s,X_{s-} - f(s,z)) - \Phi(s,X_{s-})]\nu(dz)ds + M^*_{t\wedge\tau},
\end{aligned}
$$

where

$$
M^*_t = \int_0^{t\wedge\tau}\int_{\mathbb{R}_+} [\Phi(s,X_{s-} - f(s,z)) - \Phi(s,X_{s-})]\widetilde{N}_p(dzds)
$$

is an martingale with zero mean.

Ruin Probility via Differential Equations

Thus

$$\begin{aligned}&\int_0^{t\wedge\tau}\partial_t\Phi(s,X_s)ds+\int_0^{t\wedge\tau}\partial_r\Phi(s,X_s)b(s,X_s)ds\\&+\int_0^{t\wedge\tau}\int_{\mathbb{R}_+}[\Phi(s,X_{s-}-f(s,z))-\Phi(s,X_{s-})]\nu(dz)ds\\=\;&\Phi(t\wedge\tau,X_{t\wedge\tau})-\Phi(0,x)-M^*_{t\wedge\tau}\end{aligned}$$

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= \quad & \Phi(t\wedge\tau, X_{t\wedge\tau}) - \Phi(0,x) - M^*_{t\wedge\tau} = 0.
\end{aligned}$$

(It is a continuous (local) martingale with zero mean and with bounded variation paths $\Longrightarrow$ it is a zero martingale!)

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$$\int_0^{t\wedge\tau} \partial_t\Phi(s,X_s)ds + \int_0^{t\wedge\tau} \partial_r\Phi(s,X_s)b(s,X_s)ds$$
$$+ \int_0^{t\wedge\tau}\int_{\mathbb{R}_+}[\Phi(s,X_{s-}-f(s,z)) - \Phi(s,X_{s-})]\nu(dz)ds$$
$$= \Phi(t\wedge\tau, X_{t\wedge\tau}) - \Phi(0,x) - M^*_{t\wedge\tau} = 0.$$

(It is a continuous (local) martingale with zero mean and with bounded variation paths $\Longrightarrow$ it is a zero martingale!)

Similarly, for any $t' \in [0,T)$ and $\tau' = \inf\{t \geq t' | X_t < 0\}$, one shows that

$$\int_{t'}^{t\wedge\tau'} \partial_t\Phi(s,X_s)ds + \int_{t'}^{t\wedge\tau'} \partial_r\Phi(s,X_s)b(s,X_s)ds \quad (10)$$
$$= \int_{t'}^{t\wedge\tau'}\int_{\mathbb{R}_+}[\Phi(s,X_{s-}) - \Phi(s,X_{s-}-f(s,z))]\nu(dz)ds.$$

Ruin Probility via Differential Equations

Since t' is arbitrary and $\tau' \geq t'$, we can "*differentiating*" (10) to get the following IPDE:

$$[\partial_t \Phi + \partial_r \Phi b](t, r) = \int_{\mathbb{R}_+} [\Phi(t, r) - \Phi(t, r - f(t, z))]\nu(dz). \quad (11)$$

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Remark

- Since $\Phi(t, X_t) = 0$ for $X_t < 0$, the RHS in (11) is actually

$$\int_{\{r \geq f(t,z)\}} [\Phi(t, r) - \Phi(t, r - f(t, z))]\nu(dz).$$

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$$\int_{\{r \geq f(t,z)\}} [\Phi(t,r) - \Phi(t, r - f(t,z))]\nu(dz).$$

- In the compound Poisson case $f(t,z) \equiv z$, $\nu(dz) = \lambda F_Z(dz)$, where Z is the jump size. Thus (11) becomes

$$[\partial_t \Phi + \partial_r \Phi b](t,r) = \Phi(t,r)\lambda - \lambda \int_{\{r \geq z\}} \Phi(t, r - z) F_Z(dz).$$

Special Cases

Infinite horizon case

Assume $b(t,r) = b(r)$. Denote $\psi(r) = \lim_{t\to\infty} \Psi(t,r)$ and $\varphi(r) = 1 - \psi(r)$. Then

$$\varphi'(r)b(r) = \varphi(r)\lambda - \lambda \int_{\{r \geq z\}} \varphi(r-z) F_Z(dz). \qquad (12)$$

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Example

If $b(r) = c(1+\rho) \overset{\triangle}{=} \beta$ and $Z \sim \exp\{\delta\}$ Then (12) becomes

$$\varphi'(r)\beta = \lambda \left\{ \varphi(r) - e^{-\delta r} \int_0^r \varphi(z)\delta e^{\delta z} dz \right\}. \tag{13}$$

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- Differentiating: $\varphi''(r)\beta = (\lambda - \delta\beta)\varphi'(r)$.
- Solving: $\varphi(r) = c_1 - c_2 e^{-(\delta-\lambda/\beta)r}$, where $c_1, c_2 \in \mathbb{R}$.

An Integral Equation

Denoting $\beta = c(1+\rho)$ again, and integrate (13) from 0 to x:

$$
\begin{aligned}
\frac{\beta}{\lambda}(\varphi(x) - \varphi(0)) &= \frac{\beta}{\lambda}\int_0^x \varphi'(r)dr \\
&= \int_0^x \varphi(r)dr - \int_0^x \int_0^u \varphi(u-z)F_Z(dz)du \\
&= \cdots\cdots \\
&= \int_0^x \varphi(r)dr - \int_0^x \int_0^{x-u} F_Z(dz)\varphi(u)du \\
&= \int_0^x [1 - F_Z(x-u)]\varphi(u)du.
\end{aligned}
$$

$$
\Longrightarrow \quad \varphi(x) = \varphi(0) + \frac{\lambda}{\beta}\int_0^x \varphi(x-z)\bar{F}_Z(z)dz. \tag{14}
$$

An Evidence

Recall IDE (14). By Expected Value Principle $c = \frac{dE[S_t]}{dt} = \lambda\mu$, denoting $F_I(x) = \mu^{-1}\int_0^x \bar{F}(z)dz$ (14) becomes

$$\varphi(x) = \varphi(0) + \frac{1}{(1+\rho)}\varphi * F_I(x), \tag{15}$$

where $*$ means convolution.

Lundberg bounds

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where $*$ means convolution.

Solving (15) by Laplace transforms and using the initial value $\varphi(0) = \frac{\rho}{1+\rho}$ we have

$$\varphi(x) = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}\left(\frac{1}{1+\rho}\right)^n F_I^{n*}(x). \tag{16}$$

Lundberg Bounds

Example

If $Z \sim \exp(\delta)$, then we see that

$$\psi(x) = 1 - \varphi(x) = \frac{1}{1+\rho} \exp\left\{-\frac{\rho}{\delta(1+\rho)}x\right\} \leq e^{-Rx}.$$

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Remark

A primitive method for the Lundberg bound is to consider $\psi_n(x)$, the ruin probability up to $(n+1)$-st claim. By an inductional argument one proves that, there exists an $R > 0$ such that

$$\psi_n(x) \leq e^{-Rx}, \qquad \forall n. \tag{17}$$

Letting $n \to \infty$ one derives the (upper) bound for (infinite horizon) ruin probability $\psi(x)$. The constant R is called "*Lundberg coefficient*" or "*adjustment coefficients*".

Exponential Martingale Approach (Gerber, (1973))

- Consider the classical model $X_t = x + ct - S_t$, where $ct = E[S_t] = \lambda\mu t$. Denote $Q_t = ct - S_t$ (*profit process*).

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- For any given x and $r > 0$, consider the $\mathbf{F}^p$-adapted process

$$M_t^x \triangleq \frac{e^{-r(x+Q_t)}}{e^{t\theta(r)}}, \qquad t \geq 0, \tag{18}$$

where $\theta(\cdot)$ is a function to be determined.

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  where $\theta(\cdot)$ is a function to be determined.
- **Suppose that $\{M_t^x\}$ is an $\mathbf{F}^p$-martingale(!)** Then, by optional sampling, for any given time $t_0 > 0$ and stopping time $\tau_x \triangleq \inf\{t \geq 0 : X_t = x + Q_t < 0\}$, one has

$$\begin{aligned} e^{-rx} &= M_0^x = E\left\{M_{t_0\wedge\tau_x}^x \middle| \mathscr{F}_0^p\right\} = E\left\{M_{t_0\wedge\tau_x}^x\right\} \\ &\geq E\left\{M_{\tau_x}^x \middle| \tau_x \leq t_0\right\} P\{\tau_x \leq t_0\}. \end{aligned} \tag{19}$$

Exponential Martingale Approach (Gerber, (1973))

- But on the set $\{\tau_x \leq t_0\}$ one must have $X_{\tau_x} = x + Q_{\tau_x} \leq 0$. Thus

$$
\begin{aligned}
P\{\tau_x \leq t_0\} &\leq \frac{e^{-rx}}{E\{M^x_{\tau_x}|\tau_x \leq t_0\}} \leq \frac{e^{-rx}}{E\{e^{-\tau_x\theta(r)}|\tau_x \leq t_0\}} \\
&\leq e^{-rx} \sup_{0\leq t\leq t_0} e^{t\theta(r)}.
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- Letting $t_0 \to \infty$ we obtain that

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\psi(x) \le e^{-rx} \sup_{t \ge 0} e^{t\theta(r)}. \tag{20}
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Question

How to determine θ?

Exponential Martingale Approach (Gerber, (1973))

Analysis

- Denote $\hat{f}(s) = \int_0^\infty e^{-sx} dF(x) = E[e^{-sU_1}]$. Then

$$
\begin{aligned}
E\left[e^{sS_t}\right] &= \sum_{n=0}^{\infty} E\left[e^{s\sum_{k=1}^{N_t} U_k}\Big| N_t = n\right] P(N_t = n) \\
&= \sum_{n=0}^{\infty} \hat{f}^n(-s) \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{\lambda(\hat{f}(-s)-1)t}
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\end{aligned}
$$

- Thus to make M^x a martingale, one need only choose

$$
E\left[e^{-sQ_t}\right] = e^{-sct} E\left[e^{sS_t}\right] = e^{-sct+\lambda[\hat{f}(-s)-1]t} \triangleq e^{t\theta(s)}, \quad (21)
$$

where $\theta(s) \triangleq \lambda[\hat{f}(-s) - 1] - sc$.

Exponential Martingale Approach (Gerber, (1973))

- With this choice of θ, and using (21) and the fact that Q has independent increments, we have

$$E[M_t^x|\mathscr{F}_s^p] = M_s^x E\left\{ \frac{e^{-r(Q_t-Q_s)}}{e^{(t-s)\theta(r)}} \middle| \mathscr{F}_s^p \right\} = M_s^x.$$

$\Longrightarrow$ M^x is a $\mathbf{F}^p$-martingale!

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- Recall (20). Clearly the sharp estimate of ruin probability is obtained by minimizing the RHS w.r.t. r. Namely, choosing $r^* \triangleq \sup\{r : \theta(r) \le 0\}$ would give the best estimate

$$\psi(x) \le e^{-r^* t}. \tag{22}$$

r^* is thus called *Lundberg coefficient*.

Another look at Exponential Martingales

Consider the more general model:

$$X_t = x + \int_0^t b(s, X_s)ds - \int_0^t \int_{\mathbb{R}_+} f(s,z)N_p(dsdz). \qquad (23)$$

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- For any $g \in C^{1,1}([0,T] \times \mathbb{R})$, applying Itô's formula to get

$$g(t, X_t) = g(0, x) + \int_0^t \{\partial_t g + \partial_x g b\}(s, X_s)ds$$
$$. \qquad + \int_0^t \int_{\mathbb{R}_+} [g(s, X_{s-} - f(s,z)) - g(s, X_{s-})]\nu(dz)ds + mg$$

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- Thus $M_t \stackrel{\triangle}{=} g(t, X_t)$ is a mg (or local mg) $\Longleftrightarrow$ g satisfies

$$\partial_t g + \partial_x g b + \int_{\mathbb{R}_+} [g(t, x - f(t, z)) - g(t, x)]\nu(dz) = 0. \qquad (24)$$

Another look at Exponential Martingales

In the compound Poisson case $b(t,x) = \beta$, $f \equiv z$, and $\nu(dz) = \lambda F_U(dz)$. The equation (24) becomes

$$[\partial_t g + \partial_x g]\beta + \lambda \left\{ \int_{\mathbb{R}_+} [g(t,x-z) - g(t,x)] F_U(dz) \right\} = 0.$$

If $g = g(x)$, then

$$g'(x)\beta + \lambda \left\{ \int_{\mathbb{R}_+} g(x-z) F_U(dz) - g(x) \right\} = 0. \qquad (25)$$

Setting $g(x) = \varphi(x)$ for $x \geq 0$ and $g(x) = 0$ for $x < 0$ we see that the integral becomes $\int_0^x g(x-z)F_U(dz)$ and we recover (14) for the infinite horizon ruin probability.

- Assume $g(t,x) = e^{-sx-\theta t}$, where s and θ are parameters. Then (25) reads

$$[-\theta - \beta s]g(t,x) + \lambda \left\{ \int_{\mathbb{R}_+} [e^{sz} F_U(dz) - 1] g(t,x) \right\} = 0.$$

Finite Horizon Case

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- Denoting $\hat{m}_U(s) = \int_{\mathbb{R}_+} e^{sz} F_U(dz)$, then the above becomes

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- Thus (since $g(t,x) > 0$!)

$$\theta = \theta(s) = -\beta s + \lambda [\hat{m}_U(s) - 1]. \qquad (26)$$

We obtain the *adjustment coefficient* $\theta = \theta(s)$, and

$$M_t = g(t, X_t) = \exp\{-sX_t - \theta(s)t\}$$

is a martingale!

Risk Reserve with Interests

- Consider the reserve equation with interst: $X_0 = x$

$$dX_t = [r_t X_t + c_t(1+\rho_t)]dt - \int_{\mathbb{R}_+} f(t,z) N_p(dzdt).$$

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- Denote $\Gamma_t \stackrel{\triangle}{=} e^{-\int_0^t r_s ds}$, and $\widetilde{X}_t = \Gamma_t X_t$. Then $\widetilde{X}$ satisfies

$$\widetilde{X}_t = x + \int_0^t \Gamma_s c_s(1+\rho_s)ds - \int_0^t \int_{\mathbb{R}_+} \Gamma_s f(s,z)N_p(dzds).$$

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- Assume $\beta = c(1+\rho)$ is constant, and r_t is deterministic, Then for $g \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R})$, we have

$$g(t,\widetilde{X}_t) = g(0,x) + \int_0^t [\partial_t g + \partial_x g \Gamma_s \beta](s,\widetilde{X}_s)ds$$

$$. \quad + \int_0^t \int_{\mathbb{R}_+} [g(\cdot, \cdot - \Gamma_s f) - g](s,\widetilde{X}_{s-})\nu(dz)ds + mg$$

- Thus $M_t = g(t, \widetilde{X}_t)$ is a martingale if and only if

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- Assume that $g(t, x) = a(t)e^{-sx}$, $a(t) > 0$ to be determined, and $f \equiv z$ and $\nu(dz) = \lambda F_U(dz)$, then the above becomes

$$\begin{aligned} 0 &= a'(t)e^{-sx} + \{-\beta s \Gamma_t + \lambda [\hat{m}(s\Gamma_t) - 1]\} g(t, x) \\ &= \left\{a'(t) - \theta(s\Gamma_t)a(t)\right\} e^{-sx}. \end{aligned}$$

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- Assume $a(0) = 1$. We can solve the ODE

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to get $a(t) = e^{-\int_0^t \theta(s\Gamma_u)du}$.

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- Thus $\tilde{M}_t \stackrel{\triangle}{=} g(t, \widetilde{X}_t) = \exp\{-s\widetilde{X}_t - \int_0^t \theta(s\Gamma_u)du\}$ is a mg.

Lundberg Bounds for General Models

Question:

Can we find an exponential martingale that leads to the Lundberg bound for the general reserve model (2)?

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Recall the exponential martingale

$$\widetilde{M}_t = \exp\left\{-s\Gamma_t X_t - \int_{\mathbb{R}_+} \theta(s\Gamma_u)du\right\} \stackrel{\triangle}{=} \exp\{-I_s(t, X_t) - K_t^s\}.$$

where $I_s(t,x) \stackrel{\triangle}{=} sx\Gamma_t$ and $K_t^s = \int_{\mathbb{R}_+} \theta(s\Gamma_u)du$. Define

- $\beta_t = -\int_0^t r_s ds,\ t \geq 0$
- $I_\delta(t,x) \stackrel{\triangle}{=} \delta x e^{-\int_0^t r_s ds} = \delta x \Gamma_t = \delta x e^{\beta_t},\ \delta \in \mathbb{R}.$
- $\widetilde{X}_t = e^{\beta_t} X_t = \Gamma_t X_t$ (discounted risk reserve).

Lundberg Bounds for General Models

- In general, we replace s by a parameter δ, and look for a possible exponential mg $M^\delta = \exp\{l_\delta + K^\delta\}$, where $l_\delta(t, X_t) = \delta\widetilde{X}_t$, and $\widetilde{X}$ satisfies:

$$\begin{aligned} d\tilde{X}_t &= \Gamma_t(\widetilde{b}(t, \beta_t, \widetilde{X}_t) + \eta_t)dt + \langle\, \hat{\sigma}_t, dW_t \,\rangle \\ &\quad - \int_{\mathbb{R}^+} \Gamma_t f(t, x) N_p(dtdx), \end{aligned}$$

where $\widetilde{b}(t, \beta_t, \widetilde{X}_t) = b(t, e^{-\beta_t}\widetilde{X}_t)) = b(t, X_t)$.

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  where $\widetilde{b}(t, \beta_t, \widetilde{X}_t) = b(t, e^{-\beta_t}\widetilde{X}_t)) = b(t, X_t)$.

- To "decompose K^δ, define $m_t^f(\gamma) \stackrel{\triangle}{=} \int_{\mathbb{R}_+} [e^{\gamma f(t,z)} - 1]\nu(dz)$. Then $m^f(\gamma)$ is increasing in γ and integrable for all $\gamma \le \delta_0$.

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- In compound Poisson case, $f \equiv z$ and $\nu(dz) = \lambda F_U(dz)$, then $m_t^f(\gamma) \stackrel{\triangle}{=} \lambda \int_{\mathbb{R}_+} [e^{\gamma z} - 1] F_U(dz) = \lambda(\hat{m}_U(\gamma) - 1)$, again.

Lundberg Bounds for General Models

- Now define $K_t^\delta = -V_t^\delta + \frac{1}{2}Y_t^\delta + Z_t^\delta$, where

$$V_t^\delta = \delta \int_0^t e^{\beta_s}[\widetilde{b}(s, \beta_s, \widetilde{X}_s) + \eta_s]ds;$$

$$Y_t^\delta = \delta^2 \int_0^t e^{2\beta_s}|\hat{\sigma}_s|^2 ds; \qquad Z_t^\delta \triangleq \int_0^t m_s^f(\delta e^{\beta_s})ds.$$

- Define also $Z_t^{\delta,0} \triangleq \int_0^t m_s^f(\delta)ds$, and

$$\begin{cases} \mathscr{D} = \{\delta \geq 0 : Z_t^\delta < \infty, P\text{-a.s., } \forall t \geq 0\}; \\ \mathscr{D}_0 = \{\delta \geq 0 : Z_t^{\delta,0} < \infty, P\text{-a.s.,} \forall t \geq 0\}. \end{cases}$$

- Since $\gamma \geq 0$ and $\beta_s \leq 0$, the monotonicity of $m^f(\cdot)$ shows that $\mathscr{D}_0 \subseteq \mathscr{D}$.

Main Results

Theorem (M. Sun (02))

The process $M_t^\delta \stackrel{\triangle}{=} \exp\{-\delta\widetilde{X}_t - K_t^\delta\}$, $t \geq 0$, *enjoys the following properties:*

- *For every* $\delta \in \mathscr{D}$, $\{M_t^\delta : t \geq 0\}$ *is an* $\mathbf{F}$*-local martingale.*
- *If the processes* π, σ, μ, *and* r *are all bounded and* $\mathbf{F}^W$*-adapted, and that* $f(\cdot,\cdot,\cdot)$ *is deterministic, then for every* $\delta \in \mathscr{D}_0$, $\{M_t^\delta : t \geq 0\}$ *is an* $\mathbf{F}$*-martingale.*
- *If* r *is also deterministic, then (ii) holds for all* $\delta \in \mathscr{D}$.
- *If* π *is allowed to be* $\mathbf{F}$*-adapted, then (ii) and (iii) hold for all* δ *such that* $2\delta \in \mathscr{D}$ *and* $\mathscr{D}_0$*, respectively.*

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- *If* π *is allowed to be* $\mathbf{F}$*-adapted, then (ii) and (iii) hold for all* δ *such that* $2\delta \in \mathscr{D}$ *and* $\mathscr{D}_0$*, respectively.*

Proof: Define $F^\delta(x, v, y, z) \stackrel{\triangle}{=} \exp(-\delta x + v - \frac{1}{2}y - z)$, and applying Itô's formula to $F^\delta(\widetilde{X}_t, V_t^\delta, Y_t^\delta, Z_t^\delta)$...

Main Results

Example

- **Classical Model** $\pi_t \equiv 0, r_t \equiv 0, \rho \equiv 0, \mu_t \equiv 0, \sigma_t \equiv 0,$
  - S_t is Compound Poisson
  - $K_t^\delta = t(\int_0^\infty (e^{\delta x} - 1)\lambda F(dx) - c\delta)$ $(= \theta(\delta)t!)$
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Lundberg Exponent

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- **Discounted Risk Reserve** $\pi_t = \rho_t = \mu_t = \sigma_t \equiv 0,\ r > 0$
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- **Perturbed risk reserve** $\pi_t \equiv 1,\ \rho_t = r_t = \mu_t \equiv 0,\ \sigma_t \equiv \varepsilon,$
  - $X_t = x + ct + \varepsilon W_t - S_t$
  - $K_t^\delta = t(-c\delta + \frac{1}{2}\delta^2\varepsilon^2 + \int_0^\infty (e^{\delta x} - 1)\lambda F(dx)) \stackrel{\triangle}{=} k(\delta)t$
  - $\widetilde{\delta} \stackrel{\triangle}{=} \sup\{\delta > 0 : k(\delta) = 0\}$ (Delbaen-Haezendonck (1987), ...)

Ruin Probability via "Rate Functions"

Extending the idea of the function $I_\delta(t,x) = \delta x \beta_t$, we can consider a more general "rate function": $I \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. Define

- $M^I_t \triangleq \exp\{-I(t,X_t) - K^I_t\}$, $K^I_t \triangleq -V^I_t + \frac{1}{2}Y^I_t + Z^I_t$, and
- $Z^I_t \triangleq \int_0^t \int_{\mathbb{R}^+} [\exp\{I(s,X_s) - I(s,X_s - f(s,x))\} - 1]\nu(dx)ds$
- $V^I_t \triangleq \int_0^t \{\partial_x I(s,X_s)b(s,X_s) + \partial_t I(s,X_s)\}ds$
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Definition

A function $I \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ is called a "rate function" if $Z_t^I < \infty$, $\forall t \geq 0$, P-almost surely.

Analysis

Suppose that we can find l such that M^l is a local martingale, and that $l(t,x) \le 0$, for all t and $x \le 0$.

- Let $\tau \stackrel{\triangle}{=} \inf\{t, X_t < 0\}$, and apply Optional Sampling to supermartingale (nonnegative loc mg) M_t^l:

$$
\begin{aligned}
e^{-l(0,x)} &\ge E\{e^{-l(\tau,X_\tau)-K_\tau^l}|\tau < T\}P\{\tau < T\} \\
&\ge E\Big\{\inf_{0\le t\le T} e^{-K_t^l}\Big\}\psi(x,T).
\end{aligned}
$$

- Applying Jensen's inequality we have

$$
\psi(x,T) \le \frac{e^{-l(0,x)}}{E\Big\{\inf_{0\le t\le T} e^{-K_t^l}\Big\}} \le e^{-l(0,x)} E\Big\{\sup_{0\le t\le T} e^{K_t^l}\Big\}.
$$

- One can let $T \to \infty$ to obtain the bound for $\psi(x)$.

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$$\psi(x,T) \leq e^{-I(0,x)} E \sup_{0\leq t\leq T} \exp(K_t^I),$$
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- *In the Lundberg bounds above the process* $K^I(X)$ *can be replaced by* $K^I(X^+)$, *where* $X_s^+ \stackrel{\triangle}{=} X_s \vee 0$.

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- *In the Lundberg bounds above the process $K^I(X)$ can be replaced by $K^I(X^+)$, where $X_s^+ \stackrel{\triangle}{=} X_s \vee 0$.*

Question:

How to find a rate function?

Asmussen-Nielsen Bound

Assume Compound Poisson ($f(t,x) = x$, and $v(dx) = \lambda F(dx)$), and $\pi \equiv 0$, $\mu \equiv 0$, $\sigma \equiv 0$, $r_t = r$ (constant), $\rho(t,x) \equiv \rho(x)$ is an increasing function in x. Then

- $X_t = x + \int_0^t p(X_s)ds + \int_0^t \int_{\mathbb{R}^+} x\mu(dxds)$, $t \geq 0$, where $p(x) \stackrel{\triangle}{=} rx + c(1+\rho(x))$.

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- Consider the Rate function of the form: $I(x) = \int_0^x \gamma(y)dy$, $x \geq 0$, $\gamma(\cdot) > 0$, increasing. Then

$$\begin{aligned} K_t^I &= \int_0^t \Big\{ -[\gamma p](X_s^+) + \int_{\mathbb{R}^+} \Big[e^{\int_{X_s^+ - x}^{X_s^+} \gamma(y)dy} - 1 \Big] \lambda F(dx) \Big\} ds \\ &\leq \int_0^t \Big\{ -[\gamma p](X_s^+) + \int_{\mathbb{R}^+} [e^{\gamma(X_s^+)x} - 1] \lambda F(dx) \Big\} ds. \end{aligned}$$

Asmussen-Nielsen Bound

- Let γ be the non-decreasing solution to the Lundberg equation:

$$-\gamma p(y) + \int_{\mathbb{R}^+} [e^{\gamma x} - 1]\lambda F(dx) = 0, \qquad y \geq 0.$$

(such solution exists if the so-called net profit condition: $\inf_{x \geq 0} p(x) > \lambda E[U_1]$ holds and ρ is monotone.)

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- One can show that if $p(\cdot) \in C^1$, then I can be extended so that $I(\cdot) \in C^2(\mathbb{R})$, $I(0) = 0$, and $I(x) \leq 0$ for $x < 0$.
- Thus $K_t^I(X^+) \leq 0$, $\forall t \geq 0$, and we have

$$\psi(x, T) \leq e^{-I(x)} \quad \textit{and} \quad \psi(x) \leq e^{-I(x)}.$$

This is the Asmussen and Nielsen bound (1995).

Can We Do Better?

Assume now $\rho(x) \equiv 0$, and $F(x) = 1 - e^{-\theta x}$, $x \geq 0$. Then the Asmussen-Nielsen bound tells us:

$$\psi(x) \leq e^{-\theta x}\left(1 + \frac{r}{c}x\right)^{\frac{\lambda}{r}}, \quad x \geq 0.$$

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Denote $K^I(X^+) = \int_0^t \mathscr{L}[I](X_s^+)ds$, where $\mathscr{L}$ is an ID operator:

$$\mathscr{L}[I](y) \triangleq -I'(y)[ry + c] + \int_0^\infty [e^{I(y)-I(y-x)} - 1]\lambda\theta e^{-\theta x}dx.$$

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Setting $\mathscr{L}[I](y) = 0$, we see that b must satisfy

$$e^{\theta y}[ry + c]b'(y) + \int_0^y b(z)\lambda\theta e^{-\theta z}dz + \int_y^\infty \lambda\theta e^{-\theta x}dx = \lambda e^{\theta y}b(y).$$

Can We Do Better?

Solving this equation to get

$$b(y) = C_1 \int_0^y e^{-\theta z} \left(\frac{rz}{c} + 1 \right)^{\frac{\lambda}{r} - 1} dz + C_2.$$

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Determining the constant C_1 and C_2, and working a little more to get

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But it is known that in this case $\psi(x) = e^{-I(x)}$, $x \geq 0$ (Segerdahi (1942)), we have obtained the SHARPEST bound!

Large Claim Case

It is known that in the models where large claims occur with high probability, the local adjustment coefficient method may fail.

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Example

Assume that the claim sizes U_k are of Pareto (a, b) distribution:

$$F(z) = \frac{b}{a} \int_0^z \left(\frac{a}{z}\right)^{b+1} \mathbf{1}_{[a,\infty)}(z) dz.$$

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We show that the rate function technique still works in this case!

Assume that $X_t = x + ct - \sum_{k=1}^{N_t} U_k$, where $U_k \sim Pareto(1, 2)$ and $\lambda = 1$. (i.e., $F_U(dz) = 2z^{-3}\mathbf{1}_{[1,\infty)}(z)$.) Note that the Net Profit Condition implies that $c - E[U_1] = c - 2 > 0$.

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We assume that the rate function $I \in C^2$ takes the following form:

$$I(y) = \begin{cases} \ln(y+\beta) - \ln\beta & y \geq 0, \\ 0 & y \leq -1, \end{cases}$$

Then the process $K^I(X^+)$ takes the form:

$$K^I_t = \int_0^t \Big\{ \underbrace{-\frac{c}{X^+_s + \beta} + \int_0^\infty \{e^{I(X^+_s) - I(X^+_s - x)} - 1\} F(dx)}_{\Gamma^I(X^+_s)} \Big\} ds.$$

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- First choosing $Y > 0$ such that $\frac{\ln y}{y} \leq \frac{(c-1)}{8}$, $\forall y \geq Y$.
- Then define $\beta \stackrel{\triangle}{=} \max\{Y, \frac{4}{c-1}, 2\}$ and $\varepsilon \stackrel{\triangle}{=} (\beta+1)^2$, such that $I(y) \geq -\ln(1+\varepsilon)$, for $y \in [-1, 0]$

Proportional Investments

The idea if finding Γ^I can be developed further. Consider

$$X_t = x + \int_0^t p(X_s)ds + \int_0^t \langle \alpha X_s, \sigma dW_s \rangle - \sum_{k=1}^{N_t} U_k, \ t \geq 0,$$

where $p(x) = rx + c$, $U_k \sim \exp(\theta)$, and $\alpha = (\alpha^1, \alpha^2, ..., \alpha^n)^T$.

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Purpose

Find $I \in C^2(\mathbb{R})$, such that

$$\Gamma^I(y) \triangleq -I'(y)\{ry + C\} + \frac{1}{2}(I'(y)^2 - I''(y))y^2|\sigma^T \alpha|^2 + \int_{\mathbb{R}_+} [e^{I(y)-I(y-x)} - 1]\lambda\theta e^{-\theta x} dx \leq 0,$$

and $I(y) \sim k \ln y + C$ for some constant k, C, as $y \to \infty$.

Principle of Smooth-fit

Consider the following two-parameter family:

$$I_{\beta,k}(y) = k(\ln(y+\beta) - \ln 2\beta)1_{[\beta,\infty)}(y).$$

Suppose that $r > |\sigma^T\alpha|^2/2 > 0$. Then, for $k = 2\frac{r}{|\sigma^T\alpha|^2} - 1 > 0$, one can find $\beta = \dfrac{k}{\delta}$ large enough, such that

$$\begin{aligned}\Gamma^I(y) &= -I'(y)\{ry + C\} + \frac{1}{2}(I'(y)^2 - I''(y))y^2|\sigma^T\alpha|^2 \\ &\quad + \int_{\mathbb{R}_+}[e^{I(y)-I(y-x)} - 1]\lambda\theta e^{-\theta x}dx \le 0, \quad \forall y \ge \beta.\end{aligned}$$

Consequently, $\psi(x) \le e^{-I(x)} = K(x+\beta)^{-k}$, for x large.

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Note: This result coincides with those of Nyrhinen (1999) and Kalashnikov-Norberg (2000).

Ruin Problem via Storage Processes

An important observation made by Asmussen-Petersen (1988) is that the ruin probability of the risk process:

$$X_t = x + \int_0^t b(X_s)ds - S_t,$$

where S is a compound Poisson, and $b(\cdot)$ is deterministic. Then the following relation hold:

$$P\{\tau < T\} = \psi(x, T) = P\{Y_T > x\},$$

where $Y_t \triangleq -\int_0^t b(Y_s)ds + S_T - S_{T-t}$ is called a "*storage process*".

Such a relation has proved to be very useful when Large Deviation method is used to study the asymptotics of ruin probabilities.

A Natural Extension

Consider the risk reserve process

$$X_t = x + \int_0^t b(s, \cdot, X_s)ds + \Lambda_t^\pi - S_t, \qquad 0 \le t \le T, \qquad (27)$$

where $b(t, \omega, x) = c(1 + \rho(t, x)) + r_t(\omega)x$, and

$$\Lambda_t^\pi = \int_0^t \langle \pi_s, \mu_s - r_s \mathbf{1} \rangle \, ds + \int_0^t \langle \pi_s, \sigma_s dw_s \rangle \, .$$

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Assume $b(t, x)$ is uniform Lipschitz in x, uniformly in (t, ω), then (27) has a unique solution.

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Assume $b(t, x)$ is uniform Lipschitz in x, uniformly in (t, ω), then (27) has a unique solution.

Need

A "storage" process that solves a "reflected SDE":

$$Y_t = -\int_0^t b(T - s, \cdot, Y_s)ds + \xi_t^\pi + K_t \ge 0, \qquad (28)$$

where $\xi_t^\pi \stackrel{\triangle}{=} -\Lambda_T^\pi + \Lambda_{T-t}^\pi + S_T - S_{T-t}$, $K \nearrow$, and $\int_0^\infty Y_t dK_s = 0$.

A "Reflected SDE"

Definition

A pair of processes (Y, K) is the solution of (28) if

i) $(Y, K) \in \mathbb{D}^2$ and (Y, K) satisfies (28);

ii) $Y_t \geq 0$, $\forall t \geq 0$;

iii) K is increasing, with "jump set" $\mathscr{S}_K = \{t : \Delta K_t \neq 0\}$;

iv) $\displaystyle\int_0^\infty Y_s dK_s = 0$;

v) $\Delta K_t = |Y_t + \Delta\xi_t^\pi|$, $\forall t \in \mathscr{S}_K = \{t \geq 0 : Y_t + \Delta\xi_t^\pi < 0\}$.

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Warning:

The solution of SDEDR (28) is not adapted! It is solved pathwisely as an ODE with reflection. Further, since ξ_t^π has only upward jump by definition, K is always continuous!

Remark

The reflected SDE is solved by using the solution to the "Discontinuous Skorohod Problem (DSP)" (cf. e.g., Dupuis-Ishii (90) or Ma (92)).

An important property of DSP (Dupuis-Ishii (90))

For any $Y \in D$, the *solution mapping* of DRP(Y), as a mapping $\Gamma : D \to D$ such that $\Gamma(Y) = X$, where (X, K) is the solution to DRP(Y), is *Lipschitz* under the uniform topology in $\mathbb{D}$, that is, there exists a constant $C > 0$, such that, for any $Y^1, Y^2 \in D$, it holds that

$$\sup_{0 \le s \le t} |\Gamma(Y^1)_s - \Gamma(Y^2)_s| \le C \sup_{0 \le s \le t} |Y^1_s - Y^2_s|, \quad \forall t \ge 0. \tag{29}$$

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The reflected SDE is then $Y_t = \Gamma(Z)_t = Z_t + K_t$, and Z satisfies

$$Z_t = -\int_0^t b(s, \Gamma(Z)_s, \cdot)ds + \xi_t, \qquad t \ge 0,$$

Ruin Probability via Storage Process

Let Y be the storage proc. Set $\widetilde{Y}_t = Y_{T-t}$, $J_t = K_T - K_{T-t}$, then

$$\widetilde{Y}_t = Y_T + \int_0^t b(s, \widetilde{Y}_s, \cdot)ds + \Lambda_t - S_t - J_t.$$

$$\Longrightarrow \quad X_t - \widetilde{Y}_t = x - Y_T + \int_0^t \alpha_s(X_s - Y_s)ds + J_t.$$

where $\alpha_s \triangleq \frac{b(s,X_s,\cdot)-b(s,\widetilde{Y}_s,\cdot)}{(X_s-\widetilde{Y}_s)}\mathbf{1}_{\{X_s-\widetilde{Y}_s\neq 0\}}$.

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$$X_t - \widetilde{Y}_t = (x - Y_T)e^{\int_0^t \alpha_s ds} + \int_0^t e^{\int_v^t \alpha_s ds} dJ_v \geq (x - Y_T)e^{\int_0^t \alpha_s ds}.$$

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Thus $x \geq Y_T \implies X_t \geq \widetilde{Y}_t \geq 0, \forall t \implies \tau \geq T$

$$\Longrightarrow \qquad P\{\tau < T\} \leq P\{Y_T > x\}.$$

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With some more work, one can show that the equality holds.

To consider the Large Deviation problem, we now emphasize the dependence of the coefficients on the initial reserve x:

$$dX_t = b(t, x, X_t)ds + d\Lambda_t(x) - dS_t, \quad X_0 = x, \tag{30}$$

where S_t is compound Poisson, and $d\Lambda_t(x) = \sigma_t(x)dW_t$.

Ruin Probability via Storage Process

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Example

- ("perturbed risk reserve") $b(t, x, X_t) = r_t X_t + c_t$ and $\sigma_t(x) = \varepsilon$.
- (Buy-and-hold) $\pi_t \equiv f(x)$. That is,

$$b(t, x, X_t) = r_t X_t + c(1 + \rho(t, X_t)),$$
$$\sigma_t(x) = \sigma_t^T f(x).$$

Relation with Large Deviation

Recall the Lundberg bounds

$$\psi(x,T) \le e^{-\delta x} E \sup_{0\le t\le T} \exp(\widetilde{K}_t^\delta), \tag{31}$$

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Denote the adjustment coefficient by

$$\widetilde{\delta} = \sup\{\delta \in \mathscr{D} : E \sup_{t\ge 0} \exp(\widetilde{K}_t^\delta) < \infty\},$$

$$\widetilde{\delta}_T = \sup\{\delta \in \mathscr{D} : E \sup_{0\le t\le T} \exp(K_t^\delta) < \infty\}.$$

Then for all $\varepsilon > 0$ it holds that

$$\lim_{x\to\infty} \psi(x) e^{(\tilde{\delta}-\varepsilon)x} = 0, \quad \lim_{x\to\infty} \psi(x,T) e^{(\widetilde{\delta}_T-\varepsilon)x} = 0,$$

$$\lim_{x\to\infty} \psi(x) e^{(\tilde{\delta}+\varepsilon)x} = \infty, \quad \lim_{x\to\infty} \psi(x,T) e^{(\tilde{\delta}_T+\varepsilon)x} = \infty.$$

Asymptotics via Large Deviation

- Consider the reflected “random” DE

$$Y_t(x) = -\int_0^t b(T-s, x, Y_s(x))ds + \xi_t(x) + K_t(x), \quad (33)$$

where $\xi_t(x) \triangleq -\Lambda_T(x) + \Lambda_{T-t}(x) + S_T - S_{T-t}$, and $K_t(x)$ is the *reflecting* process.

- By definition of the storage process we have

$$\psi(1/\varepsilon, T) = P\{Y_T(1/\varepsilon) > 1/\varepsilon\} = P\{\varepsilon Y_T(1/\varepsilon) > 1\}.$$

Thus the asymptotic ruin is

$$\lim_{\varepsilon\to 0} \varepsilon \log P\{\varepsilon Y_T(1/\varepsilon) > 1\} = -\tilde{\delta}_T.$$

— A problem of *(Sample-Path) Large Deviation* for the (perturbed) storage process $Y_t^\varepsilon \triangleq \varepsilon Y_t(1/\varepsilon)$!

References

- Ma, J. (1993), *Discontinuous Reflection, and a Class of Singular Stochastic Control Problems for Diffusions. Stochastic and Stochastics Reports*, Vol.44, 225–252.
- Ma, J. & Sun, X. (2003) *Ruin probabilities for insurance models involving investments*, *Scand. Actuarial J.* Vol. 3, 217-237.
- Sun, X. (2001) *Ruin Probabilities for General Insurance Models*. Ph.D Thesis, Purdue University.
- Tomasz, R. & et al. (1999) *Stochastic processes for insurance and finance*, J. Wiley, New York.