A splitting approximation scheme for the Navier-Stokes equations

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Abstract

In this paper we propose a splitting approximation scheme for Navier–Stokes equations and prove its convergence. We consider the Navier–Stokes system

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta v + (v \cdot \nabla)v + \nabla p = 0 && \text{in } Q = \Omega \times (0,T),\\ &\text{div } v = 0 && \text{in } Q,\\ &v = 0, && \text{on } \Sigma = \partial\Omega \times (0,T),\\ &v(\cdot,0) = v_0(\cdot) && \text{in } \Omega. \end{aligned} \tag{1}$$

Here $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$, v is the velocity, p is the scalar pressure and v_0 is the initial velocity.

Also we consider the Euler equations for incompressible fluid flow

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla q = 0 && \text{in } Q,\\ &\text{div } u = 0 && \text{in } Q,\\ &u \cdot N = 0, && \text{on } \Sigma\\ &u(\cdot,0) = u_0(\cdot) && \text{in } \Omega, \end{aligned} \tag{2}$$

and the Stokes system

$$\begin{aligned} &\frac{\partial w}{\partial t} - \Delta w + \nabla r = 0 && \text{in } Q,\\ &\text{div } w = 0 && \text{in } Q,\\ &w = 0, && \text{on } \Sigma\\ &w(\cdot,0) = w_0(\cdot) && \text{in } \Omega, \end{aligned} \tag{3}$$

where N is the outward normal to $\partial\Omega$ and u_0, w_0 are the initial velocities.

Let us denote by $(E(t)u_0)(\cdot)$ the solution $u(t,\cdot)$ of (2) and by $A_p = -P_p\Delta$ the Stokes operator for $p > 1$. With this notation the system (3)

may be written as an evolution equation in V_p (V_p is the standard space of free–divergence vectors for $p > 1$)

$$\frac{\partial u}{\partial t} + A_p u = 0 \text{ in } V_p. \tag{4}$$

Now we shall present the proposed splitting approximation scheme. Let $m \in \mathbb{N}^*$ and $\varepsilon = \dfrac{T}{m}$. Consider

$$\begin{aligned} &u_0 = v_0, \\ &u_{n+1} = (I + \varepsilon A_p)^{-1} E(\varepsilon) u_n, \ 0 \le n \le m-1, \end{aligned} \tag{5}$$

and define the approximate solution of (1) as

$$\begin{aligned} &u_E^\varepsilon(t_n + s) = E(s) u_n, && 0 < s \le \varepsilon, \\ &u^\varepsilon(t_n + s) = (I + \varepsilon A_p)^{-1} u_E^\varepsilon(t_n + s), && 0 < s \le \varepsilon, \ 0 \le n \le m-1, \\ &u^\varepsilon(0) = v_0. \end{aligned} \tag{6}$$

The main result is

Theorem 1 *If v_0 is free-divergence and belongs to the Sobolev space $(H^{2,p}(\Omega))^d$, $p > d$, then the approximate solution u^ε is well-defined in $(H^{2,p}(\Omega))^d$ and satisfies*

$$\sup_{0 \le t \le T} |u^\varepsilon(\cdot, t) - v(\cdot, t)|_{(L^p(\Omega))^d} \le c\varepsilon,$$

where v is the strong solution of (1) *and $c > 0$ is a constant independent of ε.*

References

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