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2-Player Zero-Sum Stochastic Differential Games

based on common work with
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Objective of the lecture

Generalization of the results of the pioneering work of Fleming and Souganidis on zero-sum two-player SDGs:

• cost functionals defined through controlled BSDEs;
• the admissible control processes can depend on events occurring before the beginning of the game.

This latter extension has the consequence that the cost functionals become random. However, by making use of Girsanov transformation we prove that the upper and the lower value functions of the game remain deterministic. This approach combined with the BSDE method allows to get in a direct way:

upper and lower value functions are deterministic
$\longrightarrow$ Dynamic Programming Principle
$\longrightarrow$ Hamilton-Jacobi-Bellman-Isaacs equations.

At the end of the lecture: some remarks on extensions of the above SDGs: SDGs defined through reflected BSDEs and so on.

Main results

The dynamics of the SDG is given by the controlled SDE

$$\left\{\begin{array}{lll} dX_s^{t,x;u,v} & = & b(s,X_s^{t,x;u,v},u_s,v_s)ds+\sigma(s,X_s^{t,x;u,v},u_s,v_s)dB_s, \\ X_t^{t,x;u,v} & = & x(\in\mathbb{R}^n). \qquad s\in[t,T]. \end{array}\right. \tag{1}$$

The cost functional (interpreted as a payoff for Player I and as a cost for Player II) is introduced by a BSDE:

$$\left\{\begin{array}{lll} -dY_s^{t,x;u,v} & = & f(s,X_s^{t,x;u,v},Y_s^{t,x;u,v},Z_s^{t,x;u,v},u_s,v_s)ds-Z_s^{t,x;u,v}dB_s, \\ Y_T^{t,x;u,v} & = & \Phi(X_T^{t,x;u,v}), \qquad s\in[t,T]. \end{array}\right. \tag{2}$$

The cost functional is given by

$$J(t,x;u,v)=Y_t^{t,x;u,v}. \tag{3}$$

We define the lower value function as follows:

$$W(t,x):=\mathrm{essinf}_{\beta\in\mathcal{B}_{t,T}}\mathrm{esssup}_{u\in\mathcal{U}_{t,T}}J(t,x;u,\beta(u)), \tag{4}$$

and the upper value function is given by

$$U(t,x) := \text{esssup}_{\alpha\in\mathcal{A}_{t,T}}\text{essinf}_{v\in\mathcal{V}_{t,T}} J(t,x;\alpha(v),v). \tag{5}$$

The main results state that W and U are deterministic continuous viscosity solutions of the Bellman–Isaacs equations

$$\begin{cases} \frac{\partial}{\partial t}W(t,x) + H^-(t,x,W,DW,D^2W) = 0, & (t,x)\in[0,T)\times\mathbb{R}^n, \\ W(T,x) = \Phi(x), & x\in\mathbb{R}^n, \end{cases} \tag{6}$$

and

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) + H^+(t,x,U,DU,D^2U) = 0, & (t,x)\in[0,T)\times\mathbb{R}^n, \\ U(T,x) = \Phi(x), & x\in\mathbb{R}^n, \end{cases} \tag{7}$$

respectively, associated with the Hamiltonians

$$H^-(t,x,y,p,X) = \sup_{u\in U}\inf_{v\in V} H(t,x,y,p,X,u,v),$$

$$H^+(t,x,y,p,X) = \inf_{v\in V}\sup_{u\in U} H(t,x,y,p,X,u,v),$$

$(t, x, y, p, X) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n$ (recall that S^n denotes the set of all $n \times n$ symmetric matrices), where

$$\begin{aligned} & H(t, x, y, p, X, u, v) = 1/2 \cdot \mathrm{tr}\left(\sigma\sigma^T(t, x, u, v) X\right) \\ + \quad & p \cdot b(t, x, u, v) + f(t, x, y, p \cdot \sigma(t, x, u, v), u, v). \end{aligned} \tag{8}$$

Preliminaries. Framework

$(\Omega, \mathcal{F}, P)$ canonical Wiener space: for a given finite time horizon $T > 0$,

- $\Omega = C_0([0,T];\mathbb{R}^d)$ (endowed with the supremum norm);
- $B_t(\omega) = \omega(t), t \in [0,T], \omega \in \Omega$ - the coordinate process;
- P - the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$: unique probability measure w.r.t. B is a standard BM;
- $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_P$;
- $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_t = \mathcal{F}_t^B = \sigma\{B_s, s \le t\} \vee \mathcal{N}_P$.

$(\Omega, \mathcal{F}, \mathbb{F}, P; B)$ - the complete, filtered probability space on which we will work.

Dynamics of the game:

Initial data: $t \in [0,T]$, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; R^d)$;

associated doubly controlled stochastic system:

$$\begin{aligned} dX_s^{t,\zeta;u,v} &= b(s, X_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)dB_s, \\ X_t^{t,\zeta;u,v} &= \zeta, \qquad s \in [t,T], \end{aligned} \tag{1}$$

Player I: $u \in \mathcal{U} =: L^0_{\mathbb{F}}(0,T;U)$;

Player II: $v \in \mathcal{V} =: L^0_{\mathbb{F}}(0,T;V)$; U, V - compact metric spaces

and where the mappings

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n, \\ \sigma &: [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n \times d}, \end{aligned}$$

are continuous over $\mathbb{R} \times U \times V$ (for simplicity); Lipschitz in x, uniformly w.r.t (t,u,v), i.e., for some $L \in \mathbb{R}_+$,

$$|\sigma(s,x,u,v)-\sigma(s,x',u,v)|,\ |b(s,x,u,v)-b(s,x',u,v)|\le L|x-x'|;$$

$$|\sigma(s,x,u,v)|,\ |b(s,x,u,v)|\le(1+|x|).$$

Existence and uniqueness of the solution $X^{t,\zeta,u,v}\in\mathcal{S}^2_{\mathbb{F}}(t,T;R^n)$;

from standard estimates: for all $p\ge 2$ there is some $C_p(=C_{p,L})\in\mathbb{R}_+$ s.t.

$$E\left[\sup_{s\in[t,T]}|X_s^{t,\zeta;u,v}-X_s^{t,\zeta';u,v}|^p \mid \mathcal{F}_t\right]\le C_p|\zeta-\zeta'|^p,\ P\text{-a.s.},$$

$$E\left[\sup_{s\in[t,T]}|X_s^{t,\zeta;u,v}|^p \mid \mathcal{F}_t\right]\le C_p(1+|\zeta|^p),\ P\text{-a.s.}$$

Definition of the associated cost functionals

The cost functional is defined with the help of a backward SDE (BSDE):

Associated with $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n), u \in \mathcal{U}$ and $v \in \mathcal{V}$, we consider the BSDE:

$$\begin{array}{rcl} dY_s^{t,\zeta;u,v} & = & -f(s, X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v}, u_s, v_s)ds + Z_s^{t,x\zeta;u,v} dB_s, \\ Y_T^{t,\zeta;u,v} & = & \Phi(X_T^{t,\zeta;u,v}), \qquad\qquad s \in [t,T], \end{array} \tag{2}$$

where

◇ Final cost: $\Phi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

◇ Running cost: $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}$, continuous; Lipschitz in (x,y,z), uniformly w.r.t (t,u,v).

Under the above assumptions: existence and uniqueness of the solution of BSDE (2):

$$(Y^{t,\zeta;u,v}, Z^{t,\zeta;u,v}) \in \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^d).$$

From standard estimates for BSDEs using the corresponding results for the controlled stochastic system: for all $p \geq 2$ there is some $C_p (= C_{p,L}) \in \mathbb{R}_+$ s.t., for any $\zeta,\ \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,\zeta;u,v} - Y_s^{t,\zeta';u,v}|^p \,|\, \mathcal{F}_t\right] \leq C_p|\zeta - \zeta'|^p, \ P\text{-a.s.};$$

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,\zeta;u,v}|^p \,|\, \mathcal{F}_t\right] \leq C_p(1 + |\zeta|^p), \ P\text{-a.s.}$$

In particular, $|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \leq C|\zeta - \zeta'|$, P-a.s.,

$|Y_t^{t,\zeta;u,v}| \leq C(1 + |\zeta|)$, P-a.s.

Let $t \in [0,T]$, $\zeta = x \in R^n$ - deterministic initial data; $u \in \mathcal{U}$, $v \in \mathcal{V}$;
associated cost functional for the game over the time interval $[t,T]$:
$J(t,x;u,v) := Y_t^{t,x;u,v} \big(\in L^2(\Omega, \mathcal{F}_t, P) \big)$.

Remark 1: (i) If $f \equiv 0 : J(t,x;u,v) = E[\Phi(X_T^{t,x;u,v}) | \mathcal{F}_t]$;
(ii) If f doesn't depend on (y, z):

$$J(t,x;u,v) = E[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s, X_s^{t,x;u,v}, u_s, v_s) ds | \mathcal{F}_t].$$

Notice: From $J(t,x,u,v) := Y_t^{t,x,u,v}$ and the standard estimates for $Y_t^{t,x,u,v}$:

$J(t,x,u,v) \in L^\infty(\Omega, \mathcal{F}_t, P)$, $(t,x,u,v) \in [0,T] \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, and:

- $|J(t,x,u,v) - J(t,x',u,v)| \leq C|x - x'|$,
- $|J(t,x,u,v)| \leq C(1 + |x|)$,

P-a.s., for all $x, x' \in \mathbb{R}^n$, $(t,u,v) \in [0,T] \times \mathcal{U} \times \mathcal{V}$;

Which kind of game shall we study?

Objective of Player I : maximization of $J(t,x,u,v)$ over $u \in \mathcal{U}$;

Objective of Player II : minimization of $J(t,x,u,v)$ over $v \in \mathcal{V}$;

the both players have the same cost functional, it's the gain for player I, the loss for player II - one speaks of "2-player zero-sum stochastic differential games";

in non-zero sum games: Player i has cost functional $J_i(t,x,u_1,u_2...)$, $i \geq 1$, the players want to maximize their cost functionals; problem of the existence and the characterization of Nash equilibrium points.

<u>Game "Control against Control"</u>?

• In general no value of the game, i.e., the result of the game depends on which player begins, and this even if Isaacs' condition is fulfilled (precision later); example: pursuit games (*Example in another slide*.)

• Games "Control against Control" with value if: $n = d$; $\sigma \in \mathbb{R}^{n\times n}(x)$ is independent of (u,v) and invertible (as matrix); $\sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is

Lipschitz (S.HAMADENE, J.-P.LEPELTIER, S.PENG 1997).

Game "Strategy against Control":

This concept has been known in the deterministic differential game theory (A.FRIEDMAN, W.H.FLEMING,..)and has been translated later by W.H.FLEMING, P.E.SOUGANIDIS (1989) to the theory of stochastic differential games.

Here: a generalization of the concept of W.H.FLEMING, P.E.SOUGANIDIS (1989); a comparison of their concept with ours: later.

Admissible controls, admissible strategies

Definition 1: (*admissible controls* for a game over the time interval $[t, T]$)

- For Player I: $\mathcal{U}_{t,T} =: L^0_{\mathbb{F}}(t, T; U)$;
- for Player II: $\mathcal{V}_{t,T} =: L^0_{\mathbb{F}}(t, T; V)$.

Notice: Different from the concept by FLEMING, SOUGANIDIS, the controls $u \in \mathcal{U}_{t,s}, v \in \mathcal{V}_{t,s}$ are not supposed to be independent of $\mathcal{F}_t$.

Definition 2: (*admissible strategies* for a game over the time interval $[t,T]$)

• For Player II: $\beta : \mathcal{U}_{t,T} \longrightarrow \mathcal{V}_{t,T}$ non anticipating, i.e.,

for any $\mathbb{F}-$ stopping time $S : \Omega \to [t,T]$ and any admissible controls $u_1, u_2 \in \mathcal{U}_{t,T}$

($u_1 = u_2$ dsdP-a.e. on $[\![t,S]\!] \Longrightarrow \beta(u_1) = \beta(u_2)$ dsdP-a.e. on $[\![t,S]\!]$).

$\mathcal{B}_{t,T} := \{\beta : \mathcal{U}_{t,T} \to \mathcal{V}_{t,T} | \beta$ is nonanticipating$\}$.

Analogously we introduce

• for Player I: $\mathcal{A}_{t,T} := \{\alpha : \mathcal{V}_{t,T} \to \mathcal{U}_{t,T} | \alpha$ is nonanticipating$\}$.

Value Functions:

Notice: From $J(t,x,u,v) := Y_t^{t,x,u,v}$ and the standard estimates for $Y_t^{t,x,u,v}$:

$J(t,x,u,v) \in L^\infty(\Omega, \mathcal{F}_t, P)$, $(t,x,u,v) \in [0,T] \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, and:

- $|J(t,x,u,v) - J(t,x',u,v)| \leq C|x-x'|$,
- $|J(t,x,u,v)| \leq C(1+|x|)$,

  P-a.s., for all $x, x' \in \mathbb{R}^n$, $(t,u,v) \in [0,T] \times \mathcal{U} \times \mathcal{V}$;

- $Y_t^{t,\zeta,u,v} = J(t,\zeta,u,v)\left(:= J(t,x,u,v)\big|_{x=\zeta}\right)$, P-a.s. (*prove it in another slides*)

The above estimates for $J(t,x,u,v)$ allow to introduce:

- Lower Value Function:

$$W(t,x) := \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u));$$

- Upper Value Function:

$$U(t,x) := \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{v \in \mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

Remarks. • Justification of the names "upper" and "lower" value functions: later we will see $W \leq U$; the proof is far from being obvious and uses the comparison principle for the associated Bellman-Isaacs equations, it will be given later.

• The esssup, essinf should be understood as ones w.r.t. a uniformly bounded, indexed family of $\mathcal{F}_t$-measurable r.v.; see: Dunford/Schwartz (1957). Consequently:

$W(t,x), U(t,x) \in L^\infty(\Omega, \mathcal{F}_t, P)$, and, for some $C \in \mathbb{R}_+$ (independent of (t,x)):

- $|W(t,x) - W(t,x')| + |U(t,x) - U(t,x')| \leq C|x-x'|$, P-a.s.,
- $|W(t,x)| + |U(t,x)| \leq C(1+|x|)$, P-a.s., for all $t \in [0,T]$, $x,x' \in \mathbb{R}^n$.

Although W, U are a priori random variables, we have:

Proposition 1: $W(t,x) = E[W(t,x)]$, $U(t,x) = E[U(t,x)]$, $(t,x) \in [0,T] \times \mathbb{R}^n$, i.e., W and U admit a deterministic version with which we identify the both functions from now on.

Corollary. $W, U : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are such that

$$|W(t,x) - W(t,x')| + |U(t,x) - U(t,x')| \leq C|x-x'|,$$

$$|W(t,x)| + |U(t,x)| \leq C(1+|x|), \text{ for all } t \in [0,T], x,x' \in \mathbb{R}^n.$$

Some Remarks preceding the proof of the Proposition 1.

1) Concept of W.H.FLEMING, P.E.SOUGANIDIS (1989):

their running cost $f(s,x,y,z)$ doesn't depend on (y,z), i.e., their cost functional is the classical one;

more essential:

• *admissible controls*: instead of $\mathcal{U}_{t,T}$: $\mathcal{U}^t_{t,T} := L^0_{\mathbb{F}^t}(t,T;U)$,

instead of $\mathcal{V}_{t,T}$: $\mathcal{V}^t_{t,T} := L^0_{\mathbb{F}^t}(t,T;V)$,

$$\mathbb{F}^t = (\mathcal{F}^t_s)_{s\in[t,T]},\ \mathcal{F}^t_s := \sigma\{B_r - B_t,\, r\in[t,s]\}\vee\mathcal{N}_P,\ s\in[t,T];$$

• *admissible strategies*: instead of $\mathcal{B}_{t,T}$: $\mathcal{B}^t_{t,T}$ - the set of all non-anticipating mappings $\beta : \mathcal{U}^t_{t,T} \longrightarrow \mathcal{V}^t_{t,T}$,

(non-anticipativity is understood in the same sense as that in the definition of $\mathcal{B}_{t,T}$); analogous definition of $\mathcal{A}^t_{t,T}$.

Their cost functional

$$J(t,x;u,v) := E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(s, X_s^{t,x,u,v}, u_s, v_s) \,|\, \mathcal{F}_t\right]$$
$$= E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(s, X_s^{t,x,u,v}, u_s, v_s)\right]$$

is automatically deterministic, and so are their upper and lower value functions:

$$\overline{W}(t,x) := \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u)), \quad \overline{U}(t,x) := \inf_{\alpha \in \mathcal{A}_{t,T}} \inf_{v \in \mathcal{V}_{t,T}} J(t,x;\alpha(v),v).$$

Our approach in comparison with theirs:

- Proof that W, U are deterministic is not evident, but later:
- Straight forward approach without approximation by discrete schemes, without further technical notions (like π-controls, r-strategies), without using the Bellman-Isaacs equation for proving the DPP:

  - Direct deduction of the DPP from the definition of W, U (with the help of Peng's notion of backward semigroups (1997));

- Direct deduction of the Bellman-Isaacs equations for W, U from the DPP (with the help of a scheme of 3 BSDEs, the so-called Peng's BSDE method developed by him for control problems);

- Adaptation of a uniqueness proof for integro-PDEs (G.BARLES, R.BUCKDAHN, E.PARDOUX) to Bellman-Isaacs equations.

2) Proof that W is deterministic for control problems (Peng, 1997):

$U \subset R^m$ compact subset; σ, b, f don't depend on v, and are supposed to be Lipschitz in all their variables (x,u) and (x,y,z,u), resp.;

$$W(t,x) := \operatorname{esssup}_{u\in \mathcal{U}_{t,T}} J(t,x,u).$$

Then:

$$|J(t,x,u)-J(t,x,u')|^2 \leq CE\left[\int_t^T |u_s-u'_s|^2 \,|\, \mathcal{F}_t\right], P\text{-a.s., } u,u' \in \mathcal{U}_{t,T}.$$

Let

$$\mathcal{U}_{t,T}^{step} := \{u = \sum_{i,k,\ell=1}^{N} I_{A_i} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} : t=t_0<t_1<\cdots<t_N=T,$$

$$\theta_{k,\ell} \in U, A_{k,\ell} \in \mathcal{F}_t, B_{k,\ell} \in \mathcal{F}^t_{t_{k_1}}, N \geq 1\};$$

then

$$W(t,x) = \operatorname{esssup}_{u \in \mathcal{U}^{step}_{t,T}} J(t,x,u).$$

On the other hand, for $u \in \mathcal{U}^{step}_{t,T}$ as above:

$$u = \sum_{i,k,\ell=1}^{N} I_{A_i} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} = \sum_{i=1}^{N} I_{A_i} \Big(\sum_{k,\ell=1}^{N} I_{B_{k,\ell}} \theta_{k,\ell} I_{(t_{k-1},t_k]} \Big)$$

$$= \sum_{i=1}^{N} I_{A_i} u^i, \qquad \text{where } u^i \in \mathcal{U}^t_{t,T},\ 1 \leq i \leq N,$$

and from the uniqueness of the solutions of the controlled forward and backward SDEs:

$$J(t,x,u) = \textstyle\sum_{i=1}^{N} I_{A_i} J(t,x,u^i) \leq \sup_{1 \leq i \leq N} J(t,x,u^i) \leq \sup_{u' \in \mathcal{U}^t_{t,T}} J(t,x,u'),$$

and, consequently, since $\mathcal{U}^t_{t,T} \subset \mathcal{U}_{t,T}$,

$$W(t,x) = \sup_{u' \in \mathcal{U}^t_{t,T}} J(t,x,u');$$

the right-hand side is deterministic and so is $W(t,x)$.

Peng's argument doesn't work for stochastic differential games:

- One cannot restrict to continuous strategies;

- W.r.t. which norm should the spaces of admissible strategies be approximable by which "admissible step strategies"?

Here new approach for the proof that W is deterministic; even continuity of the coefficients in (u,v) is not needed.

Proof the the upper and lower value functions are deterministic: main tool is a Girsanov transformation argument (*show it in another slides*).

Dynamic Programming Principle (DPP)

Some Preparation: *Stochastic Backward Semigroup*, S.Peng,1997: book on his BSDE method for stochastic control problems:

S.Peng, (1997)*BSDE and stochastic optimizations; Topics in stochastic analysis.* J.Yan, S.Peng, S.Fang and L.Wu, Chapter 2, Science Press. Beijing (in Chinese).

Given

$(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $\delta > 0 (t+\delta \leq T)$, $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$,

$\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$ - terminal condition for time horizon $t+\delta$,

we put

$$G^{t,x;u,v}_{s,t+\delta}[\eta] := \tilde{Y}_s, \ s \in [t, t+\delta],$$

where $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2_{\mathbb{F}}(t, t+\delta) \times L^2_{\mathbb{F}}(t, t+\delta; \mathbb{R}^d)$ is the unique solution of the following BSDE with time horizon $t+\delta$:

$$\begin{cases} d\tilde{Y}_s &= -f(s, X^{t,x;u,v}_s, \tilde{Y}_s, \tilde{Z}_s, u_s, v_s)ds - \tilde{Z}^{t,x;u,v}_s dB_s, \quad \in [t, t+\delta], \\ \tilde{Y}_{t+\delta} &= \eta; \end{cases}$$

$X^{t,x;u,v}$ is the solution of our doubly controlled stochastic system (the forward SDE).

Remark:

(i) (*The semigroup property*) For $0 \leq t \leq s \leq s' \leq t+\delta \leq T$,

$$G^{t,x;u,v}_{s,s'}[G^{t,x;u,v}_{s',t+\delta}[\eta]] = G^{t,x;u,v}_{s,t+\delta}[\eta].$$

(ii) $G^{t,x;u,v}_{s,T}[\Phi(X^{t,x;u,v}_T)] = Y^{t,x;u,v}_s$, P-a.s., $s \in [t,T]$.

In particular, for $s = t$,

$$G^{t,x;u,v}_{t,T}[\Phi(X^{t,x;u,v}_T)] = J(t,x;u,v), \quad \text{P-a.s..}$$

(iii) $J(t,x;u,v) = Y^{t,x;u,v}_t = G^{t,x;u,v}_{t,T}[\Phi(X^{t,x;u,v}_T)]$

$$= G^{t,x;u,v}_{t,t+\delta}[Y^{t,x;u,v}_{t+\delta}] = G^{t,x;u,v}_{t,t+\delta}[J(t+\delta, X^{t,x;u,v}_{t+\delta}; u,v)].$$

The latter relation follows from the uniqueness of the solution of the forward and the backward equations: for $\zeta = X^{t,x;u,v}_{t+h}(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$Y^{t,x;u,v}_{t+\delta} = Y^{t+h, X^{t,x;u,v}_{t+h}; u,v}_{t+h} = Y^{t+h,\zeta;u,v}_{t+h} = J(t+\delta, \zeta; u,v)$$

$$= J(t+\delta, X^{t,x;u,v}_{t+h}; u,v).$$

(iv) If f doesn't depend on (y, z) we have the classical case of conditional expectation:

$$G_{t,t+\delta}^{t,x;u,v}[\eta] = E\left[\eta + \int_t^{t+\delta} f(s, X_s^{t,x;u,v}, u_s, v_s) ds | \mathcal{F}_t\right], \text{ P-a.s.}$$

Taking now $\eta = W(t+\delta, X_{t+\delta}^{t,x;u,v})$ (resp., $U(t+\delta, X_{t+\delta}^{t,x;u,v})$) it becomes clear from the classical DPP from control problems that our DPP shall write as follows:

Theorem 2 (DPP): For any $0 \le t < t+\delta \le T$, $x \in \mathbb{R}^n$,

$$W(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})];$$

$$U(t,x) = \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{essinf}_{v \in \mathcal{V}_{t,t+\delta}} G^{t,x;\alpha(v),v}_{t,t+\delta}[U(t+\delta, X^{t,x;\alpha(v),v}_{t+\delta})].$$

Remark: If $f(x,y,z,u,v)$ is independent of (y,z) the above DPP writes:

$$W(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} E[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta}) + \int_t^{t+\delta} f(s, X^{t,x;u,\beta(u)}_s, u_s, v_s) ds | \mathcal{F}_t];$$

analogous for $U(t,x)$.

Sketch of proof: auxiliary function:

$$W_\delta(t,x) := \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta(u)}_{t+\delta})];$$

(i) $W_\delta(t,x)$ is deterministic: same Girsanov transformation argument as for $W(t,x)$.

(ii) For any $\varepsilon > 0$, and for any $\beta \in \mathcal{B}_{t,T}$, there exists some $u^\varepsilon \in \mathcal{U}_{t,T}$ such that

$$W_\delta(t,x) \leq J(t,x;u^\varepsilon,\beta(u^\varepsilon)) + \varepsilon, \ P\text{-a.s.},$$

from where: $W_\delta(t,x) \leq W(t,x)$.

(iii) For any $\varepsilon > 0$, there exists $\beta^\varepsilon \in \mathcal{B}_{t,T}$ such that $\forall u \in \mathcal{U}_{t,T}$:

$$W_\delta(t,x) \geq J(t,x;u,\beta^\varepsilon(u)) - \varepsilon, \ \text{P-a.s.},$$

from where: $W_\delta(t,x) \geq W(t,x)$. (*Show the proof in another slides.*)

Remark From the proof we see that for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\delta > 0$, with $0 < \delta \leq T - t$ and $\varepsilon > 0$, the following hold:

(a) For every $\beta \in \mathcal{B}_{t,t+\delta}$, there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that

$$W(t,x)(= W_\delta(t,x)) \leq G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})] + \varepsilon, \ P\text{-a.s.}$$

(b) There exists some $\beta^\varepsilon \in \mathcal{B}_{t,t+\delta}$ such that, for all $u \in \mathcal{U}_{t,t+\delta}$,

$$W(t,x)(=W_\delta(t,x)) \geq G_{t,t+\delta}^{t,x;u,\beta^\varepsilon(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta^\varepsilon(u)})] - \varepsilon,\ P\text{-a.s.}$$

With the help of the DPP we can prove the following

Theorem 3. $W(.,x)$ and $U(.,x)$ are $\frac{1}{2}$−Hölder continuous, for all $x \in \mathbb{R}^n$: There is some $C \in \mathbb{R}_+$ such that, for every $x \in \mathbf{R}^n,\ t,t' \in [0,T]$,

$$|W(t,x) - W(t',x)| + |U(t,x) - U(t',x)| \leq C(1+|x|)|t-t'|^{\frac{1}{2}}.$$

(*Show the proof in another slides.*)

Bellman-Isaacs equations. Existence theorem.

We consider the Hamiltonian

$$H(t,x,y,p,S,u,v)$$

$$:= \frac{1}{2}tr(\sigma\sigma^T(t,x,u,v)S) + b(t,x,u,v).p + f(t,x,y,p.\sigma(t,x,u,v),u,v),$$

$$(t,x,y,p,S,u,v) \in [0,T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n \times U \times V.$$

$$H^-(t,x,y,p,S) := \sup_{u\in U}\inf_{v\in V} H(t,x,y,p,S,u,v);$$

$$H^+(t,x,y,p,S) := \inf_{v\in V}\sup_{u\in U} H(t,x,y,p,S,u,v).$$

We will show that, in viscosity sense, we have the following Bellman-Isaacs equations:

$$\frac{\partial W}{\partial t}(t,x) + H^{-}(t,x,W,DW,D^2W) = 0, W(T,x) = \Phi(x), \qquad (3)$$

and

$$\frac{\partial U}{\partial t}(t,x) + H^{+}(t,x,U,DU,D^2U) = 0, U(T,x) = \Phi(x). \qquad (4)$$

More precisely,

Theorem 4 (Existence Theorem): $W \in C_\ell([0,T] \times \mathbf{R}^n)$ is a viscosity solution of equation (3), and $U \in C_\ell([0,T] \times \mathbf{R}^n)$ is a viscosity solution of equation (4).

(*Recall of the notion of viscosity solution if necessary*.)

We come after back to the proof of the existence theorem.

Theorem 5 (Comparison Principle): Let $u_1 \in$USC$([0,T] \times \mathbb{R}^n)$ be a viscosity subsolution of (3) (resp., of (4)) and $u_2 \in$LSC$([0,T] \times \mathbb{R}^n)$ be a viscosity supersolution of (3) (resp., of (4)). Moreover, we suppose that both functions belong to the class of measurable functions V with the following growth condition:

$\exists A > 0$ such that, uniformly in $t \in [0,T]$,

$V(t,x)\exp\{-A[\ln|x|]^2\}\left(=\frac{V(t,x)}{|x|^{A\ln|x|}}\right) \longrightarrow 0$ as $|x| \to +\infty$.

Then $u_1 \leq u_2$, on $[0,T] \times \mathbb{R}^n$.

Corollary. Let u_1 and u_2 be continuous viscosity solutions of (3) (resp., of (4)). Moreover, we suppose that both functions satisfy the above growth condition. Then $u_1 = u_2$, on $[0,T] \times \mathbb{R}^n$.

Remarks 1: • Barles, Buckdahn, Pardoux (1997) proved that this growth condition is the optimal one for the uniqueness of the (viscosity) solution of the heat equation.

• The proof of the uniqueness theorem adapts the argument of Barles, Buckdahn, Pardoux (1997) to Bellman-Isaacs equations (and, hence, also to Hamilton-Jacobi-Bellman equations).

Remarks 2: • $W \in C_\ell([0,T] \times \mathbb{R}^n)$ (resp., $U \in C_\ell([0,T] \times \mathbb{R}^n)$) is the unique viscosity solution of (3) (resp., (4)) in the class of continuous functions with the above growth condition, and so in particular in $C_p([0,T] \times \mathbb{R}^n)$.

• Notice that $H^- \leq H^+$; consequently, W is a viscosity subsolution of (4), and from the comparison principle: $W \leq U$. This justifies the names "lower value function" for W and "upper" value function for U.

• If the Isaacs' condition holds: $H^- = H^+$ on $[0,T] \times \mathbb{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$, then the equations (3) and (4) are the same, and from the uniqueness of the viscosity solution in $C_p([0,T] \times \mathbb{R}^n)$: $W = U$. One says the "game has a value".

• For the case that $f(s,x,y,z,u,v)$ doesn't depend on (y,z), W.H.FLEMING, P.E.SOUGANIDIS have got the same Bellman-Isaacs equations as we have got. From the uniqueness of the viscosity solutions in $C_p([0,T] \times \mathbb{R}^n)$:

$$\overline{W}(t,x)\big(:= \inf_{\beta \in \mathcal{B}_{t,T}^t} \sup_{u \in \mathcal{U}_{t,T}^t} J(t,x;u,\beta(u))\big) = W(t,x);$$

$$\overline{U}(t,x)\big(:= \inf_{\alpha\in\mathcal{A}_{t,T}^t} \inf_{v\in\mathcal{V}_{t,T}^t} J(t,x;\alpha(v),v)\big) = U(t,x).$$

Sketch of the proof of the existence theorem:

We prove that W is a continuous viscosity solution of the PDE

$$\frac{\partial W}{\partial t}(t,x) + H^-(t,x,W,DW,D^2W) = 0, W(T,x) = \Phi(x), \qquad (3)$$

with

$$H^-(t,x,y,p,S) := \sup_{u\in U}\inf_{v\in V} H(x,y,p,S,u,v);$$

and

$$H(x,y,p,S,u,v) := \frac{1}{2}tr(\sigma\sigma^T(x,u,v)S) + f(x,y,p.\sigma(x,u,v),u,v),$$

$(x,y,p,S,u,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U \times V$(for shortness but without restriction of the method: $b=0$; coefficients don't depend on time s).

Let $\varphi \in C^3_{\ell,b}([0,T]\times\mathbb{R}^n)$ be an arbitrary but fixed test function. We define:

$$L_{x,u,v}\varphi(s,x) = \frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x,u,v)D^2\varphi(s,x)),$$

and

$$F(s,x,y,z,u,v) := L_{x,u,v}\varphi(s,x)$$
$$+ f(s,x,y+\varphi(s,x)), z + D\varphi(s,x)\sigma(x,u,v), u, v).$$

<u>Notice</u>:

$$\frac{\partial}{\partial t}\varphi(t,x) + H^-(t,x,(\varphi, D\varphi, D^2\varphi)(t,x)) = \sup_{u\in U}\inf_{v\in V} F(t,x,0,0,u,v).$$

So we have to prove that if $W - \varphi \leq$ (resp., $\geq$)$= (W-\varphi)(t,x) = 0$ then

$\sup_{u\in U}\inf_{v\in V} F(t,x,0,0,u,v) \geq 0$ ($\longrightarrow$ subsolution)

(resp., $\sup_{u\in U}\inf_{v\in V} F(t,x,0,0,u,v) \leq 0$ ($\longrightarrow$ supersolution)).

<u>Peng's BSDE method</u>: *"Approximating BSDEs"*

<u>1st BSDE</u>: For $0 < \delta \leq T - t$ small, $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$:

$$dY_s^{1,u,v,\delta} = -F(s, X_s^{t,x,u,v}, Y_s^{1,u,v,\delta}, Z_s^{1,u,v,\delta}, u_s, v_s)ds + Z_s^{1,u,v,\delta}dB_s,$$
$$Y_{t+\delta}^{1,u,v,\delta} = 0.$$

Notice: • The BSDE admits a unique solution $(Y^{1,u,v,\delta}, Z^{1,u,v,\delta}) \in \mathcal{S}^2_{\mathbb{F}}(t,t+\delta) \times L^2_{\mathbb{F}}(t,t+\delta;\mathbb{R}^d)$.

• $Y_s^{1,u,v,\delta} = G^{t,x,u,v}_{s,t+\delta}[\varphi(t+\delta, X^{t,x,u,v}_{t+\delta})] - \varphi(s, X_s^{t,x,u,v})$, $s \in [t,t+\delta]$, P-a.s.

(*Idea of the proof: evtl. at the blackboard.*)

The 1st BSDE will translate the DPP in BSDE property. Approximation of the 1st BSDE:

2nd BSDE: For $0 < \delta \le T - t$ small, $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$:

$$dY_s^{2,u,v,\delta} = -F(s, x, Y_s^{2,u,v,\delta}, Z_s^{2,u,v,\delta}, u_s, v_s)ds + Z_s^{2,u,v,\delta} dB_s,\ s \in [t,t+\delta],$$

$$Y_{t+\delta}^{2,u,v,\delta} = 0.$$

$\Big\{$ Recall:

$$dY_s^{1,u,v,\delta} = -F(s, X_s^{t,x,u,v}, Y_s^{1,u,v,\delta}, Z_s^{1,u,v,\delta}, u_s, v_s)ds + Z_s^{1,u,v,\delta} dB_s,$$

$$Y_{t+\delta}^{1,u,v,\delta} = 0.$$

Our objective: To approximate the 1st BSDE -the key to use the DPP- by the 2nd BSDE, and the 2nd BSDE by a deterministic ordinary differential equation with terminal condition. $\Big\}$

Lemma. There is some $C \in \mathbb{R}_+$ s.t., for all $\delta \in (0, T-t]$ sufficiently small and all $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$:

$$|Y_t^{1,u,v,\delta} - Y_t^{2,u,v,\delta}| \leq C\delta^{3/2}, P\text{-a.s.}$$

(*Idea of proof at blackboard.*)

Let $F_0(s,x,y,z) = \sup_{u \in U} \inf_{v \in V} F(s,x,y,z,u,v)$.

<u>3rd BSDE</u>: For $0 < \delta \leq T-t$ small:

$$dY_s^{0,\delta} = -F_0(s,x,Y_s^{0,\delta},0)ds\big(+0dB_s\big),\ s \in [t,t+\delta],$$
$$Y_{t+\delta}^{0,\delta} = 0.$$

Lemma. $\text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} \text{essinf}_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v,\delta} = Y_t^{0,\delta}$.

(*Proof at the blackboard.*)

These 3 BSDEs allow to prove:

1)$\underline{W \text{ is a subsolution}}$: (*blackboard*)

2)$\underline{W \text{ is a supersolution}}$: (*blackboard*)

Perspectives (and work which is already done):

• 2-Person zero-sum SDG with reflection at one obstacle, at two obstacles (LI JUAN, R.B., submitted, arXiv)

• 2-Person zero-sum SDG with jumps (in redaction; LI JUAN, R.B.)

• Nonzero-sum SDGs, existence of Nash equilibrium points, Non anticipative Strategies with Delay (NAD-strategies); this concept allows to study games “NAD-strategy against NAD-strategy” (advantage: “symmetry” between both players; disadvantage: Nash equilibria can be studied only by ε-approximations): (P.CARDALIAGUET, C.RAINER, R.B., 2004)

- SDG with asymmetric information (P.CARDALIGUET, C.RAINER, submitted, web page of C.Rainer)
- Measure-valued differential games (P.CARDALIAGUET, M.QUINCAMPOIX)
- A lot of other works.