The pathwise solution of an SPDE with fractal noise

Elena Issoglio

Friedrich-Schiller Universität, Jena

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Outline

Stochastic transport equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \nabla B^H(x) \cdot \nabla u(t,x), & t \in (0,T], x \in D \\ u(0,x) = u_0(x), & x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \end{cases}$$

- $u(t,x)$: unknown concentration of the substance at time t and position x
- $D \subset \mathbb{R}^d$: bounded domain with smooth boundary
- $B^H(x) = B^H(x,\omega)$: suitable stochastic noise

$\longrightarrow$ In this session $B^H(x)$ will be a fractional Brownian field with Hurs index $0 < H < 1$.

Fractional Brownian motion ($d = 1$)

$\{B^H(x), x \in \mathbb{R}_+\}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if it is a centred Gaussian process with covariance function given by

$$\mathbb{E}(B^H_x B^H_y) = \frac{1}{2}\left(x^{2H} + y^{2H} - |x-y|^{2H}\right)$$

- homogeneous increments but not indipendent (negatively correlated if $H < 1/2$, positively if $H > 1/2$)
- there exists a version of B^H with α-Hölder continuous trajectories, for $\alpha < H$
- if $H \neq 1/2$ then B^H is not a semimartingale: Itô-type theory can not be used

The abstract Cauchy problem

- X Banach space
- A linear operator on X
- A generates a semigroup $(T(t), t \geq 0)$
- $f : [0, T) \to X$ given function

The abstract Cauchy problem is

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t) \quad , \quad t > 0 \\ u(0) = h \end{cases} \tag{1}$$

where u is a X-valued function.
We define the mild solution as the function

$$u(t) = T(t)h + \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

The stochastic transport equation as abstract Cauchy problem

- X infinite dimensional Banach space
- $h \in X$ function depending on $x \in D \subset \mathbb{R}^d$: $h(x)$

The function $u(t,x)$ is now interpreted only as function of time and takes values in X.

$$\begin{array}{rccc} \underline{u}: & [0,T] & \rightarrow & X \\ & t & \mapsto & \underline{u}(t) \end{array}$$

where $\underline{u}(t)$ is a function of x defined by

$$\begin{array}{rccc} \underline{u}(t): & D & \rightarrow & \mathbb{R} \\ & x & \mapsto & \underline{u}(t)(x) \end{array}$$

where $\underline{u}(t)(x) := u(t,x)$.

The Cauchy problem with Dirichlet conditions is now rewritten as

$$\begin{cases} \underline{u}_t = \Delta_D \underline{u} + \nabla B^H \cdot \nabla \underline{u}, & t \in (0, T] \\ \underline{u}(0) = \underline{u}_0 \end{cases}$$

where

- $\underline{u}_t$ indicates the derivative of $\underline{u}$ with respect to time
- $\underline{u}_0(x) := u(0, x) = u_0(x)$
- Δ_D is the *Dirichlet laplacian* on D: it encodes the condition $\underline{u}(t)(x) \equiv 0$ for $x \in \partial D$
- $\nabla B^H \cdot \nabla \underline{u}$ has still to be defined since ∇B^H is a distribution
- **pathwise interpretation**: fix $\omega \in \Omega$ and study the equation for almost every ω

Δ e Δ_D: probabilistic interpretation

- **Laplacian Δ on $\mathbb{R}^d$.**
  generates a semigroup $\{T_t\}_{t\geq 0}$

$$T_t u(x) = \int_{\mathbb{R}^d} p(t,x,y)u(y)\,\mathrm{d}y$$

where $p(t,x,y)$ is the heat kernel

$$p(t,x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{-\frac{|x-y|^2}{2t}\right\}$$

$<\!\!-\!\!->$ Brownian motion on $\mathbb{R}^d$ where
$p(t,x,y) = \mathbb{P}^x(B_t \in \mathrm{d}y)$ is the transition probability density function of a Brownian motion $\{B_t\}_{t\geq 0}$.

- **Laplacian Δ_D.**
  generates a semigroup $\{P_t\}_{t\geq 0}$

$$P_t u(x) = \int_D p_D(t,x,y)u(y)\,\mathrm{d}y$$

where $p_D(t,x,y) = p(t,x,y) - r(t,x,y)$ with

$$r(t,x,y) = \mathbb{E}^x[p(t-\tau_D, B_{\tau_D}, y); \tau_D < t]$$

and τ_D is the first exit time from D.
<—> killed Brownian motion (killed at exiting D)

$$\bar{B}_t = \begin{cases} B_t & \text{if } t < \tau_D \\ \zeta & \text{if } t > \tau_D. \end{cases}$$

Δ e Δ_D: analytical interpretation

- The semigroup T_t acts (for instance) on $L_2(\mathbb{R}^d)$.
  In this case we have
  $\mathrm{dom}(\Delta) = W^2(\mathbb{R}^d) \subset W^0(\mathbb{R}^d) = L_2(\mathbb{R}^d)$.

  Property: $\lambda - \Delta : H^\gamma(\mathbb{R}^d) \to H^{\gamma-2}(\mathbb{R}^d)$, for every $\gamma \in \mathbb{R}$, $\lambda > 0$.

- The semigroup P_t and its generator act on a space restricted to D which contains information on ∂D.
  —> fractional Sobolev spaces on D.

Fractional Sobolev spaces on $\mathbb{R}^d$

- **Sobolev spaces.** Let $m \in \mathbb{N}$

$$W_p^m(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \partial^\gamma f \in L_p(\mathbb{R}^d) \text{ for every } |\gamma| \leq m \right\}$$

endowed with the norm $\|f \mid W_p^m\| := \left(\sum_{|\gamma| \leq m} \|\partial^\gamma f \mid L_p\|^p \right)^{1/p}$

- **Fractional Sobolev Spaces.** Let $\alpha \in \mathbb{R}$

$$H_p^\alpha(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : ((1 + |\xi|^2)^{\alpha/2} \hat{f})^\vee \in L_p(\mathbb{R}^d) \right\}$$

endowed with the norm $\|f|H_p^\alpha(\mathbb{R}^d)\| = \|((1 + |\xi|^2)^{\alpha/2} \hat{f})^\vee\|_{L_p}$

Property: if $\alpha = m \in \mathbb{N}$ then $H_p^m(\mathbb{R}^d) = W_p^m(\mathbb{R}^d)$.

Fractional Sobolev spaces on D

Let $\alpha \in \mathbb{R}$

- define

$$H_p^\alpha(D) := \left\{ f \in \mathcal{S}'(D) : \exists g \in H_p^\alpha(\mathbb{R}^d) \text{ s.t. } g|_D = f \right\}$$

  endowed with the norm

$$\|f|H_p^\alpha(D)\| = \inf \left\{ \|g|H_p^\alpha(\mathbb{R}^d)\| \text{ s.t. } g \in H_p^\alpha(\mathbb{R}^d) \text{ and } g|_D = f \right\}$$

- define $\tilde{H}_p^\alpha(D) := \{ f \in H_p^\alpha(\mathbb{R}^d) : \mathrm{supp}(f) \subset \bar{D} \}$
  endowed with the norm $\| \cdot |H_p^\alpha(\mathbb{R}^d)\|$

—> right space for P_t and Δ_D:

$$\bar{H}^\alpha(D) := \begin{cases} \tilde{H}^\alpha(D), & \text{if } \alpha \geq 0 \\ H^\alpha(D), & \text{if } \alpha < 0 \end{cases}$$

Scale of spaces with good properties for $-3/2 < \alpha < 3/2$.

- $P_t : \bar{H}^\gamma(D) \to \bar{H}^{\gamma+2}(D)$
- $\Delta_D^\alpha : \bar{H}^\gamma(D) \to \bar{H}^{\gamma-2\alpha}(D)$

The noise ∇B^H and the term $\nabla B^H \cdot \nabla \underline{u}(s)$

- let consider a version of B^H with α-Hölder continuous paths, for $\alpha < H$.
- **Property:** if h is α-Hölder continuous on $\mathbb{R}^d$ with compact support for $0 < \alpha < 1$, then for any $\alpha' < \alpha < 1$ we have $h \in H_q^{\alpha'}(\mathbb{R}^d)$ for any $1 < q < \infty$.
- let $\psi(x) \in \mathscr{C}_c^\infty$ such that $\psi(x) \equiv 1$ for any $x \in D$. Apply Property to $\psi(x)B^H(\omega)(x)$ with a fixed $\omega \in \Omega$: $\psi B^H \in H_q^{1-\beta}(\mathbb{R}^d)$ for any $1 - \beta < H < 1$.
- substitute B^H with a deterministic function $Z \in H_q^{1-\beta}(\mathbb{R}^d)$.
- we have $\nabla Z \in H_q^{-\beta}(\mathbb{R}^d)$ with $\beta > 0$: it is a distribution –> problems while defining $\nabla Z \cdot \nabla \underline{u}(s)$.

The pointwise product in $\mathcal{S}'(\mathbb{R}^d)$

given $f, g \in \mathcal{S}'(\mathbb{R}^d)$ we define the product

$$fg := \lim_{j\to\infty} S^j f S^j g$$

if the limit exists in $\mathcal{S}'(\mathbb{R}^d)$, where

$$S^j f(x) := (\phi(\frac{\xi}{2^j})\hat{f})^\vee(x)$$

with

- $\phi(\xi) \in \mathscr{C}^\infty$, $0 \leq \phi(\xi) \leq 1$ for any $\xi \in \mathbb{R}^d$
- $\phi(\xi) = 1$ if $|\xi| \leq 1$
- $\phi(\xi) = 0$ if $|\xi| \geq 3/2$

Property

Let $1 < p, q < \infty$, $0 < \beta < \delta$ and assume $q > \max(p, d/\delta)$.
Then for any $f \in H_p^\delta(\mathbb{R}^d)$ and $g \in H_q^{-\beta}(\mathbb{R}^d)$ we have
$fg \in H_p^{-\beta}(\mathbb{R}^d)$

$$\|fg|H_p^{-\beta}(\mathbb{R}^d)\| \leq c\|f|H_p^\delta(\mathbb{R}^d)\| \cdot \|g|H_q^{-\beta}(\mathbb{R}^d)\|.$$

Application:

- $g = \nabla Z \in H_q^{-\beta}(\mathbb{R}^d)$
- $f = \nabla \underline{u}(s) \in \bar{H}_p^\delta(D) \subset H_p^\delta(\mathbb{R}^d)$ (since $\delta > 0$)
- notation: $\langle \cdot, \cdot \rangle$ for the scalar product in $\mathbb{R}^d$ combined with the poitwise product just defined.
- $\langle \nabla Z, \nabla \underline{u}(s) \rangle \in H_p^{-\beta}(\mathbb{R}^d)$

Property: Let $f, g \in \mathcal{S}'(\mathbb{R}^d)$, $\text{supp}(f) \subset \bar{D}$ then $\text{supp}(fg) \subset \bar{D}$.

- by definition of $\bar{H}_p^{\delta}(D)$ with $\delta > 0$ we have $\text{supp}(\nabla \underline{u}(s)) \subset \bar{D}$
- can apply the property: $\text{supp}(\langle \nabla Z, \nabla \underline{u}(s) \rangle) \subset \bar{D}$
- notice that $\langle \nabla Z, \nabla \underline{u}(s) \rangle \in H_p^{-\beta}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ so that $\langle \nabla Z, \nabla \underline{u}(s) \rangle \in \mathcal{S}'(D)$
- by definition of $\bar{H}_p^{-\beta}(D)$ with $\beta > 0$ (functions in $\mathcal{S}'(D)$ s.t. there exists an extension in $H_p^{-\beta}(\mathbb{R}^d)$) we have $\langle \nabla Z, \nabla \underline{u}(s) \rangle \in \bar{H}_p^{-\beta}(D)$

The mild solution

$$\begin{cases} \underline{u}_t = \Delta_D \underline{u} + \nabla Z \cdot \nabla \underline{u} & \text{for } t \in (0, T] \\ \underline{u} = u_0 \end{cases}$$

- P_t semigroup generated by $-\Delta_D$
- the boundary conditions are included in the choise of the domain of Δ_D

We say that a function $\underline{u} : [0, T] \to X$ is a **mild solution** of the problem if

$$\underline{u}(t) = P_t u_0 + \int_0^t P_{t-r} \langle \nabla \underline{u}(r), \nabla Z \rangle \, \mathrm{d}r$$

for all $t \in [0, T]$.

The operator on which we concentrate is then the following

$$I_t(\underline{u}) := \int_0^t P_{t-r}\langle \nabla \underline{u}(r), \nabla Z\rangle \, \mathrm{d}r.$$

for any fixed $\underline{u}$.

Which is the time regularity?

Let us introduce the space of all γ-Hölder continuous functions on $[0, T]$ taking values in an (infinite dimensional) Banach space $(X, \|\cdot\|_X)$:

$$C^\gamma([0,T];X) := \{h : [0,T] \to X \text{ s.t. } \|h\|_{\gamma,X} < \infty\}$$

where

$$\|h\|_{\gamma,X} := \sup_{t\in[0,T]} \|h(t)\|_X + \sup_{s<t\in[0,T]} \frac{\|h(t)-h(s)\|_X}{(t-s)^\gamma}$$

Local mapping property of I_t (I., 2009)

Let $X = \bar{H}_2^{1+\delta}(D)$ for some $0 < \beta < \delta$, $\delta + \beta < 1/2$ with $Z \in \bar{H}_q^{1-\beta}(D)$. Then we have

$$I_{(\cdot)} : C^\gamma([0,T];X) \to C^\gamma([0,T];X)$$

for all $0 < \gamma < 1/4$, and moreover for any $\underline{u} \in C^\gamma([0,T];X)$

$$\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta} \le c(T)\|\underline{u}\|_{\gamma,1+\delta}$$

where $c(T)$ is a function not depending on $\underline{u}$ and such that $\lim_{T\to 0} c(T) = 0$. □

$\Longrightarrow$ by contraction theorem it is easy to obtain existence and uniqueness of the solution $\underline{u} \in \mathcal{C}^\gamma([0,\varepsilon];X)$ with ε sufficiently small. (local solution)

How to extend the theorem to any $T < \infty$?

- Let us introduce a family of equivalent norms on $\mathcal{C}^\gamma([0,T];X)$ parametrized by a real parameter $\rho > 1$:

$$\|f\|_{\gamma,X}^{(\rho)} := \sup_{0\leq t\leq T} e^{-\rho t}\left(\|f(t)\|_X + \sup_{0\leq s<t} \frac{\|f(t)-f(s)\|_X}{(t-s)^\gamma}\right).$$

- it is easy to prove that for any $\rho > 1$

$$\|\cdot\|_{\gamma,X}^{(\rho)} \sim \|\cdot\|_{\gamma,X}$$

- **Idea:** work in the space $\mathcal{C}^\gamma([0,T];X)$ endowed with the ρ-norm and prove that I_t is a contraction for some suitable ρ which does not depend on T.

Theorem 1 (I., 2009)

Let $X = \bar{H}_2^{1+\delta}(D)$ for some $0 < \beta < \delta$, $\delta + \beta < 1/2$. Fix $Z \in H_q^{1-\beta}(\mathbb{R}^d)$. Then

$$I_{(\cdot)} : C^\gamma([0,T]; X) \to C^\gamma([0,T]; X)$$

for every $0 < 2\gamma < 1 - \delta - \beta$, and moreover for any $\underline{u} \in C^\gamma([0,T]; X)$ we have

$$\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta}^{(\rho)} \leq c(\rho)\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$$

where $c(\rho)$ is a function of ρ not depending on $\underline{u}$ and T and such that

$$\lim_{\rho\to\infty} c(\rho) = 0.$$

□

Theorem 2 (I., 2009)

Let $0 < \beta < \delta$, $\delta + \beta < 1/2$ and $0 < 2\gamma < 1 - \delta - \beta$. Moreover fix $Z \in H_q^{-\beta}(\mathbb{R}^d)$ for some $q < 2 \vee d/\delta$.
Then for every initial condition $u_0 \in \bar{H}_2^{1+\delta+2\gamma}(D)$, with $1 + \delta + 2\gamma < 3/2$, there exists a unique mild solution $u(t,x)$ for the abstract Cauchy problem

$$\begin{cases} \underline{u}_t = \Delta_D \underline{u} + \nabla \underline{u} \cdot \nabla Z & \text{for } t \in (0,T] \\ \underline{u} = u_0 \end{cases}$$

given by $u(t,\cdot) = P_t u_0 + I_t(\underline{u})$.
Moreover this solution belongs to the Hölder space $\mathscr{C}^\gamma([0,T]; \bar{H}_2^{1+\delta}(D))$ for any finite positive time T. □

Grazie.