Stratonovich Calculus for FBM with Parameter Less than $1/2$

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Contents

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Stochastic integration

We consider

$$\int_0^T \cdot \circ dB_s.$$

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Here, the stochastic integral is in the Stratonovich sense and B is a fBm with Hurst parameter $H \in (0, 1/2)$.

Contents

FBM

The process B satisfies

$$B_t = \int_0^t K(t,s)dW_s, \quad t \in [0, T].$$

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Here :

$$\begin{aligned} R_H(t,s) &= \int_0^{s\wedge t} K(t,r)K(s,r)dr \\ &= \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad t,s \in [0, T]. \end{aligned}$$

FBM

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$$B_t = \int_0^t K(t,s)dW_s, \quad t \in [0,T].$$

Here :

- $R_H(t,s) = \int_0^{s\wedge t} K(t,r)K(s,r)dr, \quad t,s \in [0,T].$
- $$K(t,s) = c_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \quad s < t,$$

  with

  $$F_1(z) = c_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1-(\theta+1)^{H-\frac{1}{2}}\right)d\theta.$$

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- $|K(t,s)| \le c((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}})$.

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- $|K(t,s)| \le c((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}}).$
- $\left|\frac{\partial K}{\partial t}(t,s)\right| \le c(t-s)^{H-\frac{3}{2}}.$

Seminorm

We consider,

$$\begin{aligned} ||\varphi||_K^2 : \;&= \int_0^T \varphi^2(s) K(T,s)^2 ds \\ &+ \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)| (t-s)^{H-\frac{3}{2}} dt \right)^2 ds, \quad \varphi \in \mathcal{E}. \end{aligned}$$

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and $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to $||\cdot||_K$.

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$\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to $||\cdot||_K$ and the space $\mathbb{D}^{1,2}(\mathcal{H}_K)$.

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$\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to $||\cdot||_K$ and the space $\mathbb{D}^{1,2}(\mathcal{H}_K)$:
A process $\{u_t; t \in [0,T]\} \in \mathbb{D}^{1,2}(\mathcal{H}_K)$ iff there exists a sequence $\{\varphi_n\}_n$ of $\mathcal{H}_K$-valued processes

$$\varphi_n = \sum_{j=0}^{n-1} F_j 1_{(t_j, t_{j+1}]},$$

where $F_j \in \mathcal{S}$ and $0 = t_0 < t_1 < \ldots < t_n = T$.

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where $F_j \in \mathcal{S}$ and $0 = t_0 < t_1 < \ldots < t_n = T$, such that

$$E\left(||u - \varphi_n||_K^2\right) + E\left(\int_0^T ||D_r(u - \varphi_n)||_K^2 dr\right) \to 0, \quad n \to \infty.$$

Divergence operator

- $|K(t,s)| \leq c((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}})$.
- $\left|\frac{\partial K}{\partial t}(t,s)\right| \leq c(t-s)^{H-\frac{3}{2}}$.

Theorem

$\mathbb{D}^{1,2}(\mathcal{H}_K) \subset Dom\,\delta$ and

$$E\left(||u-\varphi_n||_K^2\right) + E\left(\int_0^T ||D_r(u-\varphi_n)||_K^2 dr\right) \to 0$$

implies

$$E\left((\delta(u)-\delta(\varphi_n))^2\right) \to 0.$$

Contents

Stratonovich integral

Definition (Russo and Vallois)

We say that a process u with integrable paths belongs to $Dom\,\delta_S^B$ if and only if

$$(2\varepsilon)^{-1}\int_0^T u_s(B_{(s+\varepsilon\wedge T)} - B_{(s-\varepsilon\vee 0)})ds$$

converges in probability as $\varepsilon \downarrow 0$.

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converges in probability as $\varepsilon \downarrow 0$. In this case we denote this limit by

$$\delta_S^B(u) \quad \text{or} \quad \int_0^T u_s \circ dB_s.$$

Trace of the Stratonovich integral

We also need

Definition

We say that a process $u \in \mathbb{D}^{1,2}(\mathcal{H}_K)$ belongs to $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$ if the limit in probability

$$TrDu := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T \langle D^B u_s, 1_{(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]} \rangle_{\mathcal{H}} ds.$$

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exists.

Remark Remembert that $\mathcal{H}$ is the completion of $\mathcal{E}$ with respect to R_H.

Stratonovich integral

Theorem

Let $u \in \mathbb{D}_C^{1,2}(\mathcal{H}_K)$ *be such that*

$$E\left(\int_0^T u_s^2(s^{2H-1} + (T-s)^{2H-1})ds\right) < \infty,$$

and

$$E\left(\int_0^T \int_0^T (D_r u_s)^2(s^{2H-1} + (T-s)^{2H-1})dsdr\right) < \infty.$$

Then, $u \in Dom\,\delta_S^B \cap Dom\,\delta^B$ *and*

$$\delta_S^B(u) = \delta^B(u) + TrDu.$$

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Then, $u \in Dom\,\delta_S^B \cap Dom\,\delta^B$ and

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$$\delta_S^B(u) = \delta^B(u) + TrDu.$$

Proof : The Fubini theorem gives

$$\begin{aligned}
&(2\varepsilon)^{-1}\int_0^T u_s(B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0})ds \\
&= (2\varepsilon)^{-1}\int_0^T \delta^B(u_s 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]})ds \\
&\quad +(2\varepsilon)^{-1}\int_0^T \langle D_\cdot^B u_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\rangle_{\mathcal{H}} ds
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Let $u \in \mathbb{D}_C^{1,2}(\mathcal{H}_K)$. *Then,* $u \in Dom\,\delta_S^B \cap Dom\,\delta^B$ *and*

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Using that $u \in \mathbb{D}_C^{1,2}(\mathcal{H}_K)$ we get that B^ε converges to $TrDu$ in probability as $\varepsilon \downarrow 0$.

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& + (2\varepsilon)^{-1}\int_0^T \langle D^B_{\cdot} u_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\rangle_{\mathcal{H}} ds \\
= \ & \int_0^T u_r^{\varepsilon} dB_r + B^{\varepsilon}.
\end{aligned}
$$

So, we need to show that u^{ε} converges to u in the norm of $\mathbb{D}^{1,2}(\mathcal{H}_K)$ in order to see that $\int_0^T u_r^{\varepsilon} dB_r$ converges to $\delta^B(u)$ in $L^2(\Omega)$ as ε tends to zero.

Stratonovich integral

Step 1 We first assume that is a simple process of the form

$$u = \sum_{j=0}^{n-1} F_j 1_{(t_j, t_{j+1}]}.$$

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Hence, Property (i) of the kernel K, the fact that u is bounded and the dominated convergence theorem imply

$$E \int_0^T (u_s - u_s^\varepsilon)^2 K(T,s)^2 ds \longrightarrow 0 \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

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Now, using that $u_t - u_s = 0$ for $s, t \in [t_{i,} t_{i+1}]$ we obtain

$$\begin{aligned}
& \int_{t_i}^{t_{i+1}} \left(\int_s^T \left| u_t^\varepsilon - u_s^\varepsilon - (u_t - u_s) \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
\leq \; & 2 \int_{t_i}^{t_{i+1}} \left(\int_s^{t_{i+1}} \left| u_t^\varepsilon - u_s^\varepsilon \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
& + 2 \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^T \left| u_t^\varepsilon - u_s^\varepsilon - (u_t - u_s) \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
= \; & 2A_1(i, \varepsilon) + 2A_2(i, \varepsilon).
\end{aligned}$$

Stratonovich integral

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&\int_{t_i}^{t_{i+1}} \left(\int_s^T \left| u_t^\varepsilon - u_s^\varepsilon - (u_t - u_s) \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
\leq\ & 2 \int_{t_i}^{t_{i+1}} \left(\int_s^{t_{i+1}} \left| u_t^\varepsilon - u_s^\varepsilon \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
&+2 \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^T \left| u_t^\varepsilon - u_s^\varepsilon - (u_t - u_s) \right| (t-s)^{-1-\alpha} dt \right)^2 ds \\
=\ & 2A_1(i,\varepsilon) + 2A_2(i,\varepsilon).
\end{aligned}$$

$A_2(i,\varepsilon)$ goes to 0, as $\varepsilon \downarrow 0$, because of the dominated convergence theorem and the fact that u is a bounded process.

Stratonovich integral

Also we have

$$
\begin{aligned}
& A_1(i,\varepsilon) \\
\leq\ & 8\int_{t_i}^{t_i+2\varepsilon}\left(\int_s^{t_i+2\varepsilon}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds \\
& +8\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}\left(\int_s^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds \\
& +8\int_{t_i}^{t_i+2\varepsilon}\left(\int_{t_i+2\varepsilon}^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds \\
& +8\int_{t_i}^{t_{i+1}-2\varepsilon}\left(\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds.
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&\quad +8\int_{t_i}^{t_{i+1}-2\varepsilon}\left(\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|\,(t-s)^{-1-\alpha}dt\right)^2 ds.
\end{aligned}
$$

The first and second integrals converge to zero, due to the estimate

$$|u_t^\varepsilon-u_s^\varepsilon|\leq\frac{c}{\varepsilon}|t-s|.$$

Stratonovich integral

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&A_1(i,\varepsilon)\\
&\leq \; 8\int_{t_i}^{t_i+2\varepsilon}\left(\int_s^{t_i+2\varepsilon}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&\quad +8\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}\left(\int_s^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&\quad +8\int_{t_i}^{t_i+2\varepsilon}\left(\int_{t_i+2\varepsilon}^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&\quad +8\int_{t_i}^{t_{i+1}-2\varepsilon}\left(\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}|u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds.
\end{aligned}
$$

The third and fourth term of the above expression converge to zero because u_t^ε is bounded.

Stratonovich integral

Step 2 Fix $\delta > 0$ and a bounded simple $\mathcal{H}_K$-valued processes φ such that

$$E\,\|u-\varphi\|_K^2 + E\int_0^T ||D_r u - D_r\varphi||_K^2\, dr \leq \delta.$$

Stratonovich integral

Step 2 Fix $\delta > 0$ and a bounded simple $\mathcal{H}_K$-valued processes φ such that

$$E\left\|u-\varphi\right\|_K^2 + E\int_0^T \left|\left|D_r u - D_r\varphi\right|\right|_K^2 dr \leq \delta.$$

Then, Step 1 implies that for ε small enough,

$$\begin{aligned} & E\left\|u-u^\varepsilon\right\|_K^2 + E\int_0^T \left|\left|D_r\left(u-u^\varepsilon\right)\right|\right|_K^2 dr \\ \leq\ & cE\left\|u-\varphi\right\|_K^2 + cE\int_0^T \left|\left|D_r\left(u-\varphi\right)\right|\right|_K^2 dr \\ & + cE\left\|\varphi-\varphi^\varepsilon\right\|_K^2 + cE\int_0^T \left|\left|D_r\left(\varphi-\varphi^\varepsilon\right)\right|\right|_K^2 dr \\ & + cE\left\|\varphi^\varepsilon-u^\varepsilon\right\|_K^2 + cE\int_0^T \left|\left|D_r\left(\varphi^\varepsilon-u^\varepsilon\right)\right|\right|_K^2 dr \\ \leq\ & 2c\delta + cE\left\|\varphi^\varepsilon-u^\varepsilon\right\|_K^2 + cE\int_0^T \left|\left|D_r\left(\varphi^\varepsilon-u^\varepsilon\right)\right|\right|_K^2 dr. \end{aligned}$$

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$$E\,\|u - \varphi\|_K^2 + E\int_0^T ||D_r u - D_r\varphi||_K^2\,dr \le \delta.$$

We have

$$\begin{aligned}
&\int_0^T E\,(\varphi_s^\varepsilon - u_s^\varepsilon)^2 \;\; K(T,s)^2 ds \\
\le\;& \int_0^T E\left(\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}(\varphi_r - u_r)\,dr\right)^2 \;\; K(T,s)^2 ds \\
\le\;& \int_0^T E\,(\varphi_r - u_r)^2 \;\; \left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} K(T,s)^2 ds\right) dr < \delta
\end{aligned}$$

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$$E\left\|u-\varphi\right\|_K^2 + E\int_0^T \left|\left|D_r u - D_r\varphi\right|\right|_K^2 dr \le \delta.$$

We have

$$\begin{aligned}
&\int_0^T E\left(\varphi_s^\varepsilon - u_s^\varepsilon\right)^2 \ K(T,s)^2 ds \\
\le\ &\int_0^T E\left(\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}\left(\varphi_r - u_r\right)dr\right)^2 \ K(T,s)^2 ds \\
\le\ &\int_0^T E\left(\varphi_r - u_r\right)^2 \ \left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} K(T,s)^2 ds\right) dr < \delta
\end{aligned}$$

due to $(2\varepsilon)^{-1} \ \int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} K(T,t)^2 dt \le c\left[(T-r)^{-2\alpha} + r^{-2\alpha}\right].$

Stratonovich integral

Theorem

Let $u \in \mathbb{D}^{1,2}_C(\mathcal{H}_K)$. *Then,* $u \in Dom\, \delta^B_S \cap Dom\, \delta^B$ *and*

$$\delta^B_S(u) = \delta^B(u) + TrDu.$$

Remark The results of this section can be easily generalized to a centered Gaussian process of the form $B_t = \int_0^t K(t,s)dW_s$, where $K(t,s)$ is a continuously differentiable kernel in the region $\{0 < s < t < T\}$ satisfying :

- $|K(t,s)| \leq c((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}})$.
- $\left|\frac{\partial K}{\partial t}(t,s)\right| \leq c((t-s)^{H-\frac{3}{2}}$.

Example

Let F be a continuously differentiable function satisfying the growth condition

$$\max\{|F(x)|, |F'(x)|\} \leq ce^{\lambda|x|^2},$$

where c and λ are positive constants such that $\lambda < T^{-2H}$.

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$$\max\{|F(x)|, |F'(x)|\} \leq ce^{\lambda|x|^2},$$

where c and λ are positive constants such that $\lambda < T^{-2H}$. We know that if $H > \frac{1}{4}$, the process $u_t = F(B_t)$ belongs to the space $L^2(\Omega; \mathcal{H}_K)$.

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Let F be a continuously differentiable function satisfying the growth condition

$$\max\{|F(x)|, |F'(x)|\} \leq c e^{\lambda |x|^2},$$

where c and λ are positive constants such that $\lambda < T^{-2H}$.
Also

$$\begin{aligned}
&(2\varepsilon)^{-1} \int_0^T F'(B_t) \left\langle 1_{[0,t]}, 1_{[(t-\varepsilon)\vee 0, (t+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} dt \\
&= \quad (2\varepsilon)^{-1} \int_0^T F'(B_t) \left(R(t, (t+\varepsilon) \wedge T) - R(t, (t-\varepsilon) \vee 0) \right) \, dt \\
&= \quad (4\varepsilon)^{-1} \int_0^T F'(B_t)((((t+\varepsilon) \wedge T)^{2H} - \; ((t-\varepsilon) \vee 0)^{2H} \\
&\qquad + ((t+\varepsilon) \wedge T - t)^{2H} - (t - (t-\varepsilon) \vee 0)^{2H})dt \\
&\longrightarrow \quad H \int_0^T F'(B_t) t^{2H-1} dt \qquad \text{as} \qquad \varepsilon \downarrow 0.
\end{aligned}$$

Example

Let F be a continuously differentiable function satisfying the growth condition

$$\max\{|F(x)|, |F'(x)|\} \leq c e^{\lambda |x|^2},$$

where c and λ are positive constants such that $\lambda < T^{-2H}$. Then

$$\int_0^T F(B_t) \circ dB_t = \int_0^T F(B_t) dB_t + H \int_0^T F'(B_t) t^{2H-1} dt.$$

Example

The forward integral of $F(B_t)$ with respect to B defined as the limit in probability, as $\varepsilon \downarrow 0$, of

$$\varepsilon^{-1} \int_0^T F(B_t) \left(B_{(t+\varepsilon)\wedge T} - B_t \right) dt,$$

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does not exist in general. For intance, for $F(x) = x$,

$$\begin{aligned}
&\varepsilon^{-1} \int_0^T \left\langle 1_{[0,t]}, 1_{[t,(t+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} dt \\
&= \varepsilon^{-1} \int_0^T \left(R(t,(t+\varepsilon)\wedge T) - R(t,t) \right) dt \\
&= \frac{1}{2\varepsilon} \int_0^T \left(((t+\varepsilon)\wedge T)^{2H} - t^{2H} - ((t+\varepsilon)\wedge T - t)^{2H} \right) dt \\
&= \frac{1}{2} \left(T^{2H} - T\varepsilon^{2H-1} + \frac{2H-1}{2H+1}\varepsilon^{2H} \right) \to -\infty.
\end{aligned}$$

Contents

Itô's formula

(C) u and $D_r u$ are $\lambda-$Hölder continuous in the norm of the space $\mathbb{D}^{1,4}$ for some $\lambda > \alpha$, and the function

$$\gamma_r = \sup_{0\le s\le T} \|D_r u_s\|_{1,4} + \sup_{0\le s\le T} \frac{\|D_r u_t - D_r u_s\|_{1,4}}{|t-s|^{\lambda}}$$

satisfies $\int_0^T \gamma_r^p dr < \infty$ for some $p > \frac{2}{1-4\alpha}$. Here $\alpha = \frac{1}{2} - H$.
Also

$$E\int_0^T u_s^2 \left(s^{-2\alpha} + (T-s)^{-2\alpha}\right) ds < \infty,$$

and

$$E\int_0^T \int_0^T (D_r u_s)^2 \left(s^{-2\alpha} + (T-s)^{-2\alpha}\right) dsdr < \infty.$$

Itô's formula

Theorem

Suppose $\alpha < \frac{1}{4}$. Let u be an adapted process in $\mathbb{D}^{2,2}(\mathcal{H}_K)$ satisfying condition (C) and such that the following limit exists in probability

$$\int_0^T \left| (\nabla u)_s - \frac{1}{2\varepsilon} \left\langle D^B u_s, \mathbf{1}_{[(t-\varepsilon)\vee 0,(t+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} \right| ds \to 0,$$

for some process $(\nabla u)_s$ in $\mathbb{L}^{1,2}$. Define $X_t = \int_0^t u_s \circ dB_s$. Then, for all $F \in \mathcal{C}_b^2(\mathbb{R})$ the process $F'(X_s)u_s$ is Stratonovich integrable with respect to B and

$$F(X_t) = F(0) + \int_0^t F'(X_s)u_s \circ dB_s.$$

Itô's formula : Proof

Proof. We know that

$$X_t = \int_0^t u_s dB_s + \int_0^t (\nabla u)_s \, ds.$$

Itô's formula : Proof

Proof. We know that

$$X_t = \int_0^t u_s dB_s + \int_0^t (\nabla u)_s \, ds.$$

Then,

$$\begin{aligned} F(X_t) &= F(0) + \int_0^t F'(X_s) u_s dB_s \\ &+ \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,r) \left(\int_0^s D_r (K_s^* u)_\theta \, dW_\theta \right) dr \right) ds \\ &+ \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 \, dr \right) ds \\ &+ \int_0^t F'(X_s) (\nabla u)_s \, ds \\ &+ \int_0^t F''(X_s) u_s \int_0^s \left(\int_r^s D_r (\nabla u)_\theta \, d\theta \right) \frac{\partial K}{\partial s}(s,r) \, drds. \end{aligned}$$

Itô's formula : Proof

Then we only need to check that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \left\langle D^B \left(F'(X_s) u_s \right), \mathbf{1}_{[(t-\varepsilon)\vee 0,(t+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds$$

and that it is equal to

$$\int_0^t F''(X_s)\, u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,r) \left(\int_0^s D_r \left(K_s^* u \right)_\theta dW_\theta \right) dr \right) ds$$
$$+ \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s \left(K_s^* u \right)_r^2 dr \right) ds$$
$$+ \int_0^t F'(X_s) \left(\nabla u \right)_s ds$$
$$+ \int_0^t F''(X_s) u_s \int_0^s \left(\int_0^s D_r \left(\nabla u \right)_\theta d\theta \right) \frac{\partial K}{\partial s}(s,r)\, dr ds.$$

Contents

SDE

Consider the equation

$$X_t = x + \int_0^t a(X_s) \circ dB_s + \int_0^t b(X_s)\, ds.$$

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$$X_t = x + \int_0^t a(X_s) \circ dB_s + \int_0^t b(X_s)\, ds.$$

Here, $H \in (\frac{1}{4}, \frac{1}{2})$, $x \in \mathbb{R}$ and a, b are bounded and measurable functions.

SDE

Proposition

Assume that $a \in \mathcal{C}_b^2(\mathbb{R})$ and $b \in \mathcal{C}_b^1(\mathbb{R})$. Then the unique solution of above equation is given by

$$X_t = \alpha\left(B_t, Y_t\right),$$

where Y_t is the solution of

$$Y_t = x + \int_0^t \left(\frac{\partial \alpha}{\partial y}\left(B_s, Y_s\right)\right)^{-1} b\left(\alpha\left(B_s, Y_s\right)\right) ds$$

and $\alpha\left(x, y\right)$ is the solution of

$$\begin{cases} \frac{\partial \alpha}{\partial x}\left(x, y\right) = a\left(\alpha\left(x, y\right)\right) \\ \alpha\left(0, y\right) = y. \end{cases}$$

SDE : Proof

For any $\varepsilon > 0$, set

$$B_t^{\varepsilon} = \frac{1}{2\varepsilon} \int_0^t \left(B_{(s+\varepsilon)\wedge T} - B_{(s+\varepsilon)\vee 0} \right) ds$$

and

$$X_t^{\varepsilon} = \alpha \left(B_t^{\varepsilon}, Y_t \right).$$

SDE : Proof

Set $B_t^\varepsilon = \frac{1}{2\varepsilon}\int_0^t \left(B_{(s+\varepsilon)\wedge T} - B_{(s+\varepsilon)\vee 0}\right) ds$ and $X_t^\varepsilon = \alpha\left(B_t^\varepsilon, Y_t\right)$. Then,

$$
\begin{aligned}
X_t^\varepsilon & \\
= \; & x + \frac{1}{2\varepsilon}\int_0^t a\left(\alpha\left(B_s^\varepsilon, Y_s\right)\right)\left(B_{(s+\varepsilon)\wedge T} - B_{(s+\varepsilon)\vee 0}\right) ds \\
& + \int_0^t \left(\frac{\partial \alpha}{\partial y}\left(B_s^\varepsilon, Y_s\right)\right)\left(\frac{\partial \alpha}{\partial y}\left(B_s, Y_s\right)\right)^{-1} b\left(\alpha\left(B_s, Y_s\right)\right) ds \\
= \; & x + \frac{1}{2\varepsilon}\int_0^t a\left(\alpha\left(B_s, Y_s\right)\right)\left(B_{(s+\varepsilon)\wedge T} - B_{(s+\varepsilon)\vee 0}\right) ds \\
& + \frac{1}{2\varepsilon}\int_0^t \left[a\left(\alpha\left(B_s^\varepsilon, Y_s\right)\right) - a\left(\alpha\left(B_s, Y_s\right)\right)\right]\left(B_{(s+\varepsilon)\wedge T} - B_{(s+\varepsilon)\vee 0}\right) ds \\
& + \int_0^t \left(\frac{\partial \alpha}{\partial y}\left(B_s^\varepsilon, Y_s\right)\right)\left(\frac{\partial \alpha}{\partial y}\left(B_s, Y_s\right)\right)^{-1} b\left(\alpha\left(B_s, Y_s\right)\right) ds.
\end{aligned}
$$

SDE

Proposition

Assume that $a \in \mathcal{C}_b^2(\mathbb{R})$ *and* $b \in \mathcal{C}_b^1(\mathbb{R})$. *Then,*

$$X_t = \alpha\left(B_t, Y_t\right),$$

where Y_t *and* $\alpha\left(x, y\right)$ *satisfy*

$$Y_t = x + \int_0^t \left(\frac{\partial \alpha}{\partial y}\left(B_s, Y_s\right)\right)^{-1} b\left(\alpha\left(B_s, Y_s\right)\right) ds$$

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Remark Neuenkirch and Nourdin (2007) have shown that this result still hold for $H > 1/2$.