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2-Player Zero-Sum Stochastic Differential Games

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Appendix

- •1 Pursuit Games
- •2 BSDE Decoupled With Forward SDE
- •3 Proof of that the upper and lower value functions are deterministic.
- •4 Proof of DPP
- •5 Proof of Theorem 3

•1 Pursuit Games: Rabbit-Wolf.

Player I (the rabbit):

$$X_1^u(s) = x^1 + \int_0^s u_r dr,\ X_1^u(0) = x^1 \in \mathbb{R}^2, s \in [0,T]; u(\cdot) \in \overline{B_1(0)};$$

Player II (the wolf):

$$X_2^v(s) = x^2 + \int_0^s v_r dr,\ X_2^v(0) = x^2 \in \mathbb{R}^2, s \in [0,T]; v(\cdot) \in \overline{B_2(0)}.$$

Rabbit: $\mathcal{U} = L^0_{\mathbb{F}}(0,T;\overline{B_1(0)})$; Wolf: $\mathcal{V} = L^0_{\mathbb{F}}(0,T;\overline{B_2(0)})$.

Take: $f = 0,\ \Phi(x_1,x_2) = |x_1 - x_2|$. Then the cost functional is

$$J(0,x;u,v) = E[|X_1^u(T) - X_2^v(T)|].$$

Rabbit wants to maximize $J(0,x;u,v)$ via his control $u \in \mathcal{U}$;

Wolf wants to minimize $J(0,x;u,v)$ via his control $v \in \mathcal{V}$.

We consider the following functions:

$$W_0(0,x) := \text{essinf}_{v\in\mathcal{V}}\text{esssup}_{u\in\mathcal{U}}J(0,x;u,v);$$

– Player II (Wolf) begins the match, Player I (Rabbit) reacts to the choice of Player II.

$$V_0(0,x) := \text{esssup}_{u\in\mathcal{U}}\text{essinf}_{v\in\mathcal{V}}J(0,x;u,v).$$

– Player I (Rabbit) begins the match, Player II (Wolf) reacts to the choice of Player I.

1) Computation for $V_0(0,x)$. i.e., Rabbit begins the match: Suppose $|x^1-x^2|\leq T$. Rabbit chooses $u\in\mathcal{U}$, Wolf can choose $v(s)=\frac{x^1-x^2}{T}+u_s,\ s\in[0,T]$, then $v\in\mathcal{V}$. $\longrightarrow J(o,x;u,v)=0 \longrightarrow V_0(0,x)=0$. Therefore,

$$V_0(0,x)=0, \forall x=(x^1,x^2)\text{with } |x^1-x^2|\leq T.$$

2) Computation for $W_0(0,x)$. i.e., Wolf begins the match: Wolf chooses $v \in \mathcal{V}$, Rabbit can choose

$$u(s) = \begin{cases} 0, \quad \text{if } E[|x^1 - X_2^v(T)|] \geq \frac{(T\wedge 1)^2}{2},\ s \in [0,T],; \\ e(T\wedge 1),\ \text{if } E[|x^1 - X_2^v(T)|] < \frac{(T\wedge 1)^2}{2},\ |e| = 1,\ s \in [0,T]. \end{cases}$$

Then $u \in \mathcal{U}$. $\longrightarrow J(o,x;u,v) \geq \frac{(T\wedge 1)^2}{2} \longrightarrow W_0(0,x) \geq \frac{(T\wedge 1)^2}{2}$. Therefore,

$$W_0(0,x) > V_0(0,x), \forall x = (x^1,x^2) \text{with } |x^1 - x^2| \leq T.$$

$\longrightarrow$ The result of the game depends on which player begins.

Special case in which the game "control against control" doesn't depend on who begins the game: Hamadene, Lepeltier, Peng (1997).

Additional assumptions: $\sigma(t,x,u,v) = \sigma(t,x)$, independent of $(u,v) \in \mathcal{U} \times \mathcal{V}$, there exists $\sigma^{-1} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ Lipschitz and bounded.

•2 BSDE Decoupled With Forward SDE

S.Peng, (1997)*BSDE and stochastic optimizations; Topics in stochastic analysis.* J.Yan, S.Peng, S.Fang and L.Wu, Chapter 2, Science Press. Beijing (in Chinese).

In this section we give an overview over basic results which are necessary for us on BSDEs associated with forward SDEs. We consider measurable functions $b:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$, which are supposed to satisfy the following conditions:

(H4.1)

(i) $b(\cdot,0)$ and $\sigma(\cdot,0)$ are $\mathcal{F}_t$-adapted processes, and there exists some constant $C>0$ such that $|b(t,x)|+|\sigma(t,x)|\leq C(1+|x|), a.s.$, for all $0\leq t\leq T,\ x\in\mathbb{R}^n$;

(ii) b and σ are Lipschitz in x; i.e., there is some constant $C>0$

such that

$$|b(t,x)-b(t,x')|+|\sigma(t,x)-\sigma(t,x')| \le C|x-x'|, \ a.s.,$$
$$\text{for all } 0 \le t \le T, \ x, \ x' \in \mathbb{R}^n.$$

We now consider the following SDE parameterized by the initial condition $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t,\zeta} &= b(s, X_s^{t,\zeta})ds + \sigma(s, X_s^{t,\zeta})dB_s, \ s \in [t,T], \\ X_t^{t,\zeta} &= \zeta. \end{cases} \tag{1}$$

Under the assumption (H4.1), SDE (1) has a unique strong solution, and, for any $p \ge 2$, there exists $C_p \in \mathbb{R}$ such that, for any $t \in [0,T]$ and $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$E\left[\sup_{t\le s\le T} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^p | \mathcal{F}_t\right] \le C_p|\zeta-\zeta'|^p, \quad a.s., \tag{2}$$
$$E\left[\sup_{t\le s\le T} |X_s^{t,\zeta}|^p | \mathcal{F}_t\right] \le C_p(1+|\zeta|^p), \quad a.s.$$

We emphasize that the constant C_p in (2) depends only on the Lipschitz and the growth constants of b and σ. Let now be given two real-valued functions $f(t,x,y,z)$ and $\Phi(x)$ which shall satisfy the following conditions:

(H4.2)

(i) $\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is an $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n)$-measurable random variable and $f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable process such that $f(\cdot,x,y,z)$ is $\mathcal{F}_t$-adapted, for all $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$;

(ii) There exists a constant $C > 0$ such that $|f(t,x,y,z) - f(t,x',y',z')| + |\Phi(x) - \Phi(x')| \leq C(|x-x'| + |y-y'| + |z-z'|)$, $a.s.$,
for all $0 \leq t \leq T$, x, $x' \in \mathbb{R}^n$, y, $y' \in \mathbb{R}$ and z, $z' \in \mathbb{R}^d$;

(iii) f and Φ satisfy a linear growth condition; i.e., there exists some $C > 0$

such that, $dt \times$ dP-a.e., for all $x \in \mathbb{R}^n$,
$|f(t,x,0,0)| + |\Phi(x)| \leq C(1+|x|)$.

With the help of the above assumptions we can verify that the coefficient $f(s, X_s^{t,\zeta}, y, z)$ satisfies the hypotheses (1) and (2) and $\xi = \Phi(X_T^{t,\zeta}) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$. Therefore, the following BSDE possesses a unique solution:

$$\begin{cases} -dY_s^{t,\zeta} &= f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})ds - Z_s^{t,\zeta} dB_s, \ s \in [t,T], \\ Y_T^{t,\zeta} &= \Phi(X_T^{t,\zeta}). \end{cases} \tag{3}$$

Proposition 1 We suppose that the hypotheses (H4.1) and (H4.2) hold. Then, for any $0 \leq t \leq T$ and the associated initial conditions $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates:

(i) $E\left[\sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t\right] \leq C(1+|\zeta|^2), \ a.s.;$

$$\text{(ii)} \;\; E\left[\sup_{t\le s\le T}|Y_s^{t,\zeta}-Y_s^{t,\zeta'}|^2+\int_t^T|Z_s^{t,\zeta}-Z_s^{t,\zeta'}|^2ds|\mathcal{F}_t\right]\le C|\zeta-\zeta'|^2,\; a.s.$$

In particular,

$$\text{(iii)} \quad |Y_t^{t,\zeta}|\le C(1+|\zeta|),\; a.s.;\; \text{(iv)} \quad |Y_t^{t,\zeta}-Y_t^{t,\zeta'}|\le C|\zeta-\zeta'|,\; a.s., \tag{4}$$

where the constant $C>0$ depends only on the Lipschitz and the growth constants of b, σ, f, and Φ.

For the proof the reader is referred to Proposition 4.1 of Peng (1997); a similar result can be found in El Karoui, Peng, and Quenez (1997).

Let us now introduce the random field:

$$u(t,x)=Y_s^{t,x}|_{s=t},\;(t,x)\in[0,T]\times\mathbb{R}^n, \tag{5}$$

where $Y^{t,x}$ is the solution of BSDE (3) with $x\in\mathbb{R}^n$ at the place of $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$.

As a consequence of Proposition 1 we have that, for all $t \in [0,T]$, P-a.s.,

$$\begin{array}{ll} \text{(i)} & |u(t,x)-u(t,y)| \leq C|x-y| \text{ for all } x,y \in \mathbb{R}^n; \\ \text{(ii)} & |u(t,x)| \leq C(1+|x|) \text{ for all } x \in \mathbb{R}^n. \end{array} \qquad (6)$$

Remark In the general situation u is an adapted random function; that is, for any $x \in \mathbb{R}^n$, $u(\cdot,x)$ is an $\mathbb{F}$-adapted real-valued process. Indeed, recall that b, σ, and f all are $\mathbb{F}$-adapted random functions while Φ is $\mathcal{F}_T$-measurable. On the other hand, it is well known that, under the additional assumption that the functions

(H4.3) $\qquad b, \sigma, f,$ and Φ are deterministic,

u is also a deterministic function of (t,x).

The random field u and $Y^{t,\zeta}$, $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, are related by the following theorem.

Theorem 2 Under the assumptions (H4.1) and (H4.2), for any $t \in$

$[0,T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have

$$u(t,\zeta) = Y_t^{t,\zeta}, \ P\text{-a.s.} \tag{7}$$

The proof of Theorem 2 can be found in Peng (1997); we give it for the reader's convenience. It makes use of the following definition.

Definition 3 For any t $\in [0,T]$, a sequence $\{A_i\}_{i=1}^N \subset \mathcal{F}_t$ (with $1 \leq N \leq \infty$) is called a partition of $(\Omega, \mathcal{F}_t)$ if $\cup_{i=1}^N A_i = \Omega$ and $A_i \cap A_j = \phi$, whenever $i \neq j$.

Proof of Theorem 2 . We first consider the case where ζ is a simple random variable of the form

$$\zeta = \sum_{i=1}^N x_i \mathbf{1}_{A_i}, \tag{8}$$

where$\{A_i\}_{i=1}^N$ is a finite partition of $(\Omega, \mathcal{F}_t)$ and $x_i \in \mathbb{R}^n$, for $1 \leq i \leq N$.

For each i, we put $(X_s^i, Y_s^i, Z_s^i) \equiv (X_s^{t,x_i}, Y_s^{t,x_i}, Z_s^{t,x_i})$. Then X^i is the solution of the SDE

$$X_s^i = x_i + \int_t^s b(r, X_r^i)dr + \int_t^s \sigma(r, X_r^i)dB_r,\ s \in [t,T],$$

and (Y^i, Z^i) is the solution of the associated BSDE

$$Y_s^i = \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i)dr - \int_s^T Z_r^i dB_r,\ s \in [t,T].$$

The above two equations are multiplied by $\mathbf{1}_{A_i}$ and summed up with respect to i. Thus, taking into account that $\sum_i \varphi(x_i)\mathbf{1}_{A_i} = \varphi(\sum_i x_i \mathbf{1}_{A_i})$, we get

$$\sum_{i=1}^N \mathbf{1}_{A_i} X_s^i = \sum_{i=1}^N x_i \mathbf{1}_{A_i} + \int_t^s b\left(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i\right) dr + \int_t^s \sigma\left(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i\right) dB_r$$

and

$$\sum_{i=1}^{N} \mathbf{1}_{A_i} Y_s^i = \Phi\left(\sum_{i=1}^{N} \mathbf{1}_{A_i} X_T^i\right) + \int_s^T f\left(r, \sum_{i=1}^{N} \mathbf{1}_{A_i} X_r^i, \sum_{i=1}^{N} \mathbf{1}_{A_i} Y_r^i, \sum_{i=1}^{N} \mathbf{1}_{A_i} Z_r^i\right) dr - \int_s^T \sum_{i=1}^{N} \mathbf{1}_{A_i} Z_r^i dB_r.$$

Then the strong uniqueness property of the solution of the SDE and the BSDE yields

$$X_s^{t,\zeta} = \sum_{i=1}^{N} X_s^i \mathbf{1}_{A_i}, \ (Y_s^{t,\zeta}, Z_s^{t,\zeta}) = \left(\sum_{i=1}^{N} \mathbf{1}_{A_i} Y_s^i, \sum_{i=1}^{N} \mathbf{1}_{A_i} Z_s^i\right), \ s \in [t,T].$$

Finally, from $u(t,x_i) = Y_t^i, \ 1 \le i \le N$, we deduce that

$$Y_t^{t,\zeta} = \sum_{i=1}^{N} Y_t^i \mathbf{1}_{A_i} = \sum_{i=1}^{N} u(t,x_i) \mathbf{1}_{A_i} = u\left(t, \sum_{i=1}^{N} x_i \mathbf{1}_{A_i}\right) = u(t,\zeta).$$

Therefore, for simple random variables, we have the desired result.

Given a general $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ we can choose a sequence of simple random variables $\{\zeta_i\}$ which converges to ζ in $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, from the estimates (4) and (6) and the first step of the proof, we have

$$\begin{array}{rcl} E|Y_t^{t,\zeta_i} - Y_t^{t,\zeta}|^2 & \leq & CE|\zeta_i - \zeta|^2 \to 0, \ i \to \infty, \\ E|u(t,\zeta_i) - u(t,\zeta)|^2 & \leq & CE|\zeta_i - \zeta|^2 \to 0, \ i \to \infty, \\ \text{and} \qquad Y_t^{t,\zeta_i} & = & u(t,\zeta_i), \ i \geq 1. \end{array}$$

Then the proof is complete.

Remark Under (H4.1), (H4.2), and (H4.3) we know $u(t,x)$ is $\frac{1}{2}$-Hölder continuous in t: There exists a constant C such that, for every $x \in \mathbb{R}^n, \ t, \ t' \in [0,T]$,

$$|u(t,x) - u(t',x)| \leq C(1 + |x|)|t - t'|^{\frac{1}{2}}.$$

This inequality can be proved with the help of Theorem 2.

3. Proof of that the upper and lower value functions are deterministic (main tool is a Girsanov transformation argument).

Proof. Let H denote the Cameron–Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative $\dot{h}$ belongs to $L^2([0,T],\mathbb{R}^d)$.

For any $h \in H$, we define the mapping $\tau_h\omega := \omega + h,\ \omega \in \Omega$. Obviously, $\tau_h : \Omega \to \Omega$ is a bijection, and its law is given by $P \circ [\tau_h]^{-1} = \exp\{\int_0^T \dot{h}_s dB_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2 ds\}P$. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be arbitrarily fixed, and put $H_t = \{h \in H | h(\cdot) = h(\cdot \wedge t)\}$. We split now the proof in the following steps:

First step: For any $u \in \mathcal{U}_{t,T},\ v \in \mathcal{V}_{t,T},$ and $h \in H_t,\ J(t,x;u,v)\ (\tau_h) = J(t,x;u(\tau_h),v(\tau_h)),\ P$-a.s.

Indeed, we apply the Girsanov transformation to SDE (1) (with $\zeta = x$) and compare the obtained equation with the SDE obtained from (1) by substituting the transformed control processes $u(\tau_h)$ and $v(\tau_h)$ for u and v. Then from the uniqueness of the solution of (1) we get $X_s^{t,x;u,v}(\tau_h) = X_s^{t,x;u(\tau_h),v(\tau_h)}$ for any $s \in [t,T],\ P$-a.s. Furthermore, by a similar Girsanov transformation argument we get from the uniqueness of the solution of BSDE (2)

$$Y_s^{t,x;u,v}(\tau_h) = Y_s^{t,x;u(\tau_h),v(\tau_h)} \text{ for any } s \in [t,T],\ P\text{-a.s.,}$$

$$Z_s^{t,x;u,v}(\tau_h) = Z_s^{t,x;u(\tau_h),v(\tau_h)},\ dsdP\text{-a.e. on } [t,T] \times \Omega.$$

That means

$$J(t,x;u,v)(\tau_h) = J(t,x;u(\tau_h),v(\tau_h)),\ P\text{-a.s.}$$

Second step: For $\beta \in \mathcal{B}_{t,T}$, $h \in H_t$, let $\beta^h(u) := \beta(u(\tau_{-h}))(\tau_h)$, $u \in \mathcal{U}_{t,T}$. Then $\beta^h \in \mathcal{B}_{t,T}$.

Obviously, β^h maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, this mapping is nonanticipating. Indeed, let $S : \Omega \to [t,T]$ be an $\mathbb{F}$-stopping time and $u_1, u_2 \in \mathcal{U}_{t,T}$, with $u_1 \equiv u_2$ on $[\![t,S]\!]$. Then, obviously, $u_1(\tau_{-h}) \equiv u_2(\tau_{-h})$ on $[\![t,S(\tau_{-h})]\!]$ (notice that $S(\tau_{-h})$ is still a stopping time), and because $\beta \in \mathcal{B}_{t,T}$ we have $\beta(u_1(\tau_{-h})) \equiv \beta(u_2(\tau_{-h}))$ on $[\![t,S(\tau_{-h})]\!]$.

Therefore,

$$\beta^h(u_1) = \beta(u_1(\tau_{-h}))(\tau_h) \equiv \beta(u_2(\tau_{-h}))(\tau_h) = \beta^h(u_2) \text{ on } [\![t,S]\!].$$

Third step: For all $h \in H_t$ and $\beta \in \mathcal{B}_{t,T}$ we have

$$\{\text{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u))\}(\tau_h) = \text{esssup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u,\beta(u))(\tau_h)\},\ P\text{-a.s.}$$

Indeed, with the notation $I(t,x,\beta) := \text{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u))$, $\beta \in \mathcal{B}_{t,T}$, we have $I(t,x,\beta) \geq J(t,x;u,\beta(u))$, and thus $I(t,x,\beta)(\tau_h) \geq J(t,x;u,\beta(u))(\tau_h)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$. On the other hand, for any random variable ζ satisfying $\zeta \geq J(t,x;u,\beta(u))(\tau_h)$, and hence also $\zeta(\tau_{-h}) \geq J(t,x;u,\beta(u))$, P-a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau_{-h}) \geq I(t,x,\beta)$, P-a.s., i.e., $\zeta \geq I(t,x,\beta)(\tau_h)$, P-a.s. Consequently,

$$I(t,x,\beta)(\tau_h) = \text{esssup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u,\beta(u))(\tau_h)\},\ P\text{-a.s.}$$

Fourth step: $W(t,x)$ is invariant with respect to the Girsanov transformation τ_h, i.e.,

$$W(t,x)(\tau_h) = W(t,x),\ P\text{-a.s., for any } h \in H.$$

Indeed, similarly to the third step we can show that for all $h \in H_t$

$$\{\text{essinf}_{\beta \in \mathcal{B}_{t,T}} I(t,x;\beta)\}(\tau_h) = \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \{I(t,x;\beta)(\tau_h)\}, \ P\text{-a.s.}$$

Then, from the first step to the third step we have, for any $h \in H_t$,

$$\begin{aligned} W(t,x)(\tau_h) &= \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t,x;u,\beta(u))(\tau_h)\} \\ &= \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u(\tau_h),\beta^h(u(\tau_h)) \\ &= \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta^h(u)) \\ &= \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u)) \\ &= W(t,x), \ P\text{-a.s.}, \end{aligned}$$

where we have used $\{u(\tau_h) | u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T}$ and $\{\beta^h | \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$ in order to obtain both latter equalities. Therefore, for any $h \in H_t$, $W(t,x)\ (\tau_h) = W(t,x)$, P-a.s., and since $W(t,x)$ is $\mathcal{F}_t$-measurable, we have this relation even for all $h \in H$. Indeed, recall that our underlying fundamental space is $\Omega = C_0([0,T];\mathbb{R}^d)$ and that, due to the definition of the filtration, the $\mathcal{F}_t$-measurable random variable

$W(t,x)(\omega)$, $\omega \in \Omega$, depends only on the restriction of ω to the time interval $[0,t]$.

The result of the fourth step combined with the following auxiliary Lemma 1 completes the proof.

Lemma 1. Let ζ be a random variable defined over our classical Wiener space $(\Omega, \mathcal{F}_T, P)$, such that $\zeta(\tau_h) = \zeta$, P-a.s., for any $h \in H$. Then $\zeta = E\zeta$, P-a.s.

Proof. Let $h \in H$ and $A \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} & E\left[\mathbf{1}_{\{\zeta\in A\}} \exp\left\{\int_0^T \dot{h}_s dB_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2 ds\right\}\right] \\ = \quad & E\left[\mathbf{1}_{\{\zeta(\tau_{-h})\in A\}} \exp\left\{\int_0^T \dot{h}_s dB_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2 ds\right\}\right] \\ = \quad & E[\mathbf{1}_{\{\zeta\in A\}}], \end{aligned}$$

from which we deduce that

$$E\left[\mathbf{1}_{\{\zeta\in A\}}\exp\left\{\int_0^T \dot{h}_s dB_s\right\}\right] = E[\mathbf{1}_{\{\zeta\in A\}}]E\left[\exp\left\{\int_0^T \dot{h}_s dB_s\right\}\right],$$

i.e., for any $\varphi \in L^2([0,T];\mathbb{R}^d)$,

$$E\left[\mathbf{1}_{\{\zeta\in A\}}\exp\left\{\int_0^T \varphi_s dB_s\right\}\right] = E[\mathbf{1}_{\{\zeta\in A\}}]E\left[\exp\left\{\int_0^T \varphi_s dB_s\right\}\right].$$

Consequently, taking into consideration the arbitrariness of $A \in \mathcal{B}(\mathbb{R})$ and of $\varphi \in L^2([0,T];\mathbb{R}^d)$, the independence of ζ of B and hence of $\mathcal{F}_T$ follows, but this is possible only for deterministic ζ.

4 Proof of DPP

Theorem 1 *The lower value function* $W(t,x)$ *obeys the following DPP: For any* $0 \leq t < t+\delta \leq T,\ x \in \mathbb{R}^n$,

$$W(t,x) = \mathrm{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \mathrm{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})]. \quad (9)$$

Proof. To simplify notations we put

$$W_\delta(t,x) = \mathrm{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \mathrm{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})].$$

The proof that $W_\delta(t,x)$ coincides with $W(t,x)$ will be split into the following lemmas.

Lemma 1 $W_\delta(t,x)$ is deterministic.

Proof. The proof of this lemma uses the same ideas as that of Proposition 1, so it can be omitted here.

Lemma 2 $W_\delta(t,x) \leq W(t,x)$.

Proof. Let $\beta \in \mathcal{B}_{t,T}$ be arbitrarily fixed. Then, given a $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define as follows the restriction β_1 of β to $\mathcal{U}_{t,t+\delta}$:

$$\beta_1(u_1) := \beta(u_1 \oplus u_2)|_{[t,t+\delta]}, \quad u_1(\cdot) \in \mathcal{U}_{t,t+\delta},$$

where $u_1 \oplus u_2 := u_1 \mathbf{1}_{[t,t+\delta]} + u_2 \mathbf{1}_{(t+\delta,T]}$ extends $u_1(\cdot)$ to an element of $\mathcal{U}_{t,T}$. It is easy to check that $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Moreover, from the nonanticipativity property of β we deduce that β_1 is independent of the special choice of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Consequently, from the definition of $W_\delta(t,x)$,

$$W_\delta(t,x) \leq \text{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G^{t,x;u_1,\beta_1(u_1)}_{t,t+\delta}[W(t+\delta, X^{t,x;u_1,\beta_1(u_1)}_{t+\delta})], \ P\text{-a.s.} \tag{10}$$

We use the notation $I_\delta(t,x,u,v) := G^{t,x;u,v}_{t,t+\delta}[W(t+\delta, X^{t,x;u,v}_{t+\delta})]$ and notice that there exists a sequence $\{u^1_i, \ i \geq 1\} \subset \mathcal{U}_{t,t+\delta}$ such that

$$I_\delta(t,x,\beta_1) := \text{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} I_\delta(t,x,u_1,\beta_1(u_1)) = \sup_{i \geq 1} I_\delta(t,x,u^1_i,\beta_1(u^1_i)), \ P\text{-a.s.}$$

For any $\varepsilon > 0$, we put $\widetilde{\Gamma}_i := \{I_\delta(t,x,\beta_1) \leq I_\delta(t,x,u^1_i,\beta_1(u^1_i)) + \varepsilon\} \in \mathcal{F}_t, \ i \geq 1$. Then $\Gamma_1 := \widetilde{\Gamma}_1, \ \Gamma_i := \widetilde{\Gamma}_i \backslash (\cup_{l=1}^{i-1} \widetilde{\Gamma}_l) \in \mathcal{F}_t, \ i \geq 2$, form an

$(\Omega, \mathcal{F}_t)$-partition, and $u_1^\varepsilon := \sum_{i\geq 1} \mathbf{1}_{\Gamma_i} u_i^1$ belongs obviously to $\mathcal{U}_{t,t+\delta}$. Moreover, from the nonanticipativity of β_1 we have $\beta_1(u_1^\varepsilon) = \sum_{i\geq 1} \mathbf{1}_{\Gamma_i} \beta_1(u_i^1)$, and from the uniqueness of the solution of the forward-backward SDE (FBSDE), we deduce that $I_\delta(t,x,u_1^\varepsilon,\beta_1(u_1^\varepsilon)) = \sum_{i\geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x,u_i^1,\beta_1(u_i^1))$, P-a.s. Hence,

$$
\begin{aligned}
W_\delta(t,x) &\leq I_\delta(t,x,\beta_1) \leq \sum_{i\geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x,u_i^1,\beta_1(u_i^1)) + \varepsilon \qquad (11)\\
&= I_\delta(t,x,u_1^\varepsilon,\beta_1(u_1^\varepsilon)) + \varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)})] + \varepsilon, \ P\text{-a.s.}
\end{aligned}
$$

On the other hand, using the fact that $\beta_1(\cdot) := \beta(\cdot \oplus u_2) \in \mathcal{B}_{t,t+\delta}$ does not depend on $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we can define $\beta_2(u_2) := \beta(u_1^\varepsilon \oplus u_2)|_{[t+\delta,T]}$, for all $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. The such defined $\beta_2 : \mathcal{U}_{t+\delta,T} \to \mathcal{V}_{t+\delta,T}$ belongs to $\mathcal{B}_{t+\delta,T}$ since $\beta \in \mathcal{B}_{t,T}$. Therefore, from the definition of $W(t+\delta,y)$ we have, for any $y \in \mathbb{R}^n$,

$$
W(t+\delta,y) \leq \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta,y;u_2,\beta_2(u_2)), \ P\text{-a.s.}
$$

Finally, because there exists a constant $C \in \mathbb{R}$ such that

(i) $|W(t+\delta, y) - W(t+\delta, y')| \le C|y-y'|$ for any $y,\ y' \in \mathbb{R}^n$;
(ii) $|J(t+\delta, y, u_2, \beta_2(u_2)) - J(t+\delta, y', u_2, \beta_2(u_2))| \le C|y-y'|$, P-a.s., for any $u_2 \in \mathcal{U}_{t+\delta,T}$,

(12)

we can show by approximating $X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}$ that

$$W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) \le \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)),\ P\text{-a.s.}$$

To estimate the right side of the latter inequality we note that there exists some sequence $\{u_j^2,\ j \ge 1\} \subset \mathcal{U}_{t+\delta,T}$ such that

$$\begin{aligned}&\operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2))\\ &= \sup_{j\ge 1} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)),\ P\text{-a.s.}\end{aligned}$$

Then, putting $\widetilde{\Delta}_j := \{\operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \le$

$J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)) + \varepsilon\} \in \mathcal{F}_{t+\delta},\ j \geq 1$; we have with $\Delta_1 := \widetilde{\Delta}_1,\ \Delta_j := \widetilde{\Delta}_j \backslash (\cup_{l=1}^{j-1} \widetilde{\Delta}_l) \in \mathcal{F}_{t+\delta},\ j \geq 2$, an $(\Omega, \mathcal{F}_{t+\delta})$-partition and $u_2^\varepsilon := \sum_{j\geq 1} \mathbf{1}_{\Delta_j} u_j^2 \in \mathcal{U}_{t+\delta,T}$. From the nonanticipativity of β_2 we have $\beta_2(u_2^\varepsilon) = \sum_{j\geq 1} \mathbf{1}_{\Delta_j} \beta_2(u_j^2)$, and from the definition of $\beta_1,\ \beta_2$ we know that $\beta(u_1^\varepsilon \oplus u_2^\varepsilon) = \beta_1(u_1^\varepsilon) \oplus \beta_2(u_2^\varepsilon)$. Thus, again from the uniqueness of the solution of our FBSDE, we get

$$
\begin{aligned}
& J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)) = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)} \\
= & \sum_{j\geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)} \\
= & \sum_{j\geq 1} \mathbf{1}_{\Delta_j} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)),\ P\text{-a.s.}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) &\leq \operatorname{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \\
&\leq \sum_{j\geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t,x;u_1^\varepsilon\oplus u_j^2,\beta(u_1^\varepsilon\oplus u_j^2)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u_1^\varepsilon\oplus u_2^\varepsilon,\beta(u_1^\varepsilon\oplus u_2^\varepsilon)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} + \varepsilon, \ P\text{-a.s.},
\end{aligned}
\tag{13}
$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t,T}$. From (11) and (13) and Comparison Theorem for BSDEs, we have

$$\begin{aligned} W_\delta(t,x) &\leq G^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}_{t,t+\delta}[Y^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta}+\varepsilon]+\varepsilon \\ &\leq G^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}_{t,t+\delta}[Y^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta}]+(C+1)\varepsilon \\ &= G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[Y^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta}]+(C+1)\varepsilon \\ &= Y^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t}+(C+1)\varepsilon \\ &\leq \operatorname{esssup}_{u\in\mathcal{U}_{t,T}} Y^{t,x;u,\beta(u)}_{t}+(C+1)\varepsilon,\ P\text{-a.s.} \end{aligned} \tag{14}$$

Since $\beta\in\mathcal{B}_{t,T}$ has been arbitrarily chosen we have (14) for all $\beta\in\mathcal{B}_{t,T}$. Therefore,

$$W_\delta(t,x)\leq \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname{esssup}_{u\in\mathcal{U}_{t,T}} Y^{t,x;u,\beta(u)}_{t}+(C+1)\varepsilon=W(t,x)+(C+1)\varepsilon. \tag{15}$$

Finally, letting $\varepsilon\downarrow 0$, we get $W_\delta(t,x)\leq W(t,x)$.

Lemma 3 $W(t,x)\leq W_\delta(t,x)$.

Proof. We continue to use the notations introduced above, and from the definition of $W_\delta(t,x)$ we have

$$\begin{aligned} W_\delta(t,x) &= \text{essinf}_{\beta_1\in\mathcal{B}_{t,t+\delta}}\text{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}} G^{t,x;u_1,\beta_1(u_1)}_{t,t+\delta}[W(t+\delta, X^{t,x;u_1,\beta_1(u_1)}_{t+\delta})] \\ &= \text{essinf}_{\beta_1\in\mathcal{B}_{t,t+\delta}} I_\delta(t,x,\beta_1), \end{aligned}$$

and, for some sequence $\{\beta^1_i,\ i\geq 1\}\subset\mathcal{B}_{t,t+\delta}$,

$$W_\delta(t,x)=\inf_{i\geq 1} I_\delta(t,x,\beta^1_i),\ P\text{-a.s.}$$

For any $\varepsilon>0$, we let $\widetilde{\Lambda}_i := \{I_\delta(t,x,\beta^1_i)-\varepsilon\leq W_\delta(t,x)\}\in\mathcal{F}_t$, $i\geq 1$, $\Lambda_1:=\widetilde{\Lambda}_1$ and $\Lambda_i:=\widetilde{\Lambda}_i\backslash(\cup_{l=1}^{i-1}\widetilde{\Lambda}_l)\in\mathcal{F}_t$, $i\geq 2$. Then $\{\Lambda_i,\ i\geq 1\}$ is an $(\Omega,\mathcal{F}_t)$-partition, $\beta^\varepsilon_1:=\sum_{i\geq 1}\mathbf{1}_{\Lambda_i}\beta^1_i$ belongs to $\mathcal{B}_{t,t+\delta}$, and from the uniqueness of the solution of our FBSDE we conclude that $I_\delta(t,x,u_1,\beta^\varepsilon_1(u_1))=\sum_{i\geq 1}\mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta^1_i(u_1))$, P-a.s., for all $u_1(\cdot)$

$\in \mathcal{U}_{t,t+\delta}$. Hence,

$$\begin{aligned} W_\delta(t,x) &\geq \sum_{i\geq 1} \mathbf{1}_{\Lambda_i} I_\delta(t,x,\beta_i^1) - \varepsilon \\ &\geq \sum_{i\geq 1} \mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta_i^1(u_1)) - \varepsilon \\ &= I_\delta(t,x,u_1,\beta_1^\varepsilon(u_1)) - \varepsilon \\ &= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})] - \varepsilon, \\ & \quad P\text{-a.s., for all } u_1 \in \mathcal{U}_{t,t+\delta}. \end{aligned} \tag{16}$$

On the other hand, from the definition of $W(t+\delta, y)$, with the same technique as before, we deduce that, for any $y \in \mathbb{R}^n$, there exists $\beta_y^\varepsilon \in \mathcal{B}_{t+\delta,T}$ such that

$$W(t+\delta, y) \geq \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, y; u_2, \beta_y^\varepsilon(u_2)) - \varepsilon, \; P\text{-a.s.} \tag{17}$$

Let $\{O_i\}_{i\geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a decomposition of $\mathbb{R}^n$ such that $\sum_{i\geq 1} O_i = \mathbb{R}^n$ and $\text{diam}(O_i) \leq \varepsilon, \; i \geq 1$. Let y_i be an arbitrarily fixed element of

$O_i,\ i\geq 1$. Defining $[X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]:=\sum_{i\geq 1} y_i \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}$, we have

$$|X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}-[X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]|\leq \varepsilon,\ \text{everywhere on } \Omega,\ \text{for all } u_1\in\mathcal{U}_{t,t+\delta}. \tag{18}$$

Moreover, for each y_i, there exists some $\beta_{y_i}^\varepsilon\in\mathcal{B}_{t+\delta,T}$ such that (17) holds, and, clearly, $\beta_{u_1}^\varepsilon:=\sum_{i\geq 1}\mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}\beta_{y_i}^\varepsilon\in\mathcal{B}_{t+\delta,T}$.

Now we can define the new strategy $\beta^\varepsilon(u):=\beta_1^\varepsilon(u_1)\oplus\beta_{u_1}^\varepsilon(u_2),\ u\in\mathcal{U}_{t,T}$, where $u_1=u|_{[t,t+\delta]},\ u_2=u|_{(t+\delta,T]}$ (restriction of u to $[t,t+\delta]\times\Omega$ and $(t+\delta,T]\times\Omega$, resp.). Obviously, β^ε maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, β^ε is nonanticipating: Indeed, let $S:\Omega\longrightarrow[t,T]$ be an $\mathbb{F}$-stopping time and $u,u'\in\mathcal{U}_{t,T}$ be such that $u\equiv u'$ on $[\![t,S]\!]$. Decomposing $u,\ u'$ into $u_1,u_1'\in\mathcal{U}_{t,t+\delta},\ u_2,u_2'\in\mathcal{U}_{t+\delta,T}$ such that $u=u_1\oplus u_2$ and $u'=u_1'\oplus u_2'$. We have $u_1\equiv u_1'$ on $[\![t,S\wedge(t+\delta)]\!]$ from which we get $\beta_1^\varepsilon(u_1)\equiv\beta_1^\varepsilon(u_1')$ on $[\![t,S\wedge(t+\delta)]\!]$ (recall that β_1^ε is nonanticipating). On the other hand, $u_2\equiv u_2'$ on $]\!]t+\delta,S\vee(t+\delta)]\!](\subset(t+\delta,T]\times\{S>t+\delta\})$, and on $\{S>t+\delta\}$ we have $X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}=X_{t+\delta}^{t,x;u_1',\beta_1^\varepsilon(u_1')}$. Consequently, from our definition, $\beta^\varepsilon_{u_1} = \beta^\varepsilon_{u_1'}$ on $\{S > t+\delta\}$ and $\beta^\varepsilon_{u_1}(u_2) \equiv \beta^\varepsilon_{u_1'}(u_2')$ on $]\!]t+\delta, S\vee(t+\delta)]\!]$. This yields $\beta^\varepsilon(u) = \beta^\varepsilon_1(u_1)\oplus\beta^\varepsilon_{u_1}(u_2) \equiv \beta^\varepsilon_1(u_1')\oplus\beta^\varepsilon_{u_1'}(u_2') = \beta^\varepsilon(u')$ on $[\![t,S]\!]$, from which it follows that $\beta^\varepsilon \in \mathcal{B}_{t,T}$.

Let now $u\in\mathcal{U}_{t,T}$ be arbitrarily chosen and decomposed into $u_1 = u|_{[t,t+\delta]} \in \mathcal{U}_{t,t+\delta}$ and $u_2 = u|_{(t+\delta,T]} \in \mathcal{U}_{t+\delta,T}$. Then, from (16), (12)(i), (18), and Comparison Theorem, we obtain

$$
\begin{aligned}
W_\delta(t,x) &\geq G^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t,t+\delta}[W(t+\delta, X^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t+\delta})] - \varepsilon \\
&\geq G^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t,t+\delta}[W(t+\delta, [X^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t+\delta}]) - C\varepsilon] - \varepsilon \\
&\geq G^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t,t+\delta}[W(t+\delta, [X^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t+\delta}])] - C\varepsilon \\
&= G^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t,t+\delta}\left[\sum_{i\geq 1} \mathbf{1}_{\{X^{t,x;u_1,\beta^\varepsilon_1(u_1)}_{t+\delta}\in O_i\}} W(t+\delta, y_i)\right] - C\varepsilon, \quad P\text{-a.s.}
\end{aligned}
\tag{19}
$$

Furthermore, from (17), (12)(ii), (18), we have

$$
\begin{aligned}
& W_\delta(t,x) \qquad (20)\\
\geq\ & G^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[\sum_{i\geq 1}\mathbf{1}_{\{X^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}} J(t+\delta,y_i;u_2,\beta^\varepsilon_{y_i}(u_2))-\varepsilon\right]-C\varepsilon\\
\geq\ & G^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[\sum_{i\geq 1}\mathbf{1}_{\{X^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}} J(t+\delta,y_i;u_2,\beta^\varepsilon_{y_i}(u_2))\right]-C\varepsilon\\
=\ & G^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[J(t+\delta,[X^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}];u_2,\beta^\varepsilon_{u_1}(u_2))\right]-C\varepsilon\\
\geq\ & G^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}[J(t+\delta,X^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta};u_2,\beta^\varepsilon_{u_1}(u_2))-C\varepsilon]-C\varepsilon\\
\geq\ & G^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}[J(t+\delta,X^{t,x;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta};u_2,\beta^\varepsilon_{u_1}(u_2))]-C\varepsilon\\
=\ & G^{t,x;u,\beta^\varepsilon(u)}_{t,t+\delta}[Y^{t,x,u,\beta^\varepsilon(u)}_{t+\delta}]-C\varepsilon\\
=\ & Y^{t,x;u,\beta^\varepsilon(u)}_t-C\varepsilon,\ P\text{-a.s., for any } u\in\mathcal{U}_{t,T}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
W_\delta(t,x) &\geq \operatorname{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta^\varepsilon(u)) - C\varepsilon \\
&\geq \operatorname{essinf}_{\beta\in\mathcal{B}_{t,T}} \operatorname{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u)) - C\varepsilon \qquad (21) \\
&= W(t,x) - C\varepsilon, \ \ P\text{-a.s.}
\end{aligned}
$$

Finally, letting $\varepsilon \downarrow 0$ we get $W_\delta(t,x) \geq W(t,x)$. The proof is complete.

5. Proof of Theorem 3

We have already seen that the lower value function $W(t,x)$ is Lipschitz continuous in x, uniformly in t. With the help of Theorem 2 we can now also study the continuity property of $W(t,x)$ in t.

Theorem 3 The lower value function $W(t,x)$ is $\frac{1}{2}$-Hölder continuous in t: There exists a constant C such that, for every $x \in \mathbb{R}^n$, $t, t' \in [0,T]$,

$$|W(t,x) - W(t',x)| \leq C(1+|x|)|t-t'|^{\frac{1}{2}}.$$

Proof. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\delta > 0$ be arbitrarily given such that $0 < \delta \leq T - t$. Our objective is to prove the following inequality by using Remarks (a) and (b):

$$-C(1+|x|)\delta^{\frac{1}{2}} \leq W(t,x) - W(t+\delta,x) \leq C(1+|x|)\delta^{\frac{1}{2}}. \tag{22}$$

From it we obtain immediately that W is $\frac{1}{2}$-Hölder continuous in t. We will check only the second inequality in (22); the first one can be shown in a similar way. To this end we note that due to Remark (a), for an arbitrarily small $\varepsilon > 0$,

$$W(t,x) - W(t+\delta,x) \le I_\delta^1 + I_\delta^2 + \varepsilon, \tag{23}$$

where

$$\begin{aligned} I_\delta^1 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})] - G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta,x)], \\ I_\delta^2 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta,x)] - W(t+\delta,x) \end{aligned}$$

for arbitrarily chosen $\beta \in \mathcal{B}_{t,t+\delta}$ and $u^\varepsilon \in \mathcal{U}_{t,t+\delta}$ such that Remark (a) holds. On the other hand, we obtain that, for some constant C independent of the controls u^ε and $\beta(u^\varepsilon)$,

$$\begin{aligned} |I_\delta^1| &\le [CE(|W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - W(t+\delta,x)|^2|\mathcal{F}_t)]^{\frac{1}{2}} \\ &\le [CE(|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t)]^{\frac{1}{2}}, \end{aligned}$$

and since $E[|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] \le C(1+|x|^2)\delta$ we deduce that $|I_\delta^1| \le C(1+|x|)\delta^{\frac{1}{2}}$. From the definition of $G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[\cdot]$ we know that the second term I_δ^2 can be written as

$$\begin{aligned} I_\delta^2 &= E\Big[W(t+\delta,x)+\int_t^{t+\delta} f(s,X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},u_s^\varepsilon,\beta_s(u_\cdot^\varepsilon)) \\ &\quad -\int_t^{t+\delta}\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}dB_s|\mathcal{F}_t\Big]-W(t+\delta,x) \\ &= E\left[\int_t^{t+\delta} f(s,X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},u_s^\varepsilon,\beta_s(u_\cdot^\varepsilon))ds|\mathcal{F}_t\right]. \end{aligned}$$

With the help of the Schwartz inequality we then have

$$\begin{aligned} & |I_\delta^2| \\ \leq\ & \delta^{\frac{1}{2}}E\left[\int_t^{t+\delta}|f(s,X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},u_s^\varepsilon,\beta_s(u_\cdot^\varepsilon))|^2ds|\mathcal{F}_t\right]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq \delta^{\frac{1}{2}} E\Big[\int_t^{t+\delta}(|f(s,X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},0,0,u_s^\varepsilon,\beta_s(u_\cdot^\varepsilon))|+C|\tilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| \\
&+ C|\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|)^2 ds|\mathcal{F}_t\Big]^{\frac{1}{2}} \\
&\leq C\delta^{\frac{1}{2}} E\left[\int_t^{t+\delta}(|1+|X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|+|\tilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|+|\tilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|)^2 ds|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\leq C(1+|x|)\delta^{\frac{1}{2}}.
\end{aligned}
$$

Hence, from (23),

$$W(t,x)-W(t+\delta,x)\leq C(1+|x|)\delta^{\frac{1}{2}}+\varepsilon,$$

and letting $\varepsilon\downarrow 0$ we get the second inequality of (22). The proof is complete.