A Strong Uniform Approximation of FBM by Means of Transport Processes

Jorge A. León

Departamento de Control Automático
Cinvestav del IPN

Spring School "Stochastic Control in Finance", Roscoff 2010

Jointly with Johanna Garzón Merchán and Luis G. Gorostiza

Contents

Contents

Taqqu 1975

Let $\{Y_i\}$ be a sequence of stationary Gaussian random variables such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E(Y_i Y_j) \sim A n^{2H} L(n)$$

as $n \to \infty$.

Taqqu 1975

Let $\{Y_i\}$ be a sequence of stationary Gaussian random variables such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E(Y_i Y_j) \sim A n^{2H} L(n)$$

as $n \to \infty$.
Here $0 < H < 1$, $A > 0$ and L is a slowly varying function (i.e., for all $a > 0$, $\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$).

Taqqu 1975

Let $\{Y_i\}$ be a sequence of stationary Gaussian random variables such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E(Y_i Y_j) \sim A n^{2H} L(n)$$

as $n \to \infty$.
Here $0 < H < 1$, $A > 0$ and L is a slowly varying function Then

$$X_n(t) = \frac{1}{d_n} \sum_{i=1}^{\lfloor nt \rfloor} Y_i$$

converges weakly to $\sqrt{A} B_t^H$, where $d_n^2 \sim n^{2H} L(n)$.

Tommi Sottinen 2001

Let $\{\xi_i : i > 0\}$ be a sequence of i.i.d. random variables with $E[\xi_i] = 0$ and $Var[\xi_i] = 1$, and

$$B_t^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} \left(n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H\left(\frac{\lfloor nt \rfloor}{n}, s\right) ds \right) \frac{1}{\sqrt{n}} \xi_i.$$

Tommi Sottinen 2001

Let $\{\xi_i : i > 0\}$ be a sequence of i.i.d. random variables with $E[\xi_i] = 0$ and $Var[\xi_i] = 1$, and

$$B_t^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} \left(n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H \left(\frac{\lfloor nt \rfloor}{n}, s \right) ds \right) \frac{1}{\sqrt{n}} \xi_i .$$

Here, $H < 1/2$ and

$$\begin{aligned} K_H(t,s) &= d_H \left\{ (\frac{t}{s})^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \right. \\ &\quad \left. -(H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right\} \end{aligned}$$

Tommi Sottinen 2001

Let $\{\xi_i : i > 0\}$ be a sequence of i.i.d. random variables with $E[\xi_i] = 0$ and $Var[\xi_i] = 1$, and

$$B_t^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} \left(n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H\left(\frac{\lfloor nt \rfloor}{n}, s \right) ds \right) \frac{1}{\sqrt{n}} \xi_i.$$

Here, $H < 1/2$ and

$$\begin{aligned} K_H(t,s) &= d_H \Big\{ (\frac{t}{s})^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ &\quad -(H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \Big\} \end{aligned}$$

Then $B^{(n)}$ goes weakly to B^H.

Stroock 1982

The family

$$Z_\epsilon := \left\{ Z_\epsilon(t) = \frac{1}{\epsilon} \int_0^t (-1)^{N(\frac{s}{\epsilon^2})} ds, \, t \in [0, T] \right\}$$

converges weakly to the Brownian motion, as $\epsilon \to 0$.

Stroock 1982

The family

$$Z_\epsilon := \left\{ Z_\epsilon(t) = \frac{1}{\epsilon} \int_0^t (-1)^{N(\frac{s}{\epsilon^2})} ds, t \in [0, T] \right\}$$

converges weakly to the Brownian motion, as $\epsilon \to 0$.
Here $\{N(t), t \geq 0\}$ is a Poisson process

Delgado y Jolis (2000)

The family

$$X_\epsilon := \left\{ X_\epsilon(t) = \frac{1}{\epsilon} \int_0^t K_H(t,s)(-1)^{N(\frac{s}{\epsilon^2})} ds, t \in [0,T] \right\}$$

converges to B^H, as $\epsilon \to 0$.

Delgado y Jolis (2000)

The family

$$X_{\epsilon} := \left\{ X_{\epsilon}(t) = \frac{1}{\epsilon} \int_0^t K_H(t,s)(-1)^{N(\frac{s}{\epsilon^2})} ds, t \in [0,T] \right\}$$

converges to B^H, as $\epsilon \to 0$.
Here $K_H(t,s)$ is given by :

$$c_H s^{\frac{1}{2}-H} \left\{ \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right\} 1_{(0,t)}(s), \quad H > \frac{1}{2},$$

Delgado y Jolis (2000)

$$X_\epsilon := \left\{ X_\epsilon(t) = \frac{1}{\epsilon} \int_0^t K_H(t,s)(-1)^{N(\frac{s}{\epsilon^2})} ds, t \in [0,T] \right\}$$

converges to B^H, as $\epsilon \to 0$.
Here $K_H(t,s)$ is given by :

$$c_H s^{\frac{1}{2}-H} \left\{ \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right\} 1_{(0,t)}(s), \quad H > \frac{1}{2},$$

and

$$d_H \left\{ (\frac{t}{s})^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right\},$$

for $H < \frac{1}{2}$.

Szabados (2001)

i.i.d r.v.	random walks	Δt	Δx	Modification
$X_0(1), X_0(2), \cdots$	$S_0 = \sum_{k=1}^n X_0(k)$	1	1	$\tilde{S}_0(n)$
$X_1(1), X_1(2), \cdots$	$S_1 = \sum_{k=1}^n X_1(k)$	2^{-2}	2^{-1}	$\tilde{S}_1(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$X_m(1), X_m(2), \cdots$	$S_m = \sum_{k=1}^n X_m(k)$	2^{-2m}	2^{-m}	$\tilde{S}_m(n)$

$$P(\{X_n(k) = 1\}) = P(\{X_n(k) = -1\}) = \frac{1}{2}.$$

and

$$S_m(t) = S_m(\frac{j}{2^{2n}}) + 2^{2n}(t - \frac{j}{2^{2n}})X_m(j+1), \quad \frac{j}{2^{2n}} \leq t < \frac{j+1}{2^{2n}},$$

with

$$S_m(\frac{j}{2^{2n}}) = \sum_{i=1}^{j} X_m(i).$$

Szabados (2001)

i.i.d r.v.	random walks	Δt	Δx	Modification
$X_0(1), X_0(2), \cdots$	$S_0 = \sum_{k=1}^n X_0(k)$	1	1	$\tilde{S}_0(n)$
$X_1(1), X_1(2), \cdots$	$S_1 = \sum_{k=1}^n X_1(k)$	2^{-2}	2^{-1}	$\tilde{S}_1(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$X_m(1), X_m(2), \cdots$	$S_m = \sum_{k=1}^n X_m(k)$	2^{-2m}	2^{-m}	$\tilde{S}_m(n)$

Set

$$B_m(t) = 2^{-m}\tilde{S}_m(t2^{2m})$$

Theorem

$B_m \to W$ as $m \to \infty$ a.s. uniformly on compact sets.

Szabados (2001)

$$B_m(t) = 2^{-m}\tilde{S}_m(t2^{2m})$$

Set

$$B_m^H(t_k) = \sum_{r=-\infty}^{k-1} h(t_r, t_k)[B_m(t_r + \Delta t) - B_m(t_r)]$$

and

$$B_m^H(0) = 0.$$

Szabados (2001)

Set

$$B_m^H(t_k) = \sum_{r=-\infty}^{k-1} h(t_r, t_k)[B_m(t_r + \Delta t) - B_m(t_r)],$$

and

$$B_m^H(0) = 0,$$

with $\Delta t = 2^{-2m}$, $t_x = x\Delta t$, $x \in \mathbb{R}$, and

$$h(s,t) = C_H\left[(t-s)^{H-1/2} - (-s)_+^{H-1/2}\right], \quad s \le t.$$

Szabados (2001)

Set

$$B_m^H(t_k) = \sum_{r=-\infty}^{k-1} h(t_r, t_k)[B_m(t_r + \Delta t) - B_m(t_r)],$$

and

$$B_m^H(0) = 0,$$

with $\Delta t = 2^{-2m}$, $t_x = x\Delta t$, $x \in \mathbb{R}$, and

$$h(s,t) = C_H \left[(t-s)^{H-1/2} - (-s)_+^{H-1/2} \right], \quad s \leq t.$$

Theorem

For $H \in (1/4, 1)$*,*

$$B_m^H \to B^H \quad \text{as} \quad m \to \infty \quad \text{a.s. uniformly on compact sets.}$$

Szabados (2001)

Set

$$B_m^H(t_k) = \sum_{r=-\infty}^{k-1} h(t_r, t_k)[B_m(t_r + \Delta t) - B_m(t_r)],$$

and

$$B_m^H(0) = 0,$$

with $\Delta t = 2^{-2m}$, $t_x = x\Delta t$, $x \in \mathbb{R}$, and

$$h(s,t) = C_H \left[(t-s)^{H-1/2} - (-s)_+^{H-1/2} \right], \quad s \leq t.$$

Theorem

For $H \in (1/4, 1)$,

$$B_m^H \to B^H \quad as \quad m \to \infty \quad a.s.\ uniformly\ on\ compact\ sets.$$

The rate of convergence is $O(n^{-\min\{H-1/4, 1/4\} 2\log 2} \log n)$.

Contents

Transport processes

$\left\{X^{(n)}(t), t \geq 0\right\}$ is a transport process iff :

- $X^{(n)}(0) = 0$,

Transport processes

$\left\{X^{(n)}(t), t \geq 0\right\}$ is a transport process iff :

- $X^{(n)}(0) = 0$,
- $X^{(n)}$ is continuous,

Transport processes

$\left\{X^{(n)}(t), t \geq 0\right\}$ is a transport process iff :

- $X^{(n)}(0) = 0$,
- $X^{(n)}$ is continuous,
- $X^{(n)}$ is a piecewise linear function with slopes $\pm n$,

Transport processes

$\left\{X^{(n)}(t), t \geq 0\right\}$ is a transport process iff :

- $X^{(n)}(0) = 0$,
- $X^{(n)}$ is continuous,
- $X^{(n)}$ is a piecewise linear function with slopes $\pm n$,
- The slope at $0+$ is random. It is n or $-n$ with probability $1/2$,

Transport processes

$\left\{X^{(n)}(t), t \geq 0\right\}$ is a transport process iff :

- $X^{(n)}(0) = 0$,
- $X^{(n)}$ is continuous,
- $X^{(n)}$ is a piecewise linear function with slopes $\pm n$,
- The slope at $0+$ is random. It is n or $-n$ with probability $1/2$,
- The times between consecutive slope changes are independent and exponential distributed with parameter n^2.

Costruction of transport processes

We fix $(\Omega, \mathfrak{F}, P)$ y $W = (W_t)_{t\geq 0}$.

Costruction of transport processes

We fix $(\Omega, \mathfrak{F}, P)$ y $W = (W_t)_{t\geq 0}$.
Consider, for each $n > 0$, :

- A family $\{\xi_i^n,\ i = 1, 2, \dots\}$ i.i.d random variables, with distribution $\exp(2n^2)$, independent of W.

Costruction of transport processes

We fix $(\Omega, \mathfrak{F}, P)$ y $W = (W_t)_{t\geq 0}$.
Consider, for each $n > 0$, :

- A family $\{\xi_i^n,\ i = 1, 2, \ldots\}$ i.i.d random variables, with distribution $\exp(2n^2)$, independent of W.
- A family $\{\kappa_i,\ i = 1, 2, \ldots\}$ of independent random variables, independent of $\xi_i^n,\ i = 1, 2, \ldots,$ and W, such that $P(\kappa_i = \pm 1) = 1/2$

Costruction of transport processes

We fix $(\Omega, \mathfrak{F}, P)$ y $W = (W_t)_{t\geq 0}$.
Consider, for each $n > 0$, :

- A family $\{\xi_i^n,\ i = 1, 2, \dots\}$ i.i.d random variables, with distribution $\exp(2n^2)$, independent of W.
- A family $\{\kappa_i,\ i = 1, 2, \dots\}$ of independent random variables, independent of $\xi_i^n,\ i = 1, 2, \dots$ and W, such that $P(\kappa_i = \pm 1) = 1/2$

Theorem (**Skorohod**)

There exists a family $\{\sigma_i^n,\ i = 1, 2, \dots\}$ of independent and nonegative random variables such that $E\left(\sigma_j^n\right) = \frac{1}{2n^2}$, $\sigma_0^n = 0$ and

$$\left\{ W\left(\sum_{j=1}^{i} \sigma_j^n\right), i = 1, 2, \dots \right\} \stackrel{d}{=} \left\{ \sum_{j=1}^{i} \kappa_j \xi_j^n, i = 1, 2, \dots \right\}$$

Costruction of transport processes

Theorem (**Skorohod**)

There exists a family $\{\sigma_i^n,\ i = 1, 2, \ldots\}$ *of independent and nonegative random variables such that* $E\left(\sigma_j^n\right) = \frac{1}{2n^2}$, $\sigma_0^n = 0$ *and*

$$\left\{ W\left(\sum_{j=1}^{i} \sigma_j^n\right), i = 1, 2, \ldots \right\} \stackrel{d}{=} \left\{ \sum_{j=1}^{i} \kappa_j \xi_j^n, i = 1, 2, \ldots \right\}$$

Set

$$\gamma_i^n := \frac{1}{n} \underbrace{\left| W\left(\sum_{j=0}^{i} \sigma_j^n\right) - W\left(\sum_{j=0}^{i-1} \sigma_j^n\right) \right|}_{\stackrel{d}{=} \kappa_i \xi_i^n}.$$

Costruction of transport processes

Theorem (**Skorohod**)

There exists a family $\{\sigma_i^n,\ i = 1, 2, \dots\}$ *of independent and nonegative random variables such that* $E\left(\sigma_j^n\right) = \frac{1}{2n^2}$, $\sigma_0^n = 0$ *and*

$$\left\{ W\left(\sum_{j=1}^{i} \sigma_j^n\right), i = 1, 2, \dots \right\} \stackrel{d}{=} \left\{ \sum_{j=1}^{i} \kappa_j \xi_j^n, i = 1, 2, \dots \right\}$$

Set

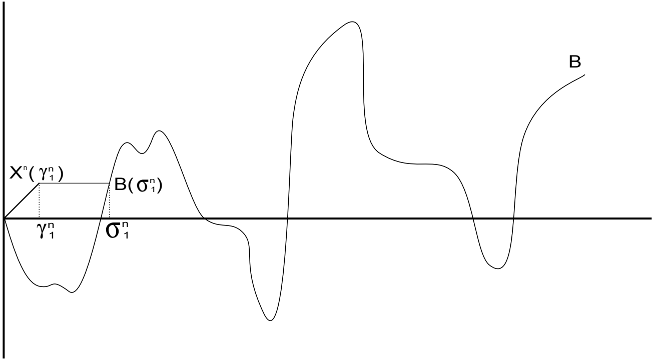
$$\gamma_i^n := \frac{1}{n} \underbrace{\left| W\left(\sum_{j=0}^{i} \sigma_j^n\right) - W\left(\sum_{j=0}^{i-1} \sigma_j^n\right) \right|}_{\stackrel{d}{=} \kappa_i \xi_i^n}.$$

Then, $\{\gamma_i^n,\ i = 1, 2, \dots\}$ is a family of i.i.d. random variables with distribution $\exp(2n^2)$.

Definition of $\mathbf{X^{(n)}}$

$$X^{(n)}\left(\sum_{j=0}^{i}\gamma_j^n\right) = W\left(\sum_{j=0}^{i}\sigma_j^n\right), \; i = 1, 2, \ldots$$

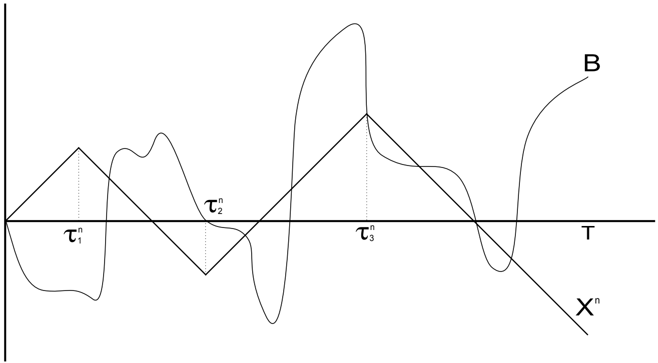
with lineal interpolation.



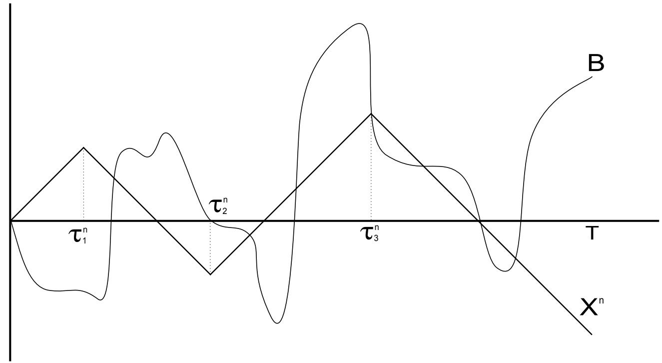
Definition of $\mathbf{X^{(n)}}$

$\tau_i^n = i$-th slope change.

The increments $\tau_i^n - \tau_{i-1}^n$, $i = 1, 2, \dots$, with $\tau_0^n = 0$, are independent with distribution $\exp(n^2)$.



Approximation of Bm

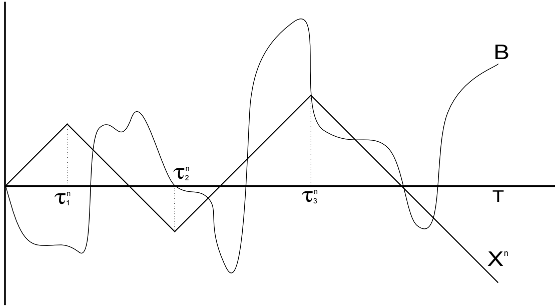


Theorem (Griego, Heath and Ruiz-Moncayo (1971))

Let $\{W(t), t \geq 0\}$ be a given Brownian motion on $(\Omega, \mathfrak{F}, P)$. Then,

$$\lim_{n\to\infty} \max_{0\leq t\leq 1} |X^{(n)}(t) - W(t)| = 0, \quad a.s.$$

Approximation of Bm



Theorem (Gorostiza y Griego (1980))

For any $q > 0$, we have

$$P\left(\max_{0\le t\le 1} |X^{(n)}(t) - W(t)| > \alpha n^{-\frac{1}{2}}(\log n)^{\frac{5}{2}}\right) = o(n^{-q}), \quad as \quad n \to \infty,$$

where $\alpha > 0$.

Contents

Mandelbrot-van Ness representation

$$B_t^H = C_H \left\{ \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW(s) + \int_0^t (t-s)^{H-1/2} dW(s) \right\},$$

where $C_H = (2H \sin \pi H \Gamma(2H))^{1/2} / \Gamma(H+1/2)$ and $W = (W(t))_{t\in\mathbb{R}}$ is a Brownian motion.

Mandelbrot-van Ness representation

$$B_t^H = C_H \left\{ \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW(s) + \int_0^t (t-s)^{H-1/2} dW(s) \right\},$$

For $a < 0$,

$$\begin{aligned} B_t^H &= C_H \left\{ \int_0^t g_t(s) dW(s) + \int_a^0 f_t(s) dW(s) + \int_{-\infty}^a f_t(s) dW(s) \right\} \\ &= C_H \left\{ \int_0^t g_t(s) dW(s) + \int_a^0 f_t(s) dW(s) + f_t(a) W(a) \right. \\ &\quad \left. - \int_{\frac{1}{a}}^0 \partial_s f_t \left(\frac{1}{v} \right) \frac{1}{v^2} W\left(\frac{1}{v} \right) dv \right\}. \end{aligned}$$

Mandelbrot-van Ness representation

For $a < 0$,

$$\begin{aligned} B_t^H &= C_H\left\{\int_0^t g_t(s)dW(s) + \int_a^0 f_t(s)dW(s) + \int_{-\infty}^a f_t(s)dW(s)\right\} \\ &= C_H\left\{\int_0^t g_t(s)dW(s) + \int_a^0 f_t(s)dW(s) + f_t(a)W(a)\right. \\ &\quad \left. - \int_{\frac{1}{a}}^0 \partial_s f_t\left(\frac{1}{v}\right)\frac{1}{v^2}W\left(\frac{1}{v}\right)dv\right\}. \end{aligned}$$

Here,

$$f_t(s) = (t-s)^{H-1/2} - (-s)^{H-1/2}, \quad s < 0 \le t \le T,$$

and

$$g_t(s) = (t-s)^{H-1/2}, \quad 0 < s < t \le T.$$

Transport processes

For $a < 0$, we introduce the following :

1. $(W_1(s))_{0 \le s \le T}$, the restriction of W on $[0, T]$.
2. $(W_2(s))_{a \le s \le 0}$, the restriction of W on $[a, 0]$.
3. $W_3(s) = \begin{cases} sW(\frac{1}{s}) & \text{si } s \in \left[\frac{1}{a}, 0\right), \\ 0 & \text{si } s = 0. \end{cases}$

Transport processes

For $a < 0$, we introduce the following :

1. $(W_1(s))_{0 \leq s \leq T}$, the restriction of W on $[0, T]$.
2. $(W_2(s))_{a \leq s \leq 0}$, the restriction of W on $[a, 0]$.
3. $W_3(s) = \begin{cases} sW(\frac{1}{s}) & \text{si } s \in \left[\frac{1}{a}, 0\right), \\ 0 & \text{si } s = 0. \end{cases}$

Then we can find

$$(X_1^{(n)}(s))_{0 \leq s \leq T}, \quad (X_2^{(n)}(s))_{a \leq s \leq 0} \quad \text{and} \quad (X_3^{(n)}(s))_{\frac{1}{a} \leq s \leq 0},$$

Transport processes

For $a < 0$, we introduce the following :

1. $(W_1(s))_{0 \le s \le T}$, the restriction of W on $[0, T]$.
2. $(W_2(s))_{a \le s \le 0}$, the restriction of W on $[a, 0]$.
3. $W_3(s) = \begin{cases} sW(\frac{1}{s}) & \text{si } s \in \left[\frac{1}{a}, 0\right), \\ 0 & \text{si } s = 0. \end{cases}$

Then we can find

$$(X_1^{(n)}(s))_{0 \le s \le T}, \quad (X_2^{(n)}(s))_{a \le s \le 0} \quad \text{and} \quad (X_3^{(n)}(s))_{\frac{1}{a} \le s \le 0},$$

such that, for any $q > 0$,

$$P\left(\sup_{b_i \le t \le c_i} |W_i(t) - X_i^{(n)}(t)| > C^{(i)} n^{-1/2} (\log n)^{5/2} \right) = o(n^{-q}),$$

as $n \to \infty$.

Approximation of FBM

Let $\varepsilon_n = -n^{-\beta/|H-1/2|}$.
For $H > 1/2$, define

$$\begin{aligned} B^{(n)}(t) &= C_H \Bigg\{ \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s) + \int_a^0 f_t(s) dX_2^{(n)}(s) \\ &\quad + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \left(-\int_{\frac{1}{a}}^{s\wedge\varepsilon_n} \partial_s f_t\left(\frac{1}{v}\right) \frac{1}{v^3} dv \right) dX_3^{(n)}(s) \Bigg\} \end{aligned}$$

Approximation of FBM

Let $\varepsilon_n = -n^{-\beta/|H-1/2|}$.
For $H > 1/2$, define

$$\begin{aligned} B^{(n)}(t) &= C_H\left\{\int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s) + \int_a^0 f_t(s) dX_2^{(n)}(s)\right. \\ &\left. + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \left(-\int_{\frac{1}{a}}^{s\wedge\varepsilon_n} \partial_s f_t\left(\frac{1}{v}\right)\frac{1}{v^3} dv\right) dX_3^{(n)}(s)\right\} \end{aligned}$$

and for $H < 1/2$, set

$$\begin{aligned} \hat{B}^{(n)}(t) &= C_H\left\{\int_0^{(t+\varepsilon_n)\vee 0} g_t(s) dX_1^{(n)}(s)\right. \\ &+ \int_{(t+\varepsilon_n)\vee 0}^t (t-\varepsilon_n - s)^{H-1/2} dX_1^{(n)}(s) + \int_a^{\varepsilon_n} f_t(s) dX_2^{(n)}(s) \\ &\left. + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \left(-\int_{\frac{1}{a}}^{s} \partial_s f_t\left(\frac{1}{v}\right)\frac{1}{v^3} dv\right) dX_3^{(n)}(s)\right\}. \end{aligned}$$

Approximation of FBM

Let $\varepsilon_n = -n^{-\beta/|H-1/2|}$.
For $H > 1/2$, define

$$\begin{aligned} B^{(n)}(t) &= C_H \Big\{ \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s) + \int_a^0 f_t(s) dX_2^{(n)}(s) \\ &\quad + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \left(-\int_{\frac{1}{a}}^{s\wedge\varepsilon_n} \partial_s f_t\left(\frac{1}{v}\right) \frac{1}{v^3} dv \right) dX_3^{(n)}(s) \Big\} \end{aligned}$$

Theorem

For any $q > 0$ and β such that $0 < H - \frac{1}{2} < \beta < \frac{1}{2}$, there exists $C > 0$ such that

$$P\left(\sup_{0\le t\le T} \left| B^H(t) - B^{(n)}(t) \right| > C n^{-(1/2-\beta)} (\log n)^{5/2} \right) = o(n^{-q})$$

as $n \to \infty$.

Approximation of FBM

For $H < 1/2$, set

$$\hat{B}^{(n)}(t) = C_H \Bigg\{ \int_0^{(t+\varepsilon_n)\vee 0} g_t(s)\, dX_1^{(n)}(s) + \int_{(t+\varepsilon_n)\vee 0}^{t} (t-\varepsilon_n - s)^{H-1/2} dX_1^{(n)}(s) + \int_a^{\varepsilon_n} f_t(s)\, dX_2^{(n)}(s) + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^{0} \left(-\int_{\frac{1}{a}}^{s} \partial_s f_t\left(\frac{1}{v}\right) \frac{1}{v^3} dv \right) dX_3^{(n)}(s) \Bigg\}.$$

Theorem

For any $q > 0$ and β such that $0 < \frac{1}{2} - H < \beta < \frac{1}{2}$, there exists $\hat{C} > 0$ such that

$$P\left(\sup_{0 \le t \le T} \left| B^H(t) - \hat{B}^{(n)}(t) \right| > \hat{C} n^{-(1/2-\beta)} (\log n)^{5/2} \right) = o(n^{-q}).$$

Approximation of FBM

Lemma

Let $H > 1/2$. Then, for any $q > 0$, there exists $C_2 > 0$ such that, for $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$,

$$P\left(\sup_{0\le t\le T} C_H \left|\int_0^t (t-s)^{H-1/2} dW_1(s) - \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s)\right| > C_2 \frac{\alpha_n}{5}\right) = o(n^{-q}) \quad as \quad n \to \infty.$$

Approximation of FBM

Lemma

Let $H > 1/2$. Then, for any $q > 0$, there exists $C_2 > 0$ such that, for $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$,

$$P\left(\sup_{0\le t\le T} C_H \left|\int_0^t (t-s)^{H-1/2} dW_1(s) - \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s)\right| > C_2 \frac{\alpha_n}{5}\right) = o(n^{-q}) \quad as \quad n \to \infty.$$

Proof : By the integration by parts formula,

$$\int_0^t (t-s)^{H-1/2} dW_1(s) = (H-1/2)\int_0^t (t-s)^{H-3/2} W_1(s) ds.$$

and

$$\int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s) = (H-1/2)\int_0^t (t-s)^{H-3/2} X_1^{(n)}(s) ds.$$

Approximation of FBM

Lemma

Let $H > 1/2$. Then, for any $q > 0$, there exists $C_2 > 0$ such that, for $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$,

$$P\left(\sup_{0\leq t\leq T} C_H \left|\int_0^t (t-s)^{H-1/2} dB_1(s) - \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s)\right| > C_2 \frac{\alpha_n}{5}\right) = o(n^{-q}) \quad \text{as} \quad n \to \infty.$$

Proof : Then

$$\left|\int_0^t (t-s)^{H-1/2} dW_1(s) - \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s)\right| \leq \sup_{0\leq s\leq t} \left|W_1(s) - X_1^{(n)}(s)\right| t^{H-1/2}.$$

Bibliography

- L. Gorostiza, Rate of convergence of uniform transport processe to Brownian motion and apprications to stochastic integrals. *Stoch.* **3**, 1980.
- R. Griego, D. Heath and A. Ruiz, Almost sure convergence of uniform transport processes to Brownian motion. *Annals of Math. Stat.* **42-3**, 1129-1131, 1971.
- T. Szabados, Strong approximation of fractional Brownian motion by moving averages of simple random walks. *Stoc. Proc. Appl.* **92**, 31-60, 2001.
- T. Sottinen, Fractional Brownian Motion, Random Walks and Binary Market Models. Finance and stochastic. *Springer-Verlag*, 2001.
- M. Taqqu, Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. and Verw. Gebiete* **31**, 287-302, 1975.

Contents

Equation

Our objective is to obtain an approximation with rate of convergence for solution of fractional stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s)ds, \qquad t \in [0, T],$$

Equation

Our objective is to obtain an approximation with rate of convergence for solution of fractional stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s) ds, \qquad t \in [0, T],$$

where $B = (B_t)_{t\in[0,T]}$ is fBm with Hurst parameter $H \in (\frac{1}{4}, 1)$, $(H \neq 1/2)$, with $\sigma \in C_b^2$, $b \in C_b^1$ and b is bounded.

Equation

Our objective is to obtain an approximation with rate of convergence for solution of fractional stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s) ds, \qquad t \in [0, T],$$

where $B = (B_t)_{t \in [0,T]}$ is fBm with Hurst parameter $H \in (\frac{1}{4}, 1)$, $(H \neq 1/2)$, with $\sigma \in C_b^2$, $b \in C_b^1$ and b is bounded.
The stochastic integral with respect to B is the forward integral for $H > \frac{1}{2}$, and the Stratonovich integral for $H < \frac{1}{2}$.

Equation

Our objective is to obtain an approximation with rate of convergence for solution of fractional stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s)ds, \qquad t \in [0, T],$$

where $B = (B_t)_{t\in[0,T]}$ is fBm with Hurst parameter $H \in (\frac{1}{4}, 1)$, $(H \neq 1/2)$, with $\sigma \in C_b^2$, $b \in C_b^1$ and b is bounded.
By Alòs, León and Nualart, and Neuenkirch and Nourdin, this equation has an unique solution X with a Doss-Sussmann type representation

$$X_t = h(Y_t, B_t),$$

Equation

Our objective is to obtain an approximation with rate of convergence for solution of fractional stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s)ds, \qquad t \in [0, T],$$

By Alòs, León and Nualart, and Neuenkirch and Nourdin, this equation has an unique solution X with a Doss-Sussmann type representation

$$X_t = h(Y_t, B_t),$$

where

$$\frac{\partial h}{\partial x_2}(x_1, x_2) = \sigma(h(x_1, x_2)), \; h(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R},$$

and

$$Y_t' = \exp\left(-\int_0^{B_t} \sigma'(h(Y_t, s))ds\right) b(h(Y_t, B_t)), \quad Y_0 = x_0.$$

Equation

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dB_s + \int_0^t b(X_s)ds, \qquad t \in [0, T],$$

By Alòs, León and Nualart, and Neuenkirch and Nourdin, this equation has an unique solution X

$$X_t = h(Y_t, B_t),$$

where

$$\frac{\partial h}{\partial x_2}(x_1, x_2) = \sigma(h(x_1, x_2)), \; h(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R},$$

and

$$\begin{aligned} Y_t' &= \exp\left(-\int_0^{B_t} \sigma'(h(Y_t, s))ds\right) b(h(Y_t, B_t)), \\ &= \left(\frac{\partial h}{\partial x_1}(Y_t, B_t)\right)^{-1} b(h(Y_t, B_t)), \quad Y_0 = x_0. \end{aligned}$$

Approximation scheme

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds.$$

Approximation scheme

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds.$$

For each $n = 1, 2, \cdots$, we take

$$-n = y^n_{-n^2} < \cdots < y^n_{-1} < y^n_0 = 0 < y^n_1 < \cdots < y^n_{n^2} = n,$$

Approximation scheme

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds.$$

For each $n = 1, 2, \cdots$, we take

$$-n = y^n_{-n^2} < \cdots < y^n_{-1} < y^n_0 = 0 < y^n_1 < \cdots < y^n_{n^2} = n,$$

where for $r_n = \frac{1}{n}$,

$$y^n_{i+1} = y_i + r_n = \frac{i+1}{n}, \quad y^n_{-(i+1)} = y_{-i} - r_n = \frac{-i-1}{n}.$$

Approximation scheme

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds.$$

For each $n = 1, 2, \cdots$, we take

$$-n = y^n_{-n^2} < \cdots < y^n_{-1} < y^n_0 = 0 < y^n_1 < \cdots < y^n_{n^2} = n,$$

where for $r_n = \frac{1}{n}$,

$$y^n_{i+1} = y_i + r_n = \frac{i+1}{n}, \quad y^n_{-(i+1)} = y_{-i} - r_n = \frac{-i-1}{n}.$$

We define functions $h^n : \mathbb{R}^2 \to \mathbb{R}$ by $h^n(x,y) = 0$ if $(x,y) \notin [-n,n] \times [-n,n]$, and for $(x,y) \in [-n,n] \times [-n,n]$ and $k = 0, 1, \cdots n^2 - 1$, as

$$\begin{aligned} h^n(x,y) &= h^n(x,y^n_k) + (y - y^n_k)\sigma(h^n(x,y^n_k)), \quad y^n_k \le y < y^n_{k+1}, \\ h^n(x,y) &= h^n(x,y^n_{-k}) - (y^n_{-k} - y)\sigma(h^n(x,y^n_{-k})), \; y^n_{-(k+1)} < y \le y^n_{-k}. \end{aligned}$$

Approximation scheme

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds.$$

$$y^n_{i+1} = y_i + \frac{1}{n} = \frac{i+1}{n}, \quad y^n_{-(i+1)} = y_{-i} - \frac{1}{n} = \frac{-i-1}{n}.$$

We define functions $h^n : \mathbb{R}^2 \to \mathbb{R}$ by $h^n(x,y) = 0$ if $(x,y) \notin [-n,n] \times [-n,n]$, and for $(x,y) \in [-n,n] \times [-n,n]$ and $k = 0, 1, \cdots n^2 - 1$, as

$$\begin{aligned} h^n(x,y) &= h^n(x,y^n_k) + (y - y^n_k)\sigma(h^n(x,y^n_k)), \quad y^n_k \le y < y^n_{k+1}, \\ h^n(x,y) &= h^n(x,y^n_{-k}) - (y^n_{-k} - y)\sigma(h^n(x,y^n_{-k})), \; y^n_{-(k+1)} < y \le y^n_{-k}, \end{aligned}$$

with

$$h^n(x,y^n_{k+1}) = h^n(x,y^n_k) + r_n\sigma(h^n(x,y^n_k)),$$
$$h^n(x,y^n_{-(k+1)}) = h^n(x,y^n_{-k}) - r_n\sigma(h^n(x,y^n_{-k})),$$

Approximation of FBM

Let $\varepsilon_n = -n^{-\beta/|H-1/2|}$.

For $H > 1/2$, define

$$B^{(n)}(t) = C_H \Big\{ \int_0^t (t-s)^{H-1/2} dX_1^{(n)}(s) + \int_a^0 f_t(s) dX_2^{(n)}(s) + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \Big(- \int_{\frac{1}{a}}^{s \wedge \varepsilon_n} \partial_s f_t\Big(\frac{1}{v}\Big) \frac{1}{v^3} dv \Big) dX_3^{(n)}(s) \Big\}$$

and for $H < 1/2$, set

$$\hat{B}^{(n)}(t) = C_H \Big\{ \int_0^{(t+\varepsilon_n)\vee 0} g_t(s) dX_1^{(n)}(s) + \int_{(t+\varepsilon_n)\vee 0}^t (t-\varepsilon_n - s)^{H-1/2} dX_1^{(n)}(s) + \int_a^{\varepsilon_n} f_t(s) dX_2^{(n)}(s) + f_t(a) X_2^{(n)}(a) + \int_{\frac{1}{a}}^0 \Big(- \int_{\frac{1}{a}}^{s} \partial_s f_t\Big(\frac{1}{v}\Big) \frac{1}{v^3} dv \Big) dX_3^{(n)}(s) \Big\}.$$

Notation

$$B^n = \begin{cases} B^{(n)} & \text{if} \quad H > \frac{1}{2}, \\ \hat{B}^{(n)} & \text{if} \quad H < \frac{1}{2}. \end{cases}$$

Notation

$$B^n = \begin{cases} B^{(n)} & \text{if } \; H > \frac{1}{2}, \\ \hat{B}^{(n)} & \text{if } \; H < \frac{1}{2}. \end{cases}$$

We have that

$$Y_t = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds.$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds.$$

Notation

$$B^n = \begin{cases} B^{(n)} & \text{if } H > \frac{1}{2}, \\ \hat{B}^{(n)} & \text{if } H < \frac{1}{2}. \end{cases}$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds.$$

That is,

$$(Y_t^n)' = f(Y_t^n, B_t^n) \qquad Y_{t_0}^n = x_0, \quad t_0 = 0,$$

where

$$f(x, y) = \exp\left(-\int_0^y \sigma'(h(x, u)du) \right) b(h(x, y)).$$

Notation

$$B^n = \begin{cases} B^{(n)} & \text{if} \quad H > \frac{1}{2}, \\ \hat{B}^{(n)} & \text{if} \quad H < \frac{1}{2}. \end{cases}$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$(Y_t^n)' = f(Y_t^n, B_t^n) \qquad Y_{t_0}^n = x_0, \quad t_0 = 0,$$

where

$$f(x, y) = \exp\left(- \int_0^y \sigma'(h(x, u)du) \right) b(h(x, y)).$$

For each $n = 1, 2, \cdots$, we define

$$f^n(x, y) = \exp\left(- \int_0^y \sigma'(h^n(x, u)du) \right) b(h^n(x, y)),$$

Euler scheme

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$(Y_t^n)' = f(Y_t^n, B_t^n) \qquad Y_{t_0}^n = x_0, \quad t_0 = 0,$$

where

$$f(x, y) = \exp\left(-\int_0^y \sigma'(h(x, u)du)\right) b(h(x, y)).$$

For each $n = 1, 2, \cdots$, we define

$$f^n(x, y) = \exp\left(-\int_0^y \sigma'(h^n(x, u)du)\right) b(h^n(x, y)),$$

The Euler scheme $(\hat{Y}^{n,m})$ for the above differential equation is defined as follows, for each $m = 1, 2, \cdots$, the partition $0 = t_0 < \cdots < t_n = T$ of $[0, T]$ with $t_{i+1} = t_i + r_m$, and $r_m = \frac{T}{m}$:

Euler scheme

$$
\begin{cases}
\hat{Y}_0^{n,m} = x_0, & \\
\hat{Y}_{t_{k+1}}^{n,m} = \hat{Y}_{t_k}^{n,m} + r_m f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n), & k = 0, \cdots, (m-1), \\
\hat{Y}_t^{n,m} = \hat{Y}_{t_k}^{n,m} + (t - t_k) f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) & \\
\qquad = \hat{Y}_{t_k}^{n,m} + \int_{t_k}^t f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) ds, & \text{if} \quad t_k \le t < t_{k+1}.
\end{cases}
$$

Euler scheme

$$\begin{cases} \hat{Y}_0^{n,m} = x_0, \\ \hat{Y}_{t_{k+1}}^{n,m} = \hat{Y}_{t_k}^{n,m} + r_m f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n), & k = 0, \cdots, (m-1), \\ \hat{Y}_t^{n,m} = \hat{Y}_{t_k}^{n,m} + (t - t_k) f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) \\ \quad\quad = \hat{Y}_{t_k}^{n,m} + \int_{t_k}^t f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) ds, & \text{if } \; t_k \le t < t_{k+1}. \end{cases}$$

$$X_t = x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \qquad t \in [0, T],$$

$$X_t = h(Y_t, B_t),$$

Euler scheme

$$\begin{cases} \hat{Y}_0^{n,m} = x_0, \\ \hat{Y}_{t_{k+1}}^{n,m} = \hat{Y}_{t_k}^{n,m} + r_m f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n), & k = 0, \cdots, (m-1), \\ \hat{Y}_t^{n,m} = \hat{Y}_{t_k}^{n,m} + (t - t_k) f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) \\ \qquad\; = \hat{Y}_{t_k}^{n,m} + \int_{t_k}^t f^n(\hat{Y}_{t_k}^{n,m}, B_{t_k}^n) ds, & \text{if} \quad t_k \le t < t_{k+1}. \end{cases}$$

$$X_t = x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \qquad t \in [0, T],$$

$$X_t = h(Y_t, B_t).$$

Here, we define the approximation of X as

$$X_t^n := h^n(\hat{Y}_t^{n,n^2}, B_t^n).$$

Euler scheme

Theorem

For any β such that $|H - 1/2| < \beta < 1/2$,

$$P\left(\limsup_{n\to\infty}\left\{\sup_{t\in[0,T]}|X_t - X_t^n| > \alpha_n\right\}\right) = 0,$$

where $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$.

Proof : Approximation of h

For fixed n, we work in the square $[-n, n] \times [-n, n]$. Let $l = n + m$ for some $m > 0$, and consider the finer partition of $[-n, n]$ given by $-n = y^l_{-nl} < \cdots < y^l_0 = 0 < \cdots < y^l_{nl} = n$, with $r_l = \frac{1}{l}$.

Proof : Approximation of h

For fixed n, we work in the square $[-n,n]\times[-n,n]$. Let $l=n+m$ for some $m>0$, and consider the finer partition of $[-n,n]$ given by $-n=y^l_{-nl}<\cdots<y^l_0=0<\cdots<y^l_{nl}=n$, with $r_l=\frac{1}{l}$.

Lemma

For all $(x,y)\in[-n,n]\times[-n,n]$ *and* $l>n$,

$$\left|h(x,y)-h^l(x,y)\right|\leq \bar{M}^2\frac{n}{l}\exp\left(\bar{M}n\right).$$

Proof : Approximation of Y

We have that

$$Y_t = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds.$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds.$$

Proof : Approximation of Y

We have that

$$Y_t = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds.$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds.$$

Proposition

We have

$$P\left(\limsup_{n\to\infty} \{ \| Y - Y^n \|_\infty > \alpha_n \} \right) = 0,$$

where $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$.

Proof : Approximation of Y^n

Proposition

Let Y^n *and* $\hat{Y}^{n,m}$ *be as above. Then*

$$P\left(\limsup_{n\to\infty}\left\{\|Y^n - \hat{Y}^{n,n^2}\|_\infty > \alpha_n\right\}\right) = 0$$

where $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$.

Proof : Approximation of Y

We have that

$$Y_t = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds.$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds.$$

Proposition

We have

$$P\left(\limsup_{n\to\infty} \{ \| Y - Y^n \|_\infty > \alpha_n \} \right) = 0,$$

where $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$.

Proof : Approximation of Y

Proposition

We have

$$P\left(\limsup_{n\to\infty}\{\|Y - Y^n\|_\infty > \alpha_n\}\right) = 0,$$

where $\alpha_n = n^{-(1/2-\beta)}(\log n)^{5/2}$.

Proof.

$$|Y_t - Y_t^n| = \left|\int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s)\right)^{-1} b(h(Y_s, B_s))ds - \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n)\right)^{-1} b(h(Y_s^n, B_s^n))ds\right|$$

$$\leq \int_0^t I_1(s)ds + \int_0^t I_2(s)ds,$$

Proof : Approximation of Y

Proof.

$$|Y_t - Y_t^n| = \left| \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds \right.$$

$$\left. - \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds \right|$$

$$\leq \int_0^t l_1(s)ds + \int_0^t l_2(s)ds,$$

where

$$l_1(s) = \left| \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} - \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} \right| \left| b(h(Y_s^n, B_s^n)) \right|,$$

and

$$l_2(s) = \left| \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} \right| \left| b(h(Y_s, B_s)) - b(h(Y_s^n, B_s^n)) \right|.$$

Proof : Approximation of Y

Proof.

$$
\begin{aligned}
|Y_t - Y_t^n| &= \left| \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s) \right)^{-1} b(h(Y_s, B_s))ds \right. \\
&\qquad \left. - \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n) \right)^{-1} b(h(Y_s^n, B_s^n))ds \right| \\
&\leq \int_0^t I_1(s)ds + \int_0^t I_2(s)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^t I_2(s)ds \\
&\leq \int_0^t \exp(M\|B\|_\infty)[M\exp(M\|B\|_\infty)|Y_s - Y_s^n| + M|B_s - B_s^n|]ds.
\end{aligned}
$$