On deviation and moment inequalities for dependent sequences and applications to intermittent maps

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joint work with J. Dedecker

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The aim

- Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real-valued r.v.'s. in $\mathbf{L}^2$.
- The aim is to find a "good" upper bound for the quantity

$$\mathbf{P}\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_i-\mathbf{E}(X_i))\Big|\ge x\Big)$$

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  for any $x>0$.
- "Good" in the sense that this upper bound implies "sharp" moment inequalities or large deviation inequalities as : for some $\alpha>0$ and any $x>0$

$$n^{\alpha}\,\mathbf{P}\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_i-\mathbf{E}(X_i))\Big|\ge nx\Big)\le C(x)\,.$$

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- Note that in the iid case and if $\mathbf{P}(|X_1-\mathbf{E}(X_1)|\geq nx\Big)\sim\frac{c}{(nx)^p}$, then
  $\liminf_{n\to\infty} n^{p-1}\,\mathbf{P}\Big(\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(X_i-\mathbf{E}(X_i))\Big|\geq nx\Big)>0.$

The Fuk-Nagaev's inequality (1971) in the independent setting

- Let $(X_i)_{i\geq 1}$ be a sequence of independent real-valued r.v.'s. in $\mathbf{L}^2$. Define $S_0 = 0$,

$$S_k = \sum_{i=1}^{k} (X_i - \mathbf{E}(X_i)) \text{ and } S_n^* = \max_{0\leq k\leq n} S_k \,,$$

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- Then, for any $v_n^2 \geq \sum_{i=1}^{n} \mathbf{E}(X_k^2)$ and any positive reals (x, y),

$$\mathbf{P}(S_n^* \geq x) \leq \exp(-y^{-2} v_n^2 h(xy/v_n^2)) + \sum_{i=1}^{n} \mathbf{P}(X_i > y)$$

where $h(u) = (1+u)\log(1+u) - u \geq \frac{u}{2}\log(1+u)$.

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- We also have: for any $\varepsilon > 0$,

$$\mathbf{P}(S_n^* \geq (1+\varepsilon)x) \leq \exp(-y^{-2}v_n^2 h(xy/v_n^2)) + \frac{1}{x\varepsilon}\sum_{i=1}^{n}\mathbf{E}((X_i - y)_+)$$

Some applications (1)

- To simplify, take the $X_i's$ identically distributed as X and such that $\mathbf{E}(X) = 0$. Set $S_k = \sum_{i=1}^{k} X_i$. Take $p \geq 2$.
- Using the fact that

$$\mathbf{E}(\max_{1\leq i\leq n} |S_k|^p) = p \int_0^\infty x^{p-1} \mathbb{P}(\max_{1\leq i\leq n} |S_k| \geq x) dx$$

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  we get
- for any $r > 0$,

$$\mathbf{E}(\max_{1\leq i\leq n} |S_k|^p) \ll \int_0^\infty x^{p-1}\Big(1+\frac{x^2}{rv_n^2}\Big)^{-r/2}dx + n\int_0^\infty x^{p-2}\mathbf{E}((|X| - x/r)_+)dx$$

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- Hence, taking $r > p$, the Rosenthal inequality follows:

$$\mathbf{E}(\max_{1\leq i\leq n} |S_k|^p) \ll v_n^p + n\mathbf{E}(|X|^p)$$

Some applications (2)

- Let $p > 2$. Still in the identically distributed case, assume now that the r.v.'s have a weak moment of order p:

$$\sup_{t>0} t^p \mathbb{P}(|X| > t) < \infty \,.$$

This condition is equivalent to:

$$\sup_{x>0} x^{p-1} \int_0^1 Q(u) \mathbf{1}_{Q(u)>x} du < \infty$$

where Q is the quantile function of $|X|$, that is the generalized inverse of $H(t) = \mathbb{P}(|X| > t)$.

- The Fuk-Nagaev inequality gives the following deviation bound: for any $x > 0$ and any $r > 0$

$$\mathbb{P}\big(\max_{1\leq i\leq n} |S_k| \geq nx\big) \ll \frac{1}{x^r n^{r/2}} + \frac{1}{x^p n^{p-1}}$$

What about $\mathbb{P}(\max_{1\le i\le n}|S_k| \ge nx)$ in the dependent setting?

- Let us consider the following Markov chain: Let $a = p - 1$ with $p > 2$. Let λ denote the Lebesgue measure on $[0,1]$. Define the probability laws ν and π by

$$\nu = (1+a)x^a\lambda \text{ and } \pi = ax^{a-1}\lambda .$$

We define now a strictly stationary Markov chain by defining its transition probabilities $K(x, A)$ as follows:

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- Then π is the unique invariant probability measure of the chain with transition probabilities $K(x,\cdot)$. Let $(Y_i)_{i\in_{i\in\mathbb{Z}}}$ be the stationary Markov chain on $[0,1]$ with transition probabilities $K(x,\cdot)$ and law π.

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- For any bounded function from $[0,1]$ to $\mathbb{R}$, set

$$X_i = f(Y_i) - \pi(f) \text{ and } S_n(f) = \sum_{i=0}^{n-1} X_i .$$

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- Y_{T_k} has law ν and the conditional distribution of τ_k given $Y_{T_k} = y$ is the geometric distribution $\mathcal{G}(1-y)$: for any $\ell \geq 0$

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- Let $N_n = \sup\{i \in \mathbb{N} : T_i \leq n\}$. Write

$$S_n(f) = \sum_{k=0}^{T_{N_n}-1} X_k + \sum_{k=T_{N_n}}^{n-1} X_k = T_0 X_0 + \sum_{k=0}^{N_n-1} \tau_k X_{T_k} + \sum_{k=T_{N_n}}^{n-1} X_k$$

A Markov chain example (2)

Recall that

$$N_n = \sup\{i \in \mathbb{N} : T_i \leq n\}$$

Setting $c = 3/(2\mathbf{E}(\tau_1))$, we have

$$\mathbb{P}\Big(\Big|\sum_{k=0}^{N_n-1} \tau_k X_{T_k}\Big| \geq nx\Big) \leq \mathbb{P}\big(N_n > [cn]+1\big) + \mathbb{P}\Big(\max_{0\leq \ell \leq [cn]} \Big|\sum_{k=0}^{\ell} \tau_k X_{T_k}\Big| \geq nx\Big)$$

$$= \mathbb{P}\big(T_{[cn]+1} \leq n\big) + \mathbb{P}\Big(\max_{0\leq \ell \leq [cn]} \Big|\sum_{k=0}^{\ell} \tau_k X_{T_k}\Big| \geq nx\Big)$$

$$\leq \mathbb{P}\Big(\sum_{k=0}^{[cn]} (\tau_k - \mathbf{E}(\tau_k)) \leq -n/2\Big) + \mathbb{P}\Big(\max_{0\leq \ell \leq [cn]} \Big|\sum_{k=0}^{\ell} \tau_k X_{T_k}\Big| \geq nx\Big)$$

A Markov chain example (3)

- We have

$$\mathbb{P}(\tau_k > \ell) = (a+1)\int_0^1 y^a(1-y)^\ell dy \ll \ell^{-(a+1)} = \ell^{-p}$$

So, using the Fuk-Nagaev inequality, we get that, for any $r > 0$,

$$\mathbb{P}(|\sum_{k=0}^{N_n-1} \tau_k X_{T_k}| \geq nx) \ll \frac{1}{x^r n^{r/2}} + \frac{1}{x^p n^{p-1}} + \frac{1}{n^{p-1}}$$

- In addition $\mathbb{P}(T_0 > \ell) = \int_0^1 (1-y)^\ell d\pi y \ll \ell^{-a} = \ell^{1-p}$
- Moreover

$$\mathbb{P}(|\sum_{k=T_{N_n}}^{n-1} X_k| \geq nx) \leq \mathbb{P}(2\|f\|_\infty \tau_{N_n} \geq nx)$$

$$\leq \mathbb{P}(N_n > [cn]+1) + ([cn]+1)\mathbb{P}(2\|f\|_\infty \tau_1 \geq nx)$$

A Markov chain example (4)

- Finally, we get that for any $r > 0$,

$$\mathbb{P}(|S_n(f)| \geq nx) \ll \frac{1}{x^r n^{r/2}} + \frac{1}{x^p n^{p-1}} \quad (*)$$

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- For this example,

$$\beta(n) := \pi\Big(\sup_{\|f\|_\infty \leq 1} \big|K^n(f) - \pi(f)\big|\Big) \ll n^{-a} = \frac{1}{n^{p-1}},$$

(see Doukhan, Massart and Rio (1994)) and the inequality can be deduced from a more general inequality due to Rio (2000) for α-mixing sequences.

A Fuk-Nagaev inequality for α-mixing sequences: Rio (2000)

- Let $(X_n)_{n\in\mathbb{Z}}$ be a strictly stationary sequence of centered real-valued r.v.'s in $\mathbb{L}^2$. let $\mathbf{X}_n = (X_k, k \geq n)$

$$\alpha(0) = 1/2 \text{ and } \alpha(n) = \sup_{\|f\|_\infty \leq 1} \|\mathbb{E}(f(\mathbf{X}_n)|\mathcal{F}_0) - \mathbb{E}(f(\mathbf{X}_n))\|_1$$

  where $\mathcal{F}_0 = \sigma(X_k, k \leq 0)$.

- For any $u \in [0, 1]$, set

$$\alpha^{-1}(u) = \min\{q \in \mathbb{N} : \alpha(q) \leq u\} = \sum_{n\geq 0} \mathbf{1}_{u<\alpha(n)}$$

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- Let Q is the quantile function of $|X_1|$, that is the generalized inverse of $H(t) = \mathbb{P}(|X_1| > t)$. So for $u \in [0,1]$,

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- Note that

$$H(t) \ll t^{-p} \iff \sup_{x>0} x^{p-1} \int_0^1 Q(u)\mathbf{1}_{Q(u)>x} du < \infty$$

- Theorem (Rio (2000)). Setting $R_n(u) = (\alpha^{-1}(u) \wedge n) Q(u)$, we have for any $x > 0$ and any $r \geq 1$,

$$\mathbb{P}\Big(\max_{1 \leq k \leq n} |S_k| \Big) \geq 4x) \leq 4\Big(1 + \frac{x^2}{v_n^2}\Big)^{-r/2} + 4nx^{-1} \int_0^1 Q(u) \mathbf{1}_{R_n(u) > x/r} du \,,$$

where $v_n^2 \geq n \sum_{k=0}^{n-1} |\mathrm{Cov}(X_0, X_k)|$.

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where $v_n^2 \geq n \sum_{k=0}^{n-1} |\mathrm{Cov}(X_0, X_k)|$.

- In the independent setting $R_n(u) = Q(u)$ and

$$\int_0^1 Q(u) \mathbf{1}_{Q(u) > x/r} du = \int_0^{H(x/r)} Q(u) du$$

$$= \int_{x/r}^{+\infty} H(t) dt = \mathbb{E}(|X_1| - x/r)_+) \,.$$

Some applications (1)

Using the fact that

$$\mathbf{E}(\max_{1\le i\le n}|S_k|^p) = p\int_0^\infty x^{p-1}\mathbb{P}(\max_{1\le i\le n}|S_k|\ge x)dx$$

we get the following Rosenthal-type inequality: for any $p\ge 2$

$$\mathbf{E}(\max_{1\le i\le n}|S_k|^p) \le a_p v_n^p + nb_p\int_0^1(\alpha^{-1}(u)\wedge n)^{p-1}Q^p(u)du$$

since, taking $r = p+1$,

$$n\int_0^\infty x^{p-2}\int_0^1 Q(u)\mathbf{1}_{R(u)>x/r}dudx$$
$$= n\frac{(p+1)^{p-1}}{p-1}\int_0^1(\alpha^{-1}(u)\wedge n)^{p-1}Q^p(u)du$$

Some applications (2)

- Let $R(u) = \alpha^{-1}(u)Q(u)$. Let $p > 2$. Assume that

$$\sup_{x>0} x^{p-1} \int_0^1 Q(u) \mathbf{1}_{R(u)>x} du < \infty \quad (*)$$

then for any $r > 0$

$$\mathbb{P}(\max_{1 \le i \le n} |S_k| \ge nx) \ll \frac{1}{x^r n^{r/2}} + \frac{1}{x^p n^{p-1}}$$

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- If $Q(u) \le C$, then $(*)$ reads as $\alpha(n) \ll \frac{1}{n^{p-1}}$. Hence the Rio's results can be applied with $X_k = f(Y_k) - \pi(f)$ where f is bounded and Y_k is the strictly Markov chain previously defined.

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- Rio's inequality is proved by using truncature, blocking arguments and coupling. In the mixing coefficients, all the past and all the future of the sequence are needed. The mixing coefficients can be replaced by coefficients allowing coupling in $\mathbb{L}^1$ (see Dedecker and Prieur (2005)).

Examples of non strong mixing processes

- In the Markov chain setting with invariant probability measure π, the *alpha*-mixing coefficients read as

$$\alpha(n) = \sup_{\|f\|_\infty \leq 1} \pi\Big(\big|K^n(f) - \pi(f)\big|\Big)$$

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- Take for instance

$$X_n = \sum_{i=0}^{\infty} \frac{\xi_{n-i}}{2^{i+1}} \,.$$

where (ξ_i) is an iid sequence of r.v.'s $\sim \mathcal{B}(1/2)$.

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- This is a Markov chain with invariant measure λ the Lebesgue measure on $[0,1]$ and transition Markov operator given by

$$K(f)(x) = \frac{1}{2}\Big(f\Big(\frac{x}{2}\Big) + f\Big(\frac{x+1}{2}\Big)\Big)$$

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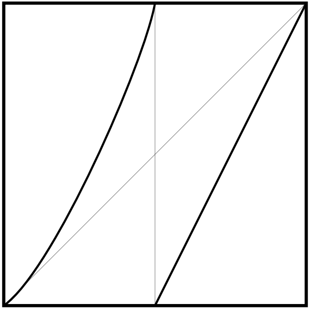
- This is a Markov chain with invariant measure λ the Lebesgue measure on $[0, 1]$ and transition Markov operator given by

$$K(f)(x) = \frac{1}{2}\Big(f\Big(\frac{x}{2}\Big) + f\Big(\frac{x+1}{2}\Big)\Big)$$

- This Markov chain is not strong mixing !

Example Let us consider a LSV map (Liverani, Saussol et Vaienti, 1999):

$$\text{for } 0 < \gamma < 1, \quad T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$



Graph of T_γ

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- We can associate a Markov chain $\mathbf{X} = (X_i)_{i \in Z}$ with invariant probability measure ν such that on the probability space $(T_\gamma, T_\gamma^2, \dots, T_\gamma^n)$ is distributed as $(X_n, X_{n-1}, \dots, X_1)$. Therefore

$$\nu\Big(\max_{1 \leq k \leq n} \Big| \sum_{i=1}^{k} (f \circ T_\gamma^i - \nu(f)) \Big| \geq x\Big) \leq \mathbb{P}\Big(2 \max_{1 \leq k \leq n} \Big| \sum_{i=1}^{k} (f(X_i) - \nu(f)) \Big| \geq x\Big)$$

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- The Markov operator of the chain is the Perron-Frobenius operator K defined as follows: for any positive measurable functions f and g,

$$\nu(f\circ T\cdot g) = \nu(f\cdot K(g))\,.$$

Dependence coefficients for the chain.

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Dependence coefficients for the chain.

- The Markov chain $\mathbf{X} = (X_i)_{i \in Z}$ with invariant probability measure ν and transition operator K is not strong mixing.
- However we have the following upper bounds: Let BV be the space of bounded variation functions f from $\mathbb{R}$ to $\mathbb{R}$ with norm $\| \cdot \|$ defined as follows:

$$\|f\| = \max(\|f\|_\infty, |f|) ,$$

where $|f| = \|df\|$. Let $B_1 = \{ f \in \mathcal{B} : |f| \leq 1 \}$. Then there exist positive constants C_1 and C_2 not depending on n such that

$$\mathbf{H}_1 : \qquad \sup_{f \in B_1} \nu\Big(\big| K^n(f) - \nu(f) \big| \Big) \leq \frac{C_1}{n^{(1-\gamma)/\gamma}}$$

and, for any function f in BV,

$$\mathbf{H}_2 : \qquad |K^n(f)| \leq C_2 |f| .$$

(See Dedecker, Gouëzel, Merlevède (2010) where GPM maps have been considered).

- Having $\mathbf{H}_1$ and $\mathbf{H}_2$ implies that there exists a constant C such that for any $k \geq 0$ and any $n \geq 1$,

$$\sup_{f,g \in B_1} \nu\Big(\big|K^n(f\,K^k(g)) - \nu(f\,K^k(g))\big|\Big) \leq \frac{C}{n^{(1-\gamma)/\gamma}}\,,$$

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- This is equivalent to say that

$$\sup_{s,t \in \mathbb{R}} \nu\Big(\big|K^n(f_t\,K^k(f_s)) - \nu(f_t\,K^k(f_s))\big|\Big) \leq \frac{C}{n^{(1-\gamma)/\gamma}}\,,$$

where $f_t(x) = \mathbf{1}_{x \leq t} - \nu(]-\infty, t])$

The α-dependent coefficients for stationary sequences.

- For any integrable random variable Z, let $Z^{(0)} = Z - \mathbb{E}(Z)$. For any random variable $V = (V_1, \cdots, V_k)$ with values in $\mathbb{R}^k$ and any σ-algebra $\mathcal{F}$, let

$$\alpha(\mathcal{F}, V) = \sup_{(x_1,\dots,x_k)\in\mathbb{R}^k} \left\| \mathbb{E}\left(\prod_{j=1}^{k} (\mathbf{1}_{V_j \le x_j})^{(0)} \Big| \mathcal{F} \right) - \mathbb{E}\left(\prod_{j=1}^{k} (\mathbf{1}_{V_j \le x_j})^{(0)} \right) \right\|_1$$

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- For a stationary sequence $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$, let

$$\alpha_{k,\mathbf{Y}}(0) = 1/2 \;, \; \alpha_{k,\mathbf{Y}}(n) = \max_{1\le l\le k} \sup_{n\le i_1\le\ldots\le i_l} \alpha(\mathcal{F}_0, (Y_{i_1}, \ldots, Y_{i_l})) \,, n > 0$$

Note that $\alpha_{1,\mathbf{Y}}(n)$ is then simply given by

$$\alpha_{1,\mathbf{Y}}(n) = \sup_{x\in\mathbb{R}} \|\mathbb{E}\left(\mathbf{1}_{Y_n\le x}|\mathcal{F}_0\right) - F(x)\|_1 \;,$$

where F is the distribution function of P_{Y_0}.

Important remarks.

- Contrary to the usual mixing case, any function of a stationary α-dependent sequence $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$ is not necessarily α-dependent (meaning that its dependency coefficients do no necessarily tend to zero). Hence, we need to impose some constraints on the observables.

Important remarks.

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- If f is monotonic on some open interval and 0 elsewhere, and if $\mathbf{X} = (f(Y_i))_{i\in\mathbb{Z}}$, then for any positive integer k,

$$\alpha_{k,\mathbf{X}}(n) \leq 2^k \alpha_{k,\mathbf{Y}}(n) \,.$$

  As a consequence, if one can prove a deviation inequality for $\sum_{k=1}^n Y_i$ with an upper bound involving the coefficients $(\alpha_{k,\mathbf{Y}}(n))_{n\geq 0}$ then it also holds for $\sum_{k=1}^n f(Y_i)$, where f is monotonic on a single interval.

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- The deviation inequality can be then extended by linearity to convex combinations of such functions.

The class of observables

- Let $H : \mathbb{R}^+ \to [0, 1]$ be a tail function so it is non-increasing, right-continuous and converges to zero at infinity.
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- $\mathcal{F}(Q,\mu)$ is the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^{L} a_\ell f_\ell$, where $\sum_{\ell=1}^{L} |a_\ell| \le 1$ and f_ℓ belongs to $\mathrm{Mon}(Q,\mu)$.

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- A function belonging to $\mathcal{F}(Q,\mu)$ is allowed to blow up at an infinite number of points.

A deviation inequality for α-dependent sequences: notations

- For $u \in [0,1]$ and $k \in \mathbb{N}^*$, let

$$\alpha_{k,\mathbf{Y}}^{-1}(u) = \min\{q \in \mathbb{N} : \alpha_{k,\mathbf{Y}}(q) \le u\} = \sum_{n=0}^{\infty} \mathbf{1}_{u < \alpha_{k,\mathbf{Y}}(n)} \,.$$

Note that $\alpha_{1,\mathbf{Y}}(n) \le \alpha_{2,\mathbf{Y}}(n)$, and consequently $\alpha_{1,\mathbf{Y}}^{-1} \le \alpha_{2,\mathbf{Y}}^{-1}$.

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- Given a positive integer n, define

$$R_n(u) = \left(\alpha_{2,\mathbf{Y}}^{-1}(u) \wedge n\right) Q(u)\,, \quad \text{for } u \in [0,1]$$

A deviation inequality for α-dependent sequences: the statement

Theorem (Dedecker & M. (2016)). For any $x > 0$, $r > 2$, $\beta \in]r-2, r[$ the following deviation bound holds

$$\mathbb{P}\left(\max_{1\le k\le n}|S_k| \ge x\right) \ll \frac{s_n^r(x)}{x^r} + \frac{n}{x}\int_0^1 Q(u)\mathbf{1}_{R_n(u)>x}du + \frac{n}{x^{1+\beta/2}}\int_0^1 R_n^{\beta/2}(u)Q(u)\mathbf{1}_{R_n(u)>x}du + \frac{n}{x^{1+r/2}}\int_0^1 R_n^{r/2}(u)Q(u)\mathbf{1}_{R_n(u)\le x}du\,.$$

where

$$s_n^2(x) = n\int_0^1 (\alpha_{1,\mathbf{Y}}^{-1}(u)\wedge n)Q^2(u)\mathbf{1}_{R_n(u)\le x}du\,,$$

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- As in the Rio's proof, we make blocks of size q with $\alpha_{2,\mathbf{Y}}(q) \leq v = R_n^{-1}(x)$ if $0 \leq v < 1/2$ and $q \leq n$.
- Setting where $U_i = \sum_{k=(i-1)q+1}^{iq} X_i$, we have

$$\max_{1 \leq k \leq n} |S_k| \leq 2qM + \max_{1 \leq 2j \leq [\frac{n}{q}]} \left| \sum_{i=1}^{j} U_{2i} \right| + \max_{1 \leq 2j-1 \leq [\frac{n}{q}]} \left| \sum_{i=1}^{j} U_{2i-1} \right|$$

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- Let $\tilde{U}_{2i} = U_{2i} - \mathbb{E}_{\mathcal{F}_{2(i-1)q}}(U_{2i})$, $\tilde{U}_{2i+1} = U_{2i+1} - \mathbb{E}_{\mathcal{G}_{(2i-1)q}}(U_{2i+1})$

$$\max_{1 \leq k \leq n} |S_k| \leq 2qM + \max_{2 \leq 2j \leq [\frac{n}{q}]} \left| \sum_{i=1}^{j} \tilde{U}_{2i} \right| + \max_{1 \leq 2j-1 \leq [\frac{n}{q}]} \left| \sum_{i=1}^{j} \tilde{U}_{2i-1} \right| + \sum_{i=1}^{[n/q]} |U_i - \tilde{U}_i|$$

A Rosenthal for stationary sequences

Theorem (M. & Peligrad (2013)). Let $p > 2$ and let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of r.v.'s in $\mathbf{L}^p$ and adapted to a stationary filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. Then for any $n \geq 1$,

$$\| \max_{1\leq j\leq n} |S_j| \|_p \ll n^{1/p} \Big(\|X_1\|_p + \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \|\mathbb{E}_0(S_k)\|_p + \Big(\sum_{k=1}^{n} \frac{1}{k^{1+2\delta/p}} \|\mathbb{E}_0(S_k^2)\|_{p/2}^{\delta} \Big)^{1/(2\delta)} \Big) ,$$

where $\delta = \min(1, 1/(p-2))$ and $\mathbb{E}_0(X) = \mathbb{E}(X|\mathcal{F}_0)$.

Remark If there exists $\beta > 2/p$ such that $n^{-\beta}\mathbb{E}(S_n^2)$ is increasing,

$$n^{\frac{1}{p}} \Big(\sum_{k=1}^{n} \frac{\|\mathbb{E}_0(S_k^2)\|_{p/2}^{\delta}}{k^{1+2\delta/p}} \Big)^{\frac{1}{2\delta}} \ll \big(\mathbb{E}(S_n^2)\big)^{\frac{1}{2}} + n^{\frac{1}{p}} \Big(\sum_{k=1}^{n} \frac{\|\mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2}^{\delta}}{k^{1+2\delta/p}} \Big)^{\frac{1}{2\delta}}$$

Application 1: a Rosenthal-type inequality

Let $p \geq 2$. Starting from

$$\mathbf{E}\big(\max_{1\leq i\leq n}|S_k|^p\big) = p\int_0^\infty x^{p-1}\mathbb{P}\big(\max_{1\leq i\leq n}|S_k| \geq x\big)dx$$

and applying the deviation inequality with

$$r-2 < \beta < 2p-2 < r < 2p$$

we get the following Rosenthal-type inequality

$$\mathbf{E}\big(\max_{1\leq i\leq n}|S_k|^p\big) \ll n^{p/2}\left(\int_0^1 (\alpha_{1,\mathbf{Y}}^{-1}(u)\wedge n)Q^2(u)du\right)^{p/2} + n\int_0^1 (\alpha_{2,\mathbf{Y}}^{-1}(u)\wedge n)^{p-1}Q^p(u)du$$

Application 2: large deviation inequalities (3)

- Let $\mathbf{Y} = (Y_i)_{i \in \mathbb{Z}}$ be a stationary sequence. Let P_{Y_0} the distribution of Y_0 and Q be a quantile function in $\mathbb{L}^1$.
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- Let $R(u) = \alpha_{2,\mathbf{Y}}^{-1}(u) Q(u)$. Let $p \geq 2$ and assume that

$$\sup_{x>0} x^{p-1} \int_0^1 Q(u) \mathbf{1}_{R(u)>x} du < \infty \quad (*)$$

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$$\mathbb{P}\left(\frac{1}{n}\max_{1\leq k\leq n}|S_k| \geq x\right) \ll \frac{1}{n^a x^{2a}} + \frac{1}{n^{p-1}x^p}\,.$$

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- For $p = 2$, any $a \in (1, 2)$, any $c \in (0, 1)$ and any $x > 0$,

$$\mathbb{P}\left(\frac{1}{n}\max_{1\leq k\leq n}|S_k| \geq x\right) \ll \frac{1}{n^{ac}x^{a(1+c)}} + \frac{1}{nx^2}\,.$$

Application 3: large deviation inequalities (2)

- If we reinforce the condition $(*)$ in the following: let $p \geq 2$ and assume
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Application to intermittent maps: the LSV map.

- Recall that

$$\text{for } 0 < \gamma < 1, \quad T(x) := T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

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- Consider the Markov chain $(Y_i)_{i \in \mathbb{Z}}$ with invariant measure ν and transition operator K and recall that

$$\nu\Big(\max_{1 \le k \le n} \big|S_k(f)\big| \ge x \Big) \le \mathbb{P}\Big(2 \max_{1 \le k \le n} \Big| \sum_{i=1}^{k} (f(Y_i) - \nu(f))\Big| \ge x \Big)$$

where $S_k(f) = \sum_{i=1}^{k} (f \circ T^i - \nu(f))$.

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$$\text{for } 0 < \gamma < 1, \quad T(x) := T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

- Consider the Markov chain $(Y_i)_{i\in\mathbb{Z}}$ with invariant measure ν and transition operator K and recall that

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where $S_k(f) = \sum_{i=1}^{k} (f \circ T^i - \nu(f))$.

- For any $k \ge 1$, there exist two positive constants C and D such that, for any $n > 0$,

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- Assume that $f \in \mathcal{F}(Q, \nu)$ and $Q(u) \ll u^{-b}$ for $b \in [0, 1)$.

Moment bounds.

- Let $p > 2$. Since, for any $b \in [0, 1/p[$

$$\int_0^1 (\alpha_{2,\mathbf{Y}}^{-1}(u) \wedge n)^{p-1} Q^p(u) du \ll \sum_{k=0}^{n} (k+1)^{p-2} \int_0^{\alpha_{2,\mathbf{Y}}(k)} Q^p(u) du$$
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- For instance our result applies if f is positive and non increasing on $(0, 1)$, with

$$f(x) \le \frac{C}{x^s} \quad \text{near 0, for some } C > 0 \text{ and } s \in [0, 1-\gamma), \text{ and}$$

$$f \text{ belongs to } \mathcal{F}(Q, \nu) \text{ with } Q(u) \ll u^{-s/(1-\gamma)}$$

Large deviations.

- Let f in $\mathcal{F}(Q,\nu)$ with $Q(u) \ll u^{-b}$ for some $b \in [0,1)$. Let $p = 1/(\gamma + b(1-\gamma))$.

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- With another deviation inequality, we have: Assume that $\gamma + b(1-\gamma) \in (1/2,1)$. Then, for any $x > 0$,

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  Moreover $\sup_{x>\varepsilon} x^p f_{b,\gamma}(x) < \infty$ for any $\varepsilon > 0$.
- When f is a bounded variation function (then $b = 0$), for any $x > 0$,

$$\nu\left(\frac{1}{n}\max_{1\le k\le n}|S_k(f)| \ge x\right) \ll \frac{f_{0,\gamma}(x)}{n^{(1-\gamma)/\gamma}}.$$

  This upper bound (with $S_n(f)$ instead of the maximum) was obtained by Melbourne (2009) when f is Hölder continuous who also proved that it is optimal.

Thank you for your attention!