Fractional Brownian motion

Jorge A. León

Departamento de Control Automático
Cinvestav del IPN

Spring School "Stochastic Control in Finance", Roscoff 2010

Contents

Contents

Stochastic integration

We consider

$$\int_0^T \cdot \, dB_s.$$

Stochastic integration

We consider

$$\int_0^T \cdot \, dB_s.$$

Here B is a fractional Brownian motion.

Contents

Fractional Brownian motion

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ if it has zero mean and covariance fuction

$$R_H(t, s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has zero mean and covariance fuction

$$R_H(t,s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B is a Brownian motion for $H = 1/2$.

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has zero mean and covariance fuction

$$R_H(t,s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has zero mean and covariance fuction

$$R_H(t,s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments :

$$E\left(|B_t - B_s|^2\right) = |t-s|^{2H}.$$

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ if it has zero mean and covariance fuction

$$R_H(t, s) = E(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments :

$$E\left(|B_t - B_s|^2\right) = |t - s|^{2H}.$$

- For any $\varepsilon \in (0, H)$ and $T > 0$, there exists $G_{\varepsilon,T}$ such that

$$|B_t - B_s| \leq G_{\varepsilon,T}|t - s|^{H-\varepsilon}, \quad t, s \in [0, T].$$

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has zero mean and covariance fuction

$$R_H(t,s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H} B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.

Properties of fBm

Definition

A Gaussian stochastic process $B = \{B_t; t \geq 0\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has zero mean and covariance fuction

$$R_H(t,s) = E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H} B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.
- The covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals.

Properties of fBm

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H}B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.
- The covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals : Let $t - s = nh$ and

$$\begin{aligned}\rho_H(n) &= E\left[(B_{t+h} - B_t)(B_{s+h} - B_s)\right] \\ &\approx h^{2H} H(2H-1)n^{2H-2} \to 0.\end{aligned}$$

Properties of fBm

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H}B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.
- The covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals : Let $t - s = nh$ and

$$\begin{aligned} \rho_H(n) &= E\left[(B_{t+h} - B_t)(B_{s+h} - B_s)\right] \\ &\approx h^{2H}H(2H-1)n^{2H-2} \to 0. \end{aligned}$$

i) If $H > 1/2$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho(n) = \infty$.

Properties of fBm

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H} B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.
- The covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals : Let $t - s = nh$ and

$$\begin{aligned} \rho_H(n) &= E\left[(B_{t+h} - B_t)(B_{s+h} - B_s)\right] \\ &\approx h^{2H} H(2H-1) n^{2H-2} \to 0. \end{aligned}$$

i) If $H > 1/2$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho(n) = \infty$.
ii) If $H < 1/2$, $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} \rho(n) < \infty$.

Properties of fBm

- B is a Brownian motion for $H = 1/2$.
- B has stationary increments.
- B is Hölder continuous for any exponent less than H.
- B is self-similar (with index H). That is, for any $a > 0$, $\{a^{-H} B_{at}; t \geq 0\}$ and $\{B_t; t \geq 0\}$ have the same distribution.
- The covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals : Let $t - s = nh$ and

$$\begin{aligned} \rho_H(n) &= E\left[(B_{t+h} - B_t)(B_{s+h} - B_s)\right] \\ &\approx h^{2H} H(2H-1) n^{2H-2} \to 0. \end{aligned}$$

i) If $H > 1/2$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho(n) = \infty$.
ii) If $H < 1/2$, $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} \rho(n) < \infty$.

- B has no bounded variation paths.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$. Let $\Pi_t = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of $[0, t]$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$. Let $\Pi_t = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of $[0, t]$. Then,

$$\begin{aligned} E\left(\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2\right) &= \sum_{i=1}^{n} |t_i - t_{i-1}|^{2H} \\ &\leq |\Pi|^{2H-1} \sum_{i=1}^{n} |t_i - t_{i-1}| \\ &= t|\Pi|^{2H-1} \to 0 \end{aligned}$$

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$. Let $\Pi_t = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of $[0, t]$. Then,

$$\begin{aligned} E\left(\sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}|^2\right) &= \sum_{i=1}^n |t_i - t_{i-1}|^{2H} \\ &\leq |\Pi|^{2H-1} \sum_{i=1}^n |t_i - t_{i-1}| \\ &= t|\Pi|^{2H-1} \to 0 \end{aligned}$$

If B were a semimartingale. Then, $B = M + V$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$. Let $\Pi_t = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of $[0, t]$. Then,

$$\begin{aligned} E\left(\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2\right) &= \sum_{i=1}^{n} |t_i - t_{i-1}|^{2H} \\ &\leq |\Pi|^{2H-1} \sum_{i=1}^{n} |t_i - t_{i-1}| \\ &= t|\Pi|^{2H-1} \to 0 \end{aligned}$$

If B were a semimartingale. Then, $B = M + V$. Thus

$$0 = [B] = [M].$$

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H > 1/2$. Let $\Pi_t = \{0 = t_0 < t_1 < \ldots < t_n\}$ be a partition of $[0, t]$. Then,

$$E\left(\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2\right) = \sum_{i=1}^{n} |t_i - t_{i-1}|^{2H} \leq |\Pi|^{2H-1} \sum_{i=1}^{n} |t_i - t_{i-1}| = t|\Pi|^{2H-1} \to 0$$

If B were a semimartingale. Then, $B = M + V$. Thus

$$0 = [B] = [M].$$

Consequently $B = V$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$*.*

Proof : (i) Case $H < 1/2$.

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H < 1/2$. We have

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H < 1/2$. We have

$$
\begin{aligned}
I_n : &= \sum_{j=1}^{n} \left| B_{j/n} - B_{(j-1)/n} \right|^2 \overset{(d)}{=} \frac{1}{n^{2H}} \sum_{j=1}^{n} \left| B_j - B_{j-1} \right|^2 \\
&= n^{1-2H} \left(\frac{1}{n} \sum_{j=1}^{n} \left| B_j - B_{j-1} \right|^2 \right) \to \infty.
\end{aligned}
$$

FBM is not a semimartingale

Theorem

B is not a semimartingale for $H \neq 1/2$.

Proof : (i) Case $H < 1/2$. We have

$$
\begin{aligned}
I_n : &= \sum_{j=1}^{n} |B_{j/n} - B_{(j-1)/n}|^2 \stackrel{(d)}{=} \frac{1}{n^{2H}} \sum_{j=1}^{n} |B_j - B_{j-1}|^2 \\
&= n^{1-2H} \left(\frac{1}{n} \sum_{j=1}^{n} |B_j - B_{j-1}|^2 \right) \to \infty.
\end{aligned}
$$

Due to, the ergodic theorem implies that

$$
\frac{1}{n} \sum_{j=1}^{n} |B_j - B_{j-1}|^2 \to E((B_1)^2) \quad \text{a.s}
$$

.

Contents

Mandelbrot-van Ness representation

$$B_t = C_H \left[\int_{\infty}^{0} \{(t-s)^{H-1/2} - (-s)^{H-1/2}\} dW_s + \int_0^t (t-s)^{H-1/2} dW_s \right].$$

Mandelbrot-van Ness representation

$$B_t = C_H \left[\int_{\infty}^{t} \{(t-s)^{H-1/2} - (-s)^{H-1/2}\} dW_s + \int_0^t (t-s)^{H-1/2} dW_s \right].$$

Here W is a Brownian motion.

Representation of fBm on an finite interval

Fix a time interval $[0, T]$ and consider the fBm $B = \{B_t; t \in [0, T]\}$.

Representation of fBm on an finite interval

Fix a time interval $[0, T]$ and consider the fBm $B = \{B_t; t \in [0, T]\}$. Then there exists a Bm $\{W_t; t \in [0, T]\}$ such that

$$B_t = \int_0^t K_H(t, s) dW_s,$$

Representation of fBm on an finite interval

Fix a time interval $[0, T]$ and consider the fBm $B = \{B_t; t \in [0, T]\}$. Then there exists a Bm $\{W_t; t \in [0, T]\}$ such that

$$B_t = \int_0^t K_H(t, s) dW_s,$$

where

- For $H > 1/2$,

$$K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du \quad s < t.$$

Representation of fBm on an finite interval

Fix a time interval $[0, T]$ and consider the fBm $B = \{B_t; t \in [0, T]\}$. Then there exists a Bm $\{W_t; t \in [0, T]\}$ such that

$$B_t = \int_0^t K_H(t, s) dW_s,$$

where

- For $H > 1/2$,

$$K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du, \quad s < t.$$

- For $H < 1/2$,

$$K_H(t, s) = c_H \left[\left(\frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}} (u - s)^{H - \frac{1}{2}} du \right], \quad s < t.$$

Contents

Wiener integrals

Let $\mathcal{E}$ be the family of the step functions of the form

$$f = \sum_{i=0}^{n} a_i I_{(t_j, t_{j+1}]}.$$

Wiener integrals

Let $\mathcal{E}$ be the family of the step functions of the form

$$f = \sum_{i=0}^{n} a_i I_{(t_j, t_{j+1}]}.$$

The Wiener integral with respect to B

$$I(f) = \sum_{i=0}^{n} a_i (B_{t_{i+1}} - B_{t_i})$$

and the space

$$\mathcal{L}(B) = \{X \in L^2(\Omega) : X = L^2(\Omega) - \lim_{n \to \infty} I(f_n), \text{ for some } \{f_n\} \subset \mathcal{E}\}.$$

Wiener integrals

$$\mathcal{L}(B) = \{X \in L^2(\Omega) : X = L^2(\Omega) - \lim_{n\to\infty} I(f_n), \text{ for some } \{f_n\} \subset \mathcal{E}\}.$$

Proposition (Pipiras and Taqqu)

Suppose that $\mathcal{H}$ is a inner product space with inner product $(\cdot, \cdot)$ such that :

i) $\mathcal{E} \subset \mathcal{H}$ *and* $(f, g) = E(I(f)I(g))$, *for* $f, g \in \mathcal{E}$.

ii) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Then $\mathcal{H}$ is isometric to $\mathcal{L}(B)$ if and only if $\mathcal{H}$ is complete.

Wiener integrals

$$\mathcal{L}(B) = \{X \in L^2(\Omega) : X = L^2(\Omega) - \lim_{n\to\infty} I(f_n), \text{ for some } \{f_n\} \subset \mathcal{E}\}.$$

Proposition (Pipiras and Taqqu)

Suppose that $\mathcal{H}$ is a inner product space with inner product $(\cdot, \cdot)$ such that :

i) $\mathcal{E} \subset \mathcal{H}$ *and* $(f, g) = E(I(f)I(g))$, *for* $f, g \in \mathcal{E}$.

ii) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Then $\mathcal{H}$ is isometric to $\mathcal{L}(B)$ if and only if $\mathcal{H}$ is complete. Moreover, the isometry is an extension of I.

Wiener integrals

Proposition (Pipiras and Taqqu)

Suppose that $\mathcal{H}$ is a inner product space with inner product $(\cdot,\cdot)$ such that :

i) $\mathcal{E} \subset \mathcal{H}$ *and* $(f,g) = E(I(f)I(g))$, *for* $f,g \in \mathcal{E}$.

ii) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Then $\mathcal{H}$ *is isometric to* $\mathcal{L}(B)$ *if and only if* $\mathcal{H}$ *is complete. Moreover, the isometry is an extension of* I.

Remarks

a) For $H < 1/2$,

$$\mathcal{H} = \{f \in L^2([0,T]) : f(s) = c_H s^{\frac{1}{2}-H}(I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\phi_f(u))(s) \quad \text{for some } \phi_f \in L^2\}$$

with $(I_{T-}^{\alpha} g)(s) = \frac{1}{\Gamma(\alpha)}\int_s^T (x-s)^{\alpha-1} g(x)dx$.

Wiener integrals

Proposition (Pipiras and Taqqu)

Suppose that $\mathcal{H}$ is a inner product space with inner product $(\cdot,\cdot)$ such that :

i) $\mathcal{E} \subset \mathcal{H}$ *and* $(f,g) = E(I(f)I(g))$, *for* $f,g \in \mathcal{E}$.

ii) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Then $\mathcal{H}$ *is isometric to* $\mathcal{L}(B)$ *if and only if* $\mathcal{H}$ *is complete. Moreover, the isometry is an extension of* I.

Remarks

a) For $H < 1/2$,

$$\mathcal{H} = \{f \in L^2([0,T]) : f(s) = c_H s^{\frac{1}{2}-H}(I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\phi_f(u))(s) \text{ for some } \phi_f \in L^2\}$$

with the inner product $(f,g) = (\phi_f, \phi_g)_{L^2([0,T])}$.

Wiener integrals

Proposition (Pipiras and Taqqu)

Suppose that $\mathcal{H}$ is a inner product space with inner product $(\cdot, \cdot)$ such that :

i) $\mathcal{E} \subset \mathcal{H}$ *and* $(f, g) = E(I(f)I(g))$, *for* $f, g \in \mathcal{E}$.

ii) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Then $\mathcal{H}$ *is isometric to* $\mathcal{L}(B)$ *if and only if* $\mathcal{H}$ *is complete. Moreover, the isometry is an extension of* I.

Remarks

b) For $H > 1/2$,

$$\mathcal{H} = \{f \in \mathcal{D}' : \exists f^* \in W^{1/2-H,2}(\mathbb{R}) \text{ with } \mathrm{supp}(f) \subset [0, T] \text{ such that } f = f^*|_{[0,T]}\}$$

with the inner product $(f, g) = c_H \int_R \mathcal{F} f^*(x)\overline{\mathcal{F} g^*(x)}|x|^{1-2H} dx$.

Wiener integrals

a) For $H < 1/2$,

$$\mathcal{H} = \{f \in L^2([0,T]) : f(s) = c_H s^{\frac{1}{2}-H}(I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \phi_f(u))(s)$$

$$\text{for some } \phi_f \in L^2\}$$

with the inner product $(f,g) = (\phi_f, \phi_g)_{L^2([0,T])}$.

b) For $H > 1/2$,

$$\mathcal{H} = \{f \in \mathcal{D}' : \exists f^* \in W^{1/2-H,2}(\mathbb{R}) \text{ with } \mathrm{supp}(f) \subset [0,T]$$

$$\text{such that } f = f^*|_{[0,T]}\}$$

with the inner product $(f,g) = c_H \int_R \mathcal{F}f^*(x)\overline{\mathcal{F}g^*(x)}|x|^{1-2H}dx$.

c) $W^{s,2}(\mathbb{R}) = \{f \in \mathcal{S} : (1+|x|^2)^{s/2}\mathcal{F}f(x) \in L^2(\mathbb{R})\}$.

Representation of Wiener integrals

Moreover, there exists an isometry $K_H^* : \mathcal{H} \to L^2([0,T])$ and a Brownian motion W such that :

1. $I(f) = \int_0^T (K_H^* f)(s) dW_s$.

Representation of Wiener integrals

Moreover, there exists an isometry $K_H^*: \mathcal{H} \to L^2([0,T])$ and a Brownian motion W such that :

1. $I(f) = \int_0^T (K_H^* f)(s) dW_s$.
2. $K_H^* I_{[0,t]} = K_H(t,\cdot)$ with

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad s < t \text{ and } H > 1/2$$

and

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \quad H < 1/2.$$

Representation of Wiener integrals

Moreover, there exists an isometry $K_H^* : \mathcal{H} \to L^2([0,T])$ and a Brownian motion W such that :

1. $I(\phi) = \int_0^T (K_H^*\phi)(s)dW_s$.
2. $K_H^* I_{[0,t]} = K_H(t,\cdot)$.
3. For $H < 1/2$,

$$K_H^* f = \phi_f,$$

with $f(s) = c_H s^{\frac{1}{2}-H}(I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\phi_f(u))(s), \ s \in [0,T]$.

Representation of Wiener integrals

Moreover, there exists an isometry $K_H^*: \mathcal{H} \to L^2([0,T])$ and a Brownian motion W such that :

1. $I(\phi) = \int_0^T (K_H^*\phi)(s)dW_s$.
2. $K_H^* I_{[0,t]} = K_H(t,\cdot)$.
3. For $H < 1/2$,
$$K_H^* f = \phi_f,$$
with $f(s) = c_H s^{\frac{1}{2}-H}(I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\phi_f(u))(s), \; s \in [0,T]$.
4. For $H > 1/2$ and $\phi \in \mathcal{E}$,
$$(K_H^*\phi)(s) = c_H s^{1/2-H}(I_{T-}^{H-\frac{1}{2}} u^{h-\frac{1}{2}}\phi(u))(s), \; s \in [0,T].$$

Contents

Derivative operator

Let $\mathcal{S}$ be the set of smooth functional of the form

$$F = f(B(\phi_1), \dots, B(\phi_n)),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\phi_i \in \mathcal{H}$.

Derivative operator

Let $\mathcal{S}$ be the set of smooth functional of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\phi_i \in \mathcal{H}$. The derivative operator is given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i.$$

Derivative operator

Let $\mathcal{S}$ be the set of smooth functional of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\phi_i \in \mathcal{H}$. The derivative operator is given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$.

Divergence operator

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Transfer principle

Let W be the Brownian motion such that

$$B_t = \int_0^t K_H(t,s)dW_s \quad t \in [0,T].$$

Then,

1. $\mathrm{Dom}\,D = \mathrm{Dom}\,D^W$ and

$$K_H^* DF = D^W F.$$

Transfer principle

Let W be the Brownian motion such that

$$B_t = \int_0^t K_H(t,s) dW_s \quad t \in [0, T].$$

Then,

1. Dom$D$=DomD^W and

$$K_H^* DF = D^W F.$$

2. $\phi \in$Domδ if and only if $K_H^* \phi \in$Domδ^W and

$$\delta(\phi) = \delta^W(K_H^* \phi).$$

Transfer principle

1. Dom$D$=DomD^W and
$$K_H^* DF = D^W F.$$
2. ϕ $\in$Domδ if and only if $K_H^*\phi$ $\in$Domδ^W and
$$\delta(\phi) = \delta^W(K_H^*\phi).$$

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Remark For $H = 1/2$, $\mathcal{H} = L^2([0, T])$.

Transfer principle

1. Dom$D$=DomD^W and $K_H^* DF = D^W F$.
2. ϕ $\in$Domδ if and only if $K_H^* \phi$ $\in$Domδ^W and

$$\delta(\phi) = \delta^W(K_H^* \phi).$$

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Remark For $H = 1/2$, $\mathcal{H} = L^2([0, T])$. So

$$E\left(F\delta^W(u)\right) = E\left(\int_0^T (D_s^W F) u_s ds\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega \times [0, T]).$$

Divergence operator

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Proposition

Let $u \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ such that $(Fu) \in L^2(\Omega; \mathcal{H})$ and $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. Then

$$F\delta(u) = \delta(Fu)$$

Divergence operator

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Proposition

Let $u \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ such that $(Fu) \in L^2(\Omega;\mathcal{H})$ and $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. Then

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}.$$

Divergence operator

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Proposition

Let $u \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ such that $(Fu) \in L^2(\Omega; \mathcal{H})$ and $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. Then

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}.$$

Proof : Let $G \in \mathcal{S}$, then

Divergence operator

The divergence operator δ is the adjoint of D. It is defined by the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Proposition

Let $u \in \mathrm{Dom}\,\delta$ *and* $F \in \mathrm{Dom}\,D$ *such that* $(Fu) \in L^2(\Omega;\mathcal{H})$ *and* $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. *Then*

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}.$$

Proof : Let $G \in \mathcal{S}$, then

$$E\left(\langle DG, Fu\rangle_{\mathcal{H}}\right) = E\left(\langle D(GF), u\rangle_{\mathcal{H}} - G\,\langle DF, u\rangle_{\mathcal{H}}\right)$$

Divergence operator

The divergence operator δ is the adjoint of D.

$$E\left(F\delta(u)\right) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \quad F \in \mathcal{S},\ u \in L^2(\Omega, \mathcal{H}).$$

Proposition

Let $u \in \mathrm{Dom}\,\delta$ *and* $F \in \mathrm{Dom}\,D$ *such that* $(Fu) \in L^2(\Omega; \mathcal{H})$ *and* $(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}) \in L^2(\Omega)$. *Then*

$$F\delta(u) = \delta(Fu) - \langle DF, u\rangle_{\mathcal{H}}.$$

Proof : Let $G \in \mathcal{S}$, then

$$\begin{aligned} E\left(\langle DG, Fu\rangle_{\mathcal{H}}\right) &= E\left(\langle D(GF), u\rangle_{\mathcal{H}} - G\,\langle DF, u\rangle_{\mathcal{H}}\right) \\ &= E\left(G\left(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}\right)\right). \end{aligned}$$