ASYMPTOTIC OF THE NUMBER OF OBSTACLES VISITED BY THE PLANAR LORENTZ PROCESS

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Abstract. *We are interested in the planar Lorentz process with a periodic configuration of strictly convex obstacles and with finite horizon. Its recurrence comes from a criteria of Conze in* [8] *or of Schmidt in* [15] *and from the central limit theorem for the billiard in the torus (*[2, 4, 20]*). Another way to prove recurrence is given by Szász and Varjú in* [19]. *Total ergodicity follows from these results (see* [17] *and* [12]*). In this paper we answer a question of Szász about the asymptotic behaviour of the number of visited cells when the time goes to infinity. It is not more difficult to study the asymptotic of the number of obstacles hit by the particle when the time goes to infinity. We give an estimate for the expectation and a result of almost sure convergence. For the simple random walk in* $\mathbb{Z}^2$, *this question has been studied by Dvoretzky and Erdös in* [10]. *We adapt the proof of Dvoretzky and Erdös. The lack of independence is compensated by a strong decorrelation result due to Chernov (*[6]*) and by some refinement (got in* [14]*) of the local limit theorem proved by Szász and Varjú in* [19].

1. Introduction

Since the early work of Sinai ([18]), billiard systems have been studied by many authors ([1, 2, 3, 4, 11]). In $\mathbb{R}^2$, we consider a finite number of open convex sets $O_1, ..., O_I$, with boundary C^3-smooth and with non null curvature. We repeat these sets $\mathbb{Z}^2$-periodically by setting $U_{a,\ell} = \ell + O_a$ for all $a \in \{1, ..., I\}$ and all $\ell \in \mathbb{Z}^2$. We suppose that the closures of the $U_{a,\ell}$ are pairwise disjoint. For any $\ell \in \mathbb{Z}^2$, we call ℓ-cell the set $\bigcup_{a=1}^{I} \partial U_{a,\ell}$. Let us consider a point particle moving in the domain $Q := \mathbb{R}^2 \setminus \bigcup_{a=1}^{I} \bigcup_{\ell\in\mathbb{Z}^2} U_{a,\ell}$ with unit speed and with elastic reflections off ∂Q. This model is a billiard model with infinite area domain. It is also called planar Lorentz process. We will consider the **finite horizon case**, i.e. we suppose that the time between two successive collisions is uniformly bounded.

We are interested in the asymptotic behaviour (when n goes to infinity) of the number N_n of cells visited before the nth reflection off ∂Q. We will also study the number N'_n of obstacles hit before the nth reflection. We prove that the expectations of $\frac{\log n}{n} N_n$ and of $\frac{\log n}{n} N'_n$ (for a natural class of initial distributions) converge to nonnull constants c and c' (as n goes to infinity). Moreover we prove that these quantities converge almost surely to c and c' (respectively).

We get analogous results for the number $\tilde{N}_t$ of cells visited before time t and for the number $\tilde{N}'_t$ of obstacles hit before time t.

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We will see a link between N_n and the number $\mathcal{T}_+$ of reflections off ∂Q before coming back to the initial cell.

1.1. **Billiard flow** $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ **and billiard transformation** (M, ν, T) **in the plane.** We call state of a particle at some time its position-speed couple. When a reflection occurs, there is coexistence of two states : one corresponding to the incident vector and one corresponding to the reflected vector. To avoid ambiguity, we will only consider reflected vectors. Hence the set of states (position-speed couples) will be :

$$\mathcal{M}_1 := \{(q, \vec{v}) \in Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1; \quad q \in \partial Q \ \Rightarrow \ \langle \vec{n}(q), \vec{v} \rangle \geq 0\},$$

where $\vec{n}(q)$ is the unit vector normal to ∂Q at $q \in \partial Q$ oriented to inward of Q. The billiard flow $(Y_t)_t$ is the flow on $\mathcal{M}_1$ such that $Y_t(q, \vec{v}) = (q_t, \vec{v}_t)$ is the state at time t of a particle with state $(q, \vec{v})$ at time 0. The billiard flow preserves the Lebesgue measure μ_1 on $\mathcal{M}_1$.

Now let us consider reflection times. Let M be the set of reflected vectors off ∂Q :

$$M := \{(q, \vec{v}) \in \partial Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1 \text{ and } \langle \vec{n}(q), \vec{v} \rangle \geq 0\}.$$

The billiard transformation T maps a state at a reflection time $x \in M$ to the state $T(x) = x'$ at the next reflection time. This transformation preserves the measure ν given by $d\nu(q, \vec{v}) = \cos(\varphi)\, dr d\varphi$, with the parametrisation (a, r, φ, ℓ) of $(q, \vec{v}) \in M$ if $q - \ell$ is the point of ∂O_a with arc length parameter r and if φ is the angular measure of $(\vec{n}(q), \vec{v})$ taken in $[-\frac{\pi}{2}; \frac{\pi}{2}]$.

We define the function $\tau : M \to [0, +\infty)$ by : $\tau(q, \vec{v}) := \min\{s > 0 \ : \ q + s\vec{v} \in \partial Q\}$. The quantity $\tau(q, \vec{v})$ corresponds to the time before the next reflection. **Here, we suppose that the billiard system has finite horizon**, i.e. $\sup \tau < +\infty$. We already know that this system is recurrent (see [8], [15] and [19]) and that it is totally ergodic (see [17] and [12]). The billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ can be represented by the special flow $(\tilde{\mathcal{M}}_1, \tilde{\mu}_1, (\tilde{Y}_t)_t)$ over (M, ν, T) with roof function τ. Let us explicit this. Let us define $\tilde{\mathcal{M}}_1 := \{(x, s) \ : \ x \in M; \ 0 \leq s < \tau(x)\}$ endowed with the measure $\tilde{\mu}_1$ given by : $d\tilde{\mu}_1(x, s) = d\nu(x) ds$. Let $(\tilde{Y}_t)_t$ be the flow defined on $\tilde{M}_1$ by $\tilde{Y}_t(x, s) = (x, s + t)$ with the identifications $(x, \tau(x)) \equiv (T(x), 0)$. Let $\Delta : \tilde{\mathcal{M}}_1 \to \mathcal{M}_1$ be given by : $\Delta((q, \vec{v}), s) = (q + s\vec{v}, \vec{v})$. We have : $Y_t = \Delta \circ \tilde{Y}_t \circ \Delta^{-1}$ and $\Delta_*(\tilde{\mu}_1) = \mu_1$.

1.2. **Billiard transformation in the torus** $(\bar{M}, \bar{\nu}, \bar{T})$**.** The billiard in the torus is obtained from the billiard in the plane by quotienting the positions by $\mathbb{Z}^2$. More precisely, let us define $\bar{M} = \{(q, \vec{v}) \in M \ : \ q \in \bigcup_{a=1}^{I} \partial O_a\}$ and $\bar{T} : \bar{M} \to \bar{M}$ with $\bar{T}(q, \vec{v}) = (q', \vec{v}')$ if there exists $\ell \in \mathbb{Z}^2$ such that $T(q, \vec{v}) = (q' + \ell, \vec{v}')$. Let $\bar{\nu}$ be the probability measure on $\bar{M}$ proportional to the restriction of ν to $\bar{M}$. We endow M with a metric d such that : $d(y, y') = |r - r'| + |\varphi - \varphi'|$, if $(a, r, \varphi, (0, 0))$ and $(a, r', \varphi', (0, 0))$ are the parametrizations of y and y' respectively. We endow $\bar{M}$ with its Borel σ-algebra $\mathcal{F}$. The study of this system is complicated by the discontinuities of the transformation $\bar{T}$. But it is known that $\bar{T}$ is C^2-regular on $\bar{M} \setminus (R_0 \cup \bar{T}^{-1}(R_0))$, where the set $R_0 := \{(q, \vec{v}) \in \bar{M} \ : \ \langle \vec{n}(q), \vec{v} \rangle = 0\}$ corresponds to vectors tangent to ∂Q.

1.3. **Results.** For any $x \in M$ and any integer $n \geq 1$, we define the number $N_n(x)$ of cells visited before the n-th reflection for a particle with initial state x :

$$N_n(x) := \# \left\{ \ell \in \mathbb{Z}^2 \ : \ \exists m = 1, ..., n \ : \ T^m(x) \in \left(\bigcup_{a=1}^{I} \partial U_{a,\ell} \right) \times \mathbb{R}^2 \right\}.$$

The number $N'_n(x)$ of obstacles visited before the n-th reflection for a particle with initial state x is given by :

$$N'_n(x) := \#\left\{(a,\ell)\in\{1,...,I\}\times\mathbb{Z}^2\ :\ \exists m=1,...,n\ :\ T^m(x)\in\partial U_{a,\ell}\times\mathbb{R}^2\right\}.$$

Our first result deals with the expectation of these quantities.

Theorem 1. *Let $H : M \to [0,\infty)$ be any measurable function such that $\int_M H\,d\nu = 1$ and such that $\int_{\bar M}\left(\sum_{\ell\in\mathbb{Z}^2} H(q+\ell,\vec v)\right)^2 d\bar\nu(q,\vec v) < +\infty$ (for example if $H=\frac{\mathbf{1}_{\bar M}}{\nu(M)}$ or if $H=\frac{\tau\mathbf{1}_{\bar M}}{\int_{\bar M}\tau\,d\nu}$). We have :*

$$\lim_{n\to+\infty}\frac{\log(n)}{n}\mathbb{E}_{H\nu}[N_n] = 2\pi\sqrt{det(\Sigma^2)}\quad and\quad \lim_{n\to+\infty}\frac{\log(n)}{n}\mathbb{E}_{H\nu}[N'_n] = 2I\pi\sqrt{det(\Sigma^2)},$$

where $\Sigma^2=\lim_{n\to+\infty}Var_{\bar\nu}\left(\frac{S_n}{\sqrt n}\right)$, where $S_n(\bar x)$ corresponds to the index of the cell at the n^{th} collision for a particle with initial state $\bar x\in\bar M$.

In section 3.1, we precise the link between this result for $H=\frac{\mathbf{1}_{\bar M}}{\nu(M)}$ and some estimations got by Dolgopyat, Szász and Varjú in [9]. For any $y\in\mathcal{M}_1$ and any $t>0$, let us define :

$$\tilde N_t(y) := \#\left\{\ell\in\mathbb{Z}^2\ :\ \exists s\in]0;t]:\ Y_s(y)\in\left(\bigcup_{a=1}^{I}\partial U_{a,\ell}\right)\times\mathbb{R}^2\right\}$$

and

$$\tilde N'_t(y) := \#\left\{(a,\ell)\in\{1,...,I\}\times\mathbb{Z}^2\ :\ \exists s\in]0;t]:\ Y_s(y)\in\partial U_{a,\ell}\times\mathbb{R}^2\right\}.$$

These quantities correspond respectively to the number of cells and of obstacles visited before time t for a particle with state y at time 0.

Corollary 2. *Let $g:\mathcal{M}_1\to[0,\infty)$ be any measurable function such that $\int_M g\,d\mu_1=1$ and such that $\int_{\bar M}\left(\sum_{\ell\in\mathbb{Z}^2}\int_0^{\tau(q,\vec v)} g(\Delta((q+\ell,\vec v),s))\,ds\right)^2 d\bar\nu(q,\vec v)<+\infty$ (for example if $g=\frac{\mathbf{1}_A}{\mu_1(A)}$ for any bounded measurable set A). We have :*

$$\lim_{t\to+\infty}\frac{\log(t)}{t}\mathbb{E}_{g\mu_1}[\tilde N_t] = \frac{2\pi\sqrt{det(\Sigma^2)}}{\int_{\bar M}\tau\,d\bar\nu}\quad and\quad \lim_{t\to+\infty}\frac{\log(t)}{t}\mathbb{E}_{g\mu_1}[\tilde N'_t] = \frac{2I\pi\sqrt{det(\Sigma^2)}}{\int_{\bar M}\tau\,d\bar\nu}.$$

Let us notice that condition $\int_{\bar M}\left(\sum_{\ell\in\mathbb{Z}^2}\int_0^{\tau(q,\vec v)} g(\Delta((q+\ell,\vec v),s))\,ds\right)^2 d\bar\nu(q,\vec v)<+\infty$ holds if we have : $\sup_{(q,\vec v)\in\mathcal{M}_1}\sum_{\ell\in\mathbb{Z}^2} g(q+\ell,\vec v)<+\infty$.

Theorem 3 (Almost everywhere convergence)**.** *For ν-almost every $x\in M$, we have :*

$$\lim_{n\to+\infty}\frac{\log(n)}{n}N_n(x) = 2\pi\sqrt{det(\Sigma^2)}\quad and\quad \lim_{n\to+\infty}\frac{\log(n)}{n}N'_n(x) = 2\pi I\sqrt{det(\Sigma^2)}.$$

For μ_1-almost every $x\in\mathcal{M}_1$, we have :

$$\lim_{t\to+\infty}\frac{\log(t)}{t}\tilde N_t = \frac{2\pi\sqrt{det(\Sigma^2)}}{\int_{\bar M}\tau\,d\bar\nu}\quad and\quad \lim_{t\to+\infty}\frac{\log(t)}{t}\tilde N'_t = \frac{2\pi I\sqrt{det(\Sigma^2)}}{\int_{\bar M}\tau\,d\bar\nu}.$$

2. Tools

As we said briefly in the abstract, we use the scheme of the proof of Dvoretzky and Erdös [10]. We will compensate the lack of independence by two ingredients : a strong decorrelation result (proposition 2.1) and an extension of the local limit theorem (proposition 2.2). Let us recall that there exist $\tilde{C} > 0$ and $\tilde{\theta} \in (0,1)$ such that, for almost every point in $\bar{M}$, there exist two unique maximal open C^1-curves $\gamma^s(x)$ and $\gamma^u(x)$ such such that :

- For all integer $n \geq 0$, $\bar{T}^n$ is C^2-regular on a neighbourhood of $\gamma^s(x)$ and the diameter of $\bar{T}^n(\gamma^s(x))$ is less than $\tilde{C}\tilde{\theta}^n$.
- For all integer $n \geq 0$, $\bar{T}^{-n}$ is C^2-regular on a neighbourhood of $\gamma^u(x)$ and the diameter of $\bar{T}^{-n}(\gamma^u(x))$ is less than $\tilde{C}\tilde{\theta}^n$.

The curves $\gamma^s(x)$ are called **stable manifolds** and the curves $\gamma^u(x)$ are called **unstable manifolds**. Let us recall that, according to Chernov [6] (see the few explanations given in appendix), we have the following result :

Proposition 2.1 (Strong decorrelation property)**.** *For any $\eta \in (0,1]$ and any integer $m \geq 0$, there exists $C_{(\eta,m)} > 0$ and $\delta_{(\eta,m)} \in (0,1)$ such that, for any measurable bounded functions $f : \bar{M} \to \mathbb{C}$ and $g : \bar{M} \to \mathbb{C}$, for any integer $n \geq 0$, we have :*

$$\left|\mathbb{E}_{\bar{\nu}}[f.g \circ \bar{T}^n] - \mathbb{E}_{\bar{\nu}}[f]\mathbb{E}_{\bar{\nu}}[g]\right| \leq C_{(\eta,m)} \left(\|f\|_\infty \|g\|_\infty + C_f^{(\eta,u,m)}\|g\|_\infty + C_g^{(\eta,s,m)}\|f\|_\infty\right) {\delta_{(\eta,m)}}^n,$$

with

$$C_f^{(\eta,u,m)} := \sup_{\gamma^u} \sup_{x,y \in \bar{T}^{-m}(\gamma^u):\ x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\eta}$$

and

$$C_g^{(\eta,s,m)} := \sup_{\gamma^s} \sup_{x,y \in \bar{T}^{m}(\gamma^s):\ x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\eta}.$$

A local limit theorem has been established by Szász and Varjú in [19]. We will use the following refinement of this result (we refer to [14] for its proof) :

Proposition 2.2. *Let any real number $p > 1$. There exists $K_0 > 0$ such that, for any positive integer k, if B is any measurable set such that, if $x \in B$ then $\gamma^s(x) \subseteq B$, if r is any positive integer and if A is any union of connected components of $\bar{M} \setminus \bigcup_{i=0}^r \bar{T}^{-i}(R_0)$, then for any $L \in \mathbb{Z}^2$, we have :*

$$\left|\bar{\nu}\left(A \cap \bar{T}^{-(k+r)}(B) \cap \{S_{k+r} - S_r = L\}\right) - \frac{\bar{\nu}(A)\bar{\nu}(B)}{\sqrt{det(\Sigma^2)}2\pi k} e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}L,L\rangle}\right| \leq$$

$$\leq K_0 \left(\frac{\bar{\nu}(B) + \bar{\nu}(A)\bar{\nu}(B)^{1/p}}{k^{3/2}}\left(\frac{\|L\|}{\sqrt{k}} + \frac{\|L\|^3}{k^{3/2}}\right)\exp^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}L,L\rangle} + \frac{\bar{\nu}(B)^{1/p}}{k^2}\right).$$

This is true in particular if A is $(S_1, ..., S_r)$-measurable.

3. First calculations, proofs of theorem 1 and of corollary 2

It is easy to see that the billiard system (M, ν, T) is a cylindrical extension of the billiard system $(\bar{M}, \bar{\nu}, \bar{T})$ by some function $\Phi : \bar{M} \to \mathbb{Z}^2$. For any $(q, \vec{v}) \in \bar{M}$ and any $\ell \in \mathbb{Z}^2$, we have $T(q + \ell, \vec{v}) = (q' + \ell + \Phi(q, \vec{v}), \vec{v}')$ with $(q', \vec{v}') = \bar{T}(q, \vec{v})$. Hence , for any non-negative integer n, we have : $T^n(q+\ell,\vec{v}) = \left(q_n+\ell+\sum_{k=0}^{n-1}\Phi(\bar{T}^k(q,\vec{v})),\vec{v}_n\right)$ with $(q_n,\vec{v}_n)=\bar{T}^n(q,\vec{v})$. We have : $\mathbb{E}_{\bar{\nu}}[\Phi]=0$. Let us consider the asymptotic covariance matrix Σ^2 associated to Φ :

$$\Sigma^2 := \lim_{n\to+\infty} Cov_{\bar{\nu}}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\Phi\circ\bar{T}^k\right).$$

Let us notice that, since the billiard in the plane (M,ν,T) is ergodic, the matrix Σ^2 is invertible. Let $(q,\vec{v})$ be in $\bar{M}$ and ℓ in $\mathbb{Z}^2$. We have

$$N_n(q+\ell,\vec{v}) = \#\left\{\sum_{k=0}^{m-1}\Phi(\bar{T}^k(q,\vec{v}));\ m=1,...,n\right\}.$$

For any $x\in\bar{M}$ and any nonnegative integer k, let $\mathcal{I}_k(x)$ be the index of the obstacle (taken in $\{1,...,I\}$) on which $\bar{T}^k(x)$ is. We have :

$$N'_n(q+\ell,\vec{v}) = \#\left\{\left(\sum_{k=0}^{m-1}\Phi(\bar{T}^k(q,\vec{v})),\mathcal{I}_m(q,\vec{v})\right);\ m=1,...,n\right\}.$$

Hence, $N_n(q+\ell,\vec{v})$ and $N'_n(q+\ell,\vec{v})$ do not depend on $\ell\in\mathbb{Z}^2$. For any non negative integer m, let us define :

$$S_m := \sum_{k=0}^{m-1}\Phi\circ\bar{T}^\ell \text{ and } X_m := \Phi\circ\bar{T}^{m-1}.$$

The random variable S_n corresponds to the index of the cell at the n^{th} reflection off ∂Q if we start from the $(0,0)$-cell. We have :

$$N_n(x) = \#\{m=1,...,n\ :\ \forall k=m+1,...,n,\ S_m(x)\neq S_k(x)\} = \sum_{m=1}^n \mathbf{1}_{B_{n-m}}\circ\bar{T}^m,$$

with $B_p := \left\{x\in\bar{M}\ :\ \forall k=1,...,p,\ S_k(x)\neq(0,0)\right\}$. Moreover we have :

$$N'_n(x) = \#\{m=1,...,n\ :\ \forall k=m+1,...,n,\ (S_m(x),\mathcal{I}_m(x))\neq(S_k(x),\mathcal{I}_k(x))\} = \sum_{m=1}^n \mathbf{1}_{B'_{n-m}}\circ\bar{T}^m,$$

with $B'_p := \left\{x\in\bar{M}\ :\ \forall k=1,...,p,\ (S_k(x),\mathcal{I}_k)\neq((0,0),\mathcal{I}_0)\right\}$. Let us consider $\bar{H}:\bar{M}\to[0,+\infty)$ given by : $\bar{H}(q,\vec{v})=\nu(\bar{M})\sum_{\ell\in\mathbb{Z}^2}H(q+\ell,\vec{v})$. We have :

$$(1)\qquad \mathbb{E}_{H\nu}\left[N_n\right] = \sum_{m=1}^n \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B_{n-m}}\circ\bar{T}^m\times\bar{H}\right] \text{ and } \mathbb{E}_{H\nu}\left[N'_n\right] = \sum_{m=1}^n \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B'_{n-m}}\circ\bar{T}^m\times\bar{H}\right].$$

3.1. Estimation of $\bar{\nu}(B_n)$ and of $\mathbb{E}_{\bar{\nu}}[N_n]$.

Proposition 3.1. *We have :*

$$\lim_{n\to+\infty}\log(n)\bar{\nu}(B_n) = 2\pi\sqrt{det(\Sigma^2)} \quad and \quad \lim_{n\to+\infty}\log(n)\bar{\nu}(B'_n) = 2\pi I\sqrt{det(\Sigma^2)}.$$

Let us consider the first return time $\mathcal{T}_+$ into the initial cell : $\mathcal{T}_+ := \min\left\{m\geq 1\ :\ \sum_{k=0}^{m-1}\Phi\circ\bar{T}^k=(0,0)\right\}$. We have $\bar{\nu}(\mathcal{T}_+>n)=\bar{\nu}(B_n)$ and :

$$\mathbb{E}_{\bar{\nu}}[\min(\mathcal{T}_+,(n+1))] = \sum_{m=0}^n\bar{\nu}(\mathcal{T}_+>m) = 1+\sum_{m=1}^n\bar{\nu}(B_m) = 1+\mathbb{E}_{\bar{\nu}}\left[N_n\right].$$

Analogously, if $\mathcal{T}'_+$ is the first return time into the initial obstacle, we have : $\bar{\nu}(\mathcal{T}'_+ > n) = \bar{\nu}(B'_n)$.

Hence proposition 3.1 for B_n is linked with estimations for the expectation of the first return time into the initial cell. Such estimations have already been done by Dolgopyat, Szász and Varjú in [9]. In this section, we detail the proof for B'_n.

Proof of proposition 3.1. We will use an idea of Dvoretzky and Erdös. First it is easy to see that :

$$1 = \sum_{k=0}^{n} \bar{\nu}\left(\{S_k = (0,0) \text{ and } \forall m = k+1, ..., n,\ S_m - S_k \neq (0,0)\}\right).$$

Hence we have : $1 = \sum_{k=0}^{n} \bar{\nu}\left(\{S_k = (0,0)\} \cap \bar{T}^{-k}(B_{n-k})\right)$. Let $a = 1, ..., I$. In the same way, if we write $B'_p(a) = \{\mathcal{I}_0 = a\} \cap B'_p$, then we have :

$$\bar{\nu}(\{\mathcal{I}_0 = a\}) = \sum_{k=0}^{n} \bar{\nu}\left(\{\mathcal{I}_0 = a\} \cap \{S_k = (0,0)\} \cap \bar{T}^{-k}(B'_{n-k}(a))\right).$$

We replace the independence property used by Dvoretzky and Erdös by our proposition 2.2. According to this result, we know that there exists $C > 0$ such that for all non negative integers n and k with $2 \leq k \leq n-1$, we have :

$$\left| \bar{\nu}\left(\{S_k = (0,0)\} \cap \bar{T}^{-k}(B_{n-k})\right) - \frac{\bar{\nu}(B_{n-k})}{2\pi\sqrt{det(\Sigma^2)}k} \right| \leq \frac{C}{k^2}$$

and

$$\left| \bar{\nu}\left(\{\mathcal{I}_0 = a\} \cap \{S_k = (0,0)\} \cap \bar{T}^{-k}(B'_{n-k}(a))\right) - \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_{n-k}(a))}{2\pi\sqrt{det(\Sigma^2)}k} \right| \leq \frac{C}{k^2}$$

For any $\varepsilon > 0$, we consider an integer ℓ_ε such that : $\sum_{k \geq \ell_\varepsilon} \frac{C}{k^2} < \varepsilon$.

- To get an upper bound, we write :

$$\begin{aligned} \bar{\nu}(\{\mathcal{I}_0 = a\}) &\geq \sum_{k=\ell_\varepsilon}^{n} \bar{\nu}\left(\{\mathcal{I}_0 = a\} \cap \{S_k = (0,0)\} \cap \bar{T}^{-k}(B'_{n-k}(a))\right) \\ &\geq \sum_{k=\ell_\varepsilon}^{n} \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_{n-k}(a))}{2\pi\sqrt{det(\Sigma^2)}k} - \varepsilon \\ &\geq \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_n(a))}{2\pi\sqrt{det(\Sigma^2)}} \sum_{k=\ell_\varepsilon}^{n} \frac{1}{k} - \varepsilon. \end{aligned}$$

  Hence we have :

$$\limsup_{n \to +\infty} \log(n)\bar{\nu}(B'_n(a)) \leq 2\pi\sqrt{det(\Sigma^2)}.$$

- To get a lower bound, we write :

$$\bar{\nu}(\{\mathcal{I}_0 = a\}) \leq \sum_{k=0}^{\ell_\varepsilon - 1} \bar{\nu}(B'_{n-k}(a)) + \varepsilon + \sum_{k=\ell_\varepsilon}^{m-1} \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_{n-k}(a))}{2\pi\sqrt{det(\Sigma^2)}k} + \sum_{k=m}^{n} \frac{1}{2\pi\sqrt{det(\Sigma^2)}k}.$$

  Hence we have :

$$\bar{\nu}(\{\mathcal{I}_0 = a\}) - \varepsilon \leq \bar{\nu}\left(B'_{n-m}(a)\right)\left(\ell_\varepsilon + \frac{\sum_{k=\ell_\varepsilon}^{m-1} \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})}{k}}{2\pi\sqrt{det(\Sigma^2)}}\right) + \frac{\sum_{k=m}^{n} \frac{1}{k}}{2\pi\sqrt{det(\Sigma^2)}}.$$

Let us consider an integer $q \geq e^2$. Applying the previous inequality with $n := q\lfloor \log(q) \rfloor$ and with $m := q\left(\lfloor \log(q) \rfloor - 1\right)$, we get :

$$\bar{\nu}(\{\mathcal{I}_0 = a\}) - \varepsilon \leq \bar{\nu}\left(B'_q(a)\right)\left(\ell_\varepsilon + \bar{\nu}(\{\mathcal{I}_0 = a\})\frac{\log(q) + \log(\log(q))}{2\pi\sqrt{det(\Sigma^2)}}\right) + \frac{-\log\left(1 - \frac{1}{\lfloor \log(q) \rfloor}\right)}{2\pi\sqrt{det(\Sigma^2)}}.$$

Hence we have : $\lim_{n\to+\infty} \log(n)\bar{\nu}(B'_n(a)) = 2\pi\sqrt{det(\Sigma^2)}$. Let us notice that the same argument gives the same estimate for B_n instead of $B'_n(a)$ and that we have : $\bar{\nu}(B'_n) = \sum_{a=1}^{I} \bar{\nu}(B'_n(a))$. □

Since we have $\mathbb{E}_{\bar{\nu}}[N_n] = \sum_{m=1}^{n} \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B_{n-m}}\right]$ and $\mathbb{E}_{\bar{\nu}}[N'_n] = \sum_{m=1}^{n} \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B'_{n-m}}\right]$, we easily get theorem 1 for $H = \frac{\mathbf{1}_{\bar{M}}}{\nu(\bar{M})}$:

Corollary 3.2. *We have :*

$$\mathbb{E}_{\bar{\nu}}[N_n] \sim_{n\to+\infty} 2\pi\sqrt{det(\Sigma^2)}\frac{n}{\log(n)} \quad \text{and} \quad \mathbb{E}_{\bar{\nu}}[N'_n] \sim_{n\to+\infty} 2\pi I\sqrt{det(\Sigma^2)}\frac{n}{\log(n)}.$$

3.2. **Speed of convergence of $\frac{\log(n)}{n}\mathbb{E}_{\bar{\nu}}[N_n]$ and of $\frac{\log(n)}{n}\mathbb{E}_{\bar{\nu}}[N'_n]$.** To estimate the variances of N_n and of N'_n, we need the following more precise estimates :

Proposition 3.3. *For all $p > 1$, we have :*

$$\mathbb{E}_{\bar{\nu}}[N_n] = \frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)} + O\left(\frac{n}{(\log(n))^{1+\frac{1}{p}}}\right) \quad \text{and} \quad \mathbb{E}_{\bar{\nu}}[N'_n] = \frac{2I\pi\sqrt{det(\Sigma^2)}n}{\log(n)} + O\left(\frac{n}{(\log(n))^{1+\frac{1}{p}}}\right).$$

Proof. Let $a \in \{1, ..., I\}$. Here again, we adapt the argument of Dvoretzky and Erdös by replacing the independence property by our extension of the local limit (proposition 2.2). Let p be any real number satisfying $p > 1$. Let us take $L_n := \lceil (\log(n))^2 \rceil$. For n large enough, we have : $1 < L_n \leq n$ and :

$$\begin{aligned}
\bar{\nu}(\{\mathcal{I}_0 = a\}) &\geq \sum_{k=L_n}^{n} \frac{\bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_{n-k}(a))}{2\pi\sqrt{det(\Sigma^2)}k} - K_0 \sum_{k=L_n}^{n} \frac{1}{k^2} \\
&\geq \bar{\nu}(\{\mathcal{I}_0 = a\})\bar{\nu}(B'_n(a))\frac{\log(n) - \log(L_n - 1)}{2\pi\sqrt{det(\Sigma^2)}} - \frac{2K_0}{L_n - 1}.
\end{aligned}$$

Hence we have :

$$\begin{aligned}
\bar{\nu}(B'_n(a)) &\leq 2\pi\sqrt{det(\Sigma^2)}\frac{1 + \frac{2K_0}{\bar{\nu}(\{\mathcal{I}_0 = a\})(L_n - 1)}}{\log(n) - \log(L_n - 1)} \\
&\leq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n) - \log(L_n - 1)} + O\left(\frac{1}{L_n \log(n)}\right) \\
&\leq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{\log(L_n)}{(\log(n))^2}\right) + O\left(\frac{1}{L_n \log(n)}\right) \\
&\leq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{\log(\log(n))}{(\log(n))^2}\right).
\end{aligned}$$

On the other hand, there exist $K'_0 > 0$ and $C'_0 > 0$ such that, for $1 \leq L \leq m - 2 \leq m \leq n$, we have :

$$\bar\nu(\{\mathcal{I}_0=a\}) = \sum_{k=0}^{n} \bar\nu(\{\mathcal{I}_0=a\}\cap\{S_k=(0,0)\}\cap \bar T^{-k}(B'_{n-k}(a)))$$

$$\leq \sum_{k=0}^{L}\frac{K'_0\bar\nu(B'_{n-k}(a))^{\frac{1}{p}}}{k+1} + \sum_{k\geq L+1}\frac{K'_0}{k^2} + \sum_{k=L+1}^{m-1}\frac{\bar\nu(\{\mathcal{I}_0=a\})\bar\nu(B'_{n-k}(a))}{2\pi\sqrt{det(\Sigma^2)}k} + \sum_{k=m}^{n}\frac{1}{2\pi\sqrt{det(\Sigma^2)}k}$$

$$\leq \bar\nu(B'_{n-m}(a))\left(C'_0\log(L)\left(\log(n-m)\right)^{1-\frac{1}{p}} + \frac{\log(m)\bar\nu(\{\mathcal{I}_0=a\})}{2\pi\sqrt{det(\Sigma^2)}}\right) + \frac{2K'_0}{L} + \frac{\log((n+1)/m)}{2\pi\sqrt{det(\Sigma^2)}}.$$

Hence we have :

$$\bar\nu(B'_{n-m}(a)) \geq \frac{\bar\nu(\{\mathcal{I}_0=a\}) - \frac{2K'_0}{L} - \frac{\log((n+1)/m)}{2\pi\sqrt{det(\Sigma^2)}}}{C'_0\log(L)(\log(n-m))^{1-\frac{1}{p}} + \frac{\log(m)\bar\nu(\{\mathcal{I}_0=a\})}{2\pi\sqrt{det(\Sigma^2)}}}.$$

Let q be an integer large enough. We take : $n := q\lfloor\log(q)\rfloor$ and $m := q(\lfloor\log(q)\rfloor - 1)$ and $L := \left\lfloor\left(\frac{\log(q)}{\log(\log(q))}\right)^2\right\rfloor$. We have :

$$\begin{aligned}
\bar\nu(B'_q(a)) &\geq \frac{\bar\nu(\{\mathcal{I}_0=a\}) - \frac{2K'_0}{L} + O\left(\frac{1}{\log(q)}\right)}{C'_0\log(L)(\log(q))^{1-\frac{1}{p}} + \bar\nu(\{\mathcal{I}_0=a\})\frac{\log(q)+\log(\log(q))}{2\pi\sqrt{det(\Sigma^2)}}} \\
&\geq \frac{\bar\nu(\{\mathcal{I}_0=a\})}{C'_0\log(L)(\log(q))^{1-\frac{1}{p}} + \bar\nu(\{\mathcal{I}_0=a\})\frac{(\log(q)+\log(\log(q)))}{2\pi\sqrt{det(\Sigma^2)}}} + O\left(\frac{\log(\log(q))}{(\log(q))^2}\right) \\
&\geq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(q)} + O\left(\frac{1}{(\log(q))^{1+\frac{1}{p}}}\right).
\end{aligned}$$

Hence, for all $p>1$, we have : $\bar\nu(B'_n(a)) = \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{1}{(\log(n))^{1+\frac{1}{p}}}\right)$ and so :

$$\bar\nu(B'_n) = \frac{2\pi I\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{1}{(\log(n))^{1+\frac{1}{p}}}\right),$$

which gives the estimate of $\mathbb{E}_{\bar\nu}[N'_n]$. With the same proof, we can get the same estimate for $\bar\nu(B_n)$ than the one got for $\bar\nu(B'_n(a))$. From which, we easily get the estimate of $\mathbb{E}_{\bar\nu}[N_n]$. □

3.3. **Estimation of the variance of N_n.**

Proposition 3.4. *For any $p>1$, we have :*

$$Var_{\bar\nu}(N_n) = O\left(\frac{n^2}{(\log(n))^{2+\frac{1}{p}}}\right) \quad \textit{and} \quad Var_{\bar\nu}(N'_n) = O\left(\frac{n^2}{(\log(n))^{2+\frac{1}{p}}}\right).$$

Proof. We follow the proof of Dvoretzky and Erdös. We use the estimation of $\mathbb{E}_{\bar\nu}[N_n]$ obtained in proposition 3.3 and we replace the independence property by the strong decorrelation property coming from Chernov's calculations (our proposition 2.1). Let us recall that $N_n = \sum_{m=1}^{n}\mathbf{1}_{B_{n-m}}\circ$

$\bar{T}^m$. Let us take $m_n := \lfloor\sqrt{n}\rfloor$. We have :

$$\begin{aligned}
\mathbb{E}_{\bar\nu}[(N_n)^2] &= \sum_{i,j=1}^{n} \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{n-i}}\circ\bar{T}^i\mathbf{1}_{B_{n-j}}\circ\bar{T}^j\right]\\
&\le 2n(m_n+1)+2\sum_{1\le i\le i+m_n+1\le j\le n}\mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{j-m_n-i}}\circ\bar{T}^i\mathbf{1}_{B_{n-j}}\circ\bar{T}^j\right]\\
&\le 2n(m_n+1)+2\sum_{1\le i\le i+m_n+1\le j\le n}\mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{j-m_n-i}}\circ\bar{T}^{-(j-m_n-i)}\mathbf{1}_{B_{n-j}}\circ\bar{T}^{m_n}\right]\\
&\le 2n(m_n+1)+2\sum_{1\le i\le i+m_n+1\le j\le n}\bar\nu\left(B_{j-i-m_n}\right)\bar\nu\left(B_{n-j}\right)\\
&\qquad +2C_{(1,0)}\sum_{1\le i\le i+m_n\le j\le n}{\delta_{(1,0)}}^{m_n}\\
&\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\sum_{i'=1}^{n}\bar\nu(B_{i'})\sum_{j'=0}^{n-i'}\bar\nu(B_{j'}),
\end{aligned}$$

according to proposition 2.1, since $\mathbf{1}_{B_{j-m_n-i}}\circ\bar{T}^{-(j-m_n-i)}$ is constant along the unstable manifolds of $\bar{T}$ and since $\mathbf{1}_{B_{n-j}}$ is constant along the stable manifolds of $\bar{T}$. Hence we have :

$$\begin{aligned}
Var_{\bar\nu}(N_n) &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\sum_{i=1}^{n}\bar\nu(B_i)\sum_{j=1}^{n-i}\bar\nu(B_j)-\sum_{i'=1}^{n}\sum_{j'=1}^{n}\bar\nu(B_{i'})\bar\nu(B_{j'})\\
&\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+n+2\sum_{i=1}^{n}\bar\nu(B_i)\left(\sum_{j=1}^{n-i}\bar\nu(B_j)-\sum_{j'=i}^{n}\bar\nu(B_{j'})\right).
\end{aligned}$$

Let us recall that if $j\le j'$ then $\bar\nu(B_{j'})\le\bar\nu(B_j)$. Hence, if $j>\lfloor n/2\rfloor$, $\bar\nu(B_j)-\bar\nu(B_{n-j})\le 0$. Therefore, for any $i=1,...,n$, we have :

$$\begin{aligned}
\sum_{j=1}^{n-i}\bar\nu(B_j)-\sum_{j'=i}^{n}\bar\nu(B_{j'}) &= -\bar\nu(B_n)+\sum_{j=1}^{n-i}\left(\bar\nu(B_j)-\bar\nu(B_{n-j})\right)\\
&\le -\bar\nu(B_n)+\sum_{j=1}^{\max(n-i,\lfloor n/2\rfloor)}\left(\bar\nu(B_j)-\bar\nu(B_{n-j})\right)\\
&\le -\bar\nu(B_n)+\sum_{j=1}^{\lfloor n/2\rfloor}\left(\bar\nu(B_j)-\bar\nu(B_{n-j})\right).
\end{aligned}$$

Therefore, thanks to proposition 3.3, we get :

$$\begin{aligned}
Var_{\bar\nu}(N_n) &\le 2n(m_n+2)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\sum_{i=1}^{n}\bar\nu(B_i)\left(\sum_{j=1}^{\lfloor n/2\rfloor}\bar\nu(B_j)-\sum_{j'=n-\lfloor n/2\rfloor}^{n}\bar\nu(B_{j'})\right)\\
&\le 2n(m_n+2)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\mathbb{E}_{\bar\nu}[N_n]\left(\mathbb{E}_{\bar\nu}[N_{\lfloor n/2\rfloor}]+\mathbb{E}_{\bar\nu}[N_{n-\lfloor n/2\rfloor}]-\mathbb{E}_{\bar\nu}[N_n]\right)\\
&\le 2n(m_n+2)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+O\left(\frac{n}{\log(n)}\right)\times O\left(\frac{n}{(\log(n))^{1+\frac{1}{p}}}\right).
\end{aligned}$$

This gives the estimate of $Var_{\bar\nu}(N_n)$. In the same way we get the estimate of $Var_{\bar\nu}(N'_n)$. □

3.4. **End of the proof of theorem 1.** From formula (1) (just before section 3.1) and corollary 3.2, it suffices to prove that :

$$\left|\sum_{m=1}^n \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{n-m}}\circ \bar T^m(\bar H-1)\right]\right| = o\left(\frac{n}{\log(n)}\right)$$

and that

$$\left|\sum_{m=1}^n \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B'_{n-m}}\circ \bar T^m(\bar H-1)\right]\right| = o\left(\frac{n}{\log(n)}\right).$$

Let us prove the first point, the proof of the second one following exactly the same scheme. We have :

$$\begin{aligned}\left|\sum_{m=1}^n \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{n-m}}\circ \bar T^m(\bar H-1)\right]\right| &= \left|\mathbb{E}_{\bar\nu}\left[\sum_{m=1}^n \left(\mathbf{1}_{B_{n-m}}\circ \bar T^m - \bar\nu(B_{n-m})\right)\bar H\right]\right|\\ &\le \|\bar H\|_{L^2(\bar\nu)}\|N_n-\mathbb{E}_{\bar\nu}[N_n]\|_{L^2(\bar\nu)}\\ &\le \|\bar H\|_{L^2(\bar\nu)}\sqrt{Var_{\bar\nu}(N_n)} = o\left(\frac{n}{\log(n)}\right),\end{aligned}$$

according to proposition 3.4. □

3.5. **Proof fo corollary 2.** We prove the result for $\tilde N_t$, the proof for $\tilde N'_t$ being analogous. We have $\tilde N_t(\Delta(y,s)) = N_{n(t+s,y)}(y)$ where Δ is the map defined in section 1.1 and with $n(t,\cdot)$ the number of reflections before time t : $n(t,x) := \max\{m\ge 0\ :\ \sum_{k=0}^{m-1}\tau\circ T^k(x)\le t\}$. Moreover, we have :

$$\begin{aligned}\left|N_{n(t+s,y)}(y) - N_{\left\lfloor\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right\rfloor}(y)\right| &\le 1+\left|n(t+s,y)-\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right|\\ &\le 1+\frac{s}{\min\tau}+\left|n(t,y)-\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right|\end{aligned}$$

and therefore :

$$\begin{aligned}\int_{\tilde{\mathcal M}_1}\left|N_{n(t+s,y)}(y) - N_{\left\lfloor\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right\rfloor}(y)\right| g(\Delta(y,s))\,d\tilde\mu_1(y,s) &\le 1+\frac{\max\tau}{\min\tau}+\left\|n(t,\cdot)-\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right\|_{L^1(h\bar\nu)}\\ &\le 1+\frac{\max\tau}{\min\tau}+\|h\|_{L^2(\bar\nu)}\left\|n(t,\cdot)-\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right\|_{L^2(\bar\nu)},\end{aligned}$$

with $h(q,\vec v) = \nu(\bar M)\sum_{\ell\in\mathbb{Z}^2}\int_0^{\tau(q,\vec v)} g(\Delta((q+\ell,\vec v),s))\,ds$. We conclude by the using the fact that : $\left\|n(t,\cdot)-\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right\|_{L^2(\bar\nu)} = O(\sqrt{t})$ (see for example lemma 4.1 of [13]). □

4. Proof of theorem 3

The results for the flow are consequences of the results for the transformation. Indeed we have $\tilde N_t(\Delta(y,s)) = N_{n(t+s,y)}(y)$ and we know that $(y,s)\mapsto \frac{n(t+s,y)}{t}$ converges $\tilde\mu_1$-almost everywhere to $\frac{1}{\int\tau\,d\bar\nu}$ as t goes to $+\infty$.

To prove the results for the transformation, it is enough to prove that it is true for $\bar\nu$-almost every point in $\bar M$. Let us prove this. The sketch of our proof follows sections 3 and 5 of

[10]. We will refer to [10] for some details of the proof. We insist on the adaptation to do to the proof of Dvoretzki and Erdös of [10]. Let us consider some fixed real number p satisfying $1 < p < \sqrt{3/2}$. In the following n will be such that : $n \geq 1$, $\log(n) \geq 1$ and $\log(\log(n)) \geq 1$ and $\log(n/\log(n)^2) \geq ((\log(n))/2)$. According to the beginning of section 5 of [10], it suffices to prove that :

$$\exists\delta > 0,\ \forall\varepsilon > 0,\ \ \bar{\nu}\left(\left\{\left|N_n - \frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right| > 2\pi\sqrt{det(\Sigma^2)}\varepsilon\frac{n}{\log(n)}\right\}\right) = O\left((\log(n))^{-1-\delta}\right).$$

and that the same holds for

$$N'_n(a) := \#\{\ell \in \mathbb{Z}^2\ :\ \exists m = 1, ..., n\ :\ T^m(x) \in \partial U_{a,\ell} \times \mathbb{R}^2\}$$

instead of N_n. As in [10], we consider $L = L_n$ with $L = O(\log(n))$ and $L_n \to +\infty$.

(1) First we estimate : $\bar{\nu}\left(\left\{N_n > (1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)$. For all $i = 1, ..., L$, we define, :

$$A_i := \left\{N_{\lfloor in/L\rfloor} - N_{\lfloor (i-1)n/L\rfloor} > \left(1+\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}.$$

and :

$$\tilde{A}_i := \left\{N_{\lfloor in/L\rfloor} - N_{\lfloor (i-1)n/L\rfloor} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}.$$

As noticed by Dvoretzky and Erdös in [10], we have :

$$\boxed{\left\{N_n > (1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\} \subseteq \left(\bigcup_{1\leq i<j\leq L}(A_i\cap A_j)\right) \cup \left(\bigcup_{1\leq i\leq L}\tilde{A}_i\right)}.$$

(a) We estimate $\bar{\nu}(A_i)$ by the Markov inequality.
Let us notice that, if $a \leq b$, $N_b - N_a$ is less than the number $N_{b-a}\circ\bar{T}^a$ of cells visited between the $(a+1)$th reflection and the bth reflection. Hence, we have :

$$\begin{aligned}\bar{\nu}(A_i) &\leq \bar{\nu}\left(\left\{N_{\lfloor n/L\rfloor+1} > \left(1+\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}\right)\\ &\leq \bar{\nu}\left(\left\{N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}\right)\\ &\leq \frac{\mathbb{E}_{\bar{\nu}}\left[\left(N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2\right]}{\left(\frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2} = O\left(\frac{1}{(\log(n))^{\frac{1}{p}}}\right).\end{aligned}$$

(b) We estimate $\sum_{1\leq i<j\leq L}\bar{\nu}(A_i\cap A_j)$. Here A_i and A_j are not independent (contrarily to [10]). We will use the strong decorrelation property (proposition 2.1). Let $A' :=$

$\left\{N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} - \sqrt{n}\right\}$. We have :

$$\begin{aligned}
\sum_{1\le i<j\le L} \bar\nu(A_i\cap A_j) &\le \sum_{1\le i<j\le L} \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{A_i}\mathbf{1}_{A'}\circ \bar T^{\lfloor\sqrt{n}\rfloor+\left\lfloor\frac{(j-1)n}{L}\right\rfloor}\right]\\
&\le \sum_{1\le i<j\le L} \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{A_i}\circ \bar T^{-\lfloor in/L\rfloor}\times\right.\\
&\qquad\left.\times\mathbf{1}_{A'}\circ \bar T^{\lfloor (j-1)n/L\rfloor-\lfloor in/L\rfloor+\lfloor\sqrt{n}\rfloor}\right]\\
&\le \left(\sum_{1\le i<j\le L}\bar\nu\left(A_i\right)\bar\nu\left(A'\right)\right) + L^2 C_{(1,0)}\delta_{(1,0)}{}^{\sqrt{n}-1},
\end{aligned}$$

according to proposition 2.1 since $\mathbf{1}_{A_i}\circ \bar T^{-\lfloor in/L\rfloor}$ is constant along the unstable manifolds and $\mathbf{1}_{A'}$ is constant along the stable manifolds. Let us notice that, for n large enough, we have :

$$A' \subseteq \left\{N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{\varepsilon}{4}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}.$$

Hence, we get an estimation for $\bar\nu(A')$ analogous to the one obtained for $\bar\nu(A_j)$. We get :

$$\boxed{\sum_{1\le i<j\le L}\bar\nu\left(A_i\cap A_j\right) = O\left(\frac{L^2}{(\log(n))^{2/p}}\right)}. \tag{2}$$

(c) We estimate $\bar\nu(\tilde A_i)$ as we did for $\bar\nu(A_i)$:

$$\begin{aligned}
\bar\nu(\tilde A_i) &\le \bar\nu\left(\left\{N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\left(\frac{\varepsilon L}{2}-1\right)\right\}\right)\\
&= O\left(\frac{1}{L^2\log(n)^{\frac{1}{p}}}\right).
\end{aligned}$$

From this, we get :

$$\boxed{\bar\nu\left(\bigcup_{i=1}^{L}\tilde A_i\right) \le O\left(\frac{1}{L\log(n)^{\frac{1}{p}}}\right)}. \tag{3}$$

(d) Hence, with the choice $L = \left\lfloor(\log(n))^{\frac{1}{3p}}\right\rfloor$ (here only), we get :

$$\bar\nu\left(\left\{N_n > (1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right) = O\left((\log(n))^{-4/(3p)}\right)$$

and we have $4/(3p) > 1$.

(2) Second, we estimate : $\bar\nu\left(\left\{N_n < (1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)$. From now, we will take : $L = \lfloor\log(\log(n))\rfloor$ (for $n\ge 3$). For any $i=1,...,L$, we set M_i the number of cells visited between times $\lfloor(i-1)n/L\rfloor+1$ and $\lfloor in/L\rfloor$. We define $D_i := \{M_i < \left(1-\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\}$. For any $1\le i<j\le L$, we consider the number $M_{i,j}$ of points

in common in $\{S_{\lfloor n(i-1)/L\rfloor+1},...S_{\lfloor in/L\rfloor}\}$ and in $\{S_{\lfloor n(j-1)/L\rfloor+1},...S_{\lfloor nj/L\rfloor}\}$. Let us take $\eta\in(0,1/2)$. Let us define :

$$C_{i,j} := \left\{ M_{i,j} > \frac{n\log(\log(n))}{L(\log(n))^{1+\eta}} \right\}.$$

As noticed by Dvoretzky and Erdös in [10], we have :

$$\boxed{\left\{N_n < (1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\} \subseteq \left(\bigcup_{i<j}(D_i\cap D_j)\right) \cup \left(\bigcup_{i,j,i',j'\ :\ \#\{i,j,i',j'\}=4} (C_{i,j}\cap C_{i',j'})\right)}.$$

(a) We estimate $\bar\nu(D_i)$ thanks to the Markov inequality. We have :

$$\begin{aligned}\bar\nu(D_i) &\le \bar\nu\left(M_i - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)\\ &\le \mathbb{E}_{\bar\nu}\left[\left(M_i - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2\right]\frac{4L^2\log^2(n)}{\varepsilon^2 4\pi^2 det(\Sigma^2)n^2}\\ &\le O\left(\frac{n^2}{L^2(\log(n/L))^{2+\frac{1}{p}}}\right)\frac{4L^2\log^2(n)}{\varepsilon^2 4\pi^2 det(\Sigma^2)n^2}\\ &\le O\left(\frac{1}{(\log(n))^{\frac{1}{p}}}\right).\end{aligned}$$

We want to estimate $\sum_{1\le i<j\le L}\bar\nu(D_i\cap D_j)$. The events D_i and D_j are not independent here but, according to proposition 2.1, we have :

$$\sum_{1\le i<j\le L}\bar\nu(D_i\cap D_j) \le \sum_{1\le i<j\le L}\mathbb{E}_{\bar\nu}[\mathbf{1}_{D_i}\mathbf{1}_{D'_j}\circ\bar T^{\lfloor\sqrt{n}\rfloor}],$$

with $D'_j := \left\{M_j - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{2}2\pi\sqrt{det(\Sigma^2)}\frac{n}{L\log(n)} + \sqrt{n}\right\}$. Let us notice that, for n large enough, we have :

$$D'_j \subseteq \left\{M_j - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{4}2\pi\sqrt{det(\Sigma^2)}\frac{n}{L\log(n)}\right\}.$$

Hence we can estimate $\bar\nu(D'_j)$ as we have estimated $\bar\nu(D_i)$. According to proposition 2.1, since $\mathbf{1}_{D_i}\circ\bar T^{-\lfloor in/L\rfloor}$ is constant on each γ^u and since $\mathbf{1}_{D'_j}\circ\bar T^{-\lfloor in/L\rfloor}$ is constant on each γ^s, we have :

$$\sum_{1\le i<j\le L}\bar\nu(D_i\cap D_j) \le \sum_{1\le i<j\le L}\bar\nu(D_i)\bar\nu(D'_j) + \sum_{1\le i<j\le L}C_{(1,0)}\delta_{(1,0)}{}^{\sqrt{n}-1}$$

and so :

$$\boxed{\sum_{1\le i<j\le L}\bar\nu(D_i\cap D_j) = O\left(\frac{L^2}{(\log(n))^{2/p}}\right)}.$$

(b) We will use the following notations :

$$\xi_{a,b} := \{\forall q=a,...,b-1,\ S_q\ne S_b\} \text{ and } \zeta_{a,b} := \{\forall q=a+1,...,b,\ S_q\ne S_a\}.$$

We have :

$$M_{i,j}=\sum_{k=\lfloor(i-1)n/L\rfloor+1}^{\lfloor in/L\rfloor}\sum_{\ell=\lfloor(j-1)n/L\rfloor+1}^{\lfloor jn/L\rfloor}\mathbf{1}_{\xi_{\lfloor(i-1)n/L\rfloor+1,k}\cap\{S_k=S_\ell\}\cap\zeta_{\ell,\lfloor jn/L\rfloor}}.$$

Let us define :

$$n_{(i,-)}:=\lfloor(i-1)n/L\rfloor+\lceil n/(\log(n))^2\rceil \ \text{ and } \ n_{(i,+)}:=\lfloor in/L\rfloor-\lfloor n/(\log(n))^2\rfloor.$$

(c) Let us prove that we have :

$$\sup_{i<j}\mathbb{E}_{\bar\nu}[M_{i,j}]=O\left(\frac{n\log(\log(n))}{L(\log(n))^2}\right). \tag{4}$$

Let $i<j$. According to proposition 2.2, we have :
$\mathbb{E}_{\bar\nu}[M_{i,j}]\le O\left(\frac{n}{(\log(n))^2}\right)+\sum_{k=n_{(i,-)}}^{n_{(i,+)}}\sum_{\ell=n_{(j,-)}}^{n_{(j,+)}}\bar\nu\left(\xi_{\lfloor(i-1)n/L\rfloor+1,k}\cap\{S_k=S_\ell\}\cap\zeta_{\ell,\lfloor jn/L\rfloor}\right)$

$$\begin{aligned}\le\ & K_0'\sum_{k=n_{(i,-)}}^{n_{(i,+)}}\sum_{\ell=n_{(j,-)}}^{n_{(j,+)}}\left[\frac{1}{\ell-k}\frac{1}{\log\left(k-\lfloor(i-1)n/L\rfloor\right)}\frac{1}{\log\left(\lfloor jn/L\rfloor-\ell\right)}+\right.\\ &\left.+\frac{1}{(\ell-k)^2}\frac{1}{(\log(\lfloor jn/L\rfloor-\ell))^{1/p}}\right]\\ \le\ & K_0''\left[\frac{n}{L}\frac{\log\left(\frac{n}{n/\log^2(n)}\right)}{\log^2(n)}+\frac{n}{L}\frac{\log^2(n)}{n}\frac{1}{\log^{1/p}(n)}\right]=O\left(\frac{n\log(\log(n))}{L\log^2(n)}\right).\end{aligned}$$

(d) According to the Markov inequality, we have :

$$\boxed{\bar\nu\left(C_{i,j}\right)=O\left(\frac{1}{(\log(n))^{1-\eta}}\right)}.$$

(e) Let us prove that :

$$\boxed{\sup_{1\le i<j<i'<j'\le L}\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}]=O\left(\frac{n^2\log(\log(n))^2}{L^2(\log(n))^4}\right)}. \tag{5}$$

Let $1\le i<j<i'<j'\le L$. Let us define :

$$\tilde M_{i,j}:=\sum_{k=n_{(i,-)}}^{n_{(i,+)}}\sum_{\ell=n_{(j,-)}}^{n_{(j,+)}}\mathbf{1}_{\xi_{n_{(i,-)},k}\cap\{S_k=S_\ell\}\cap\zeta_{\ell,n_{(j,+)}}}.$$

We have $M_{i,j}\le\tilde M_{i,j}+O\left(\frac{n}{(\log(n))^2}\right)$. Moreover, according to proposition 2.1, since $\tilde M_{i,j}\circ\bar T^{-n_{(j,+)}}$ is constant along the unstable manifolds and since $\tilde M_{i',j'}\circ\bar T^{-n_{(i',-)}}$ is constant along the stable manifolds, we have :

$$\mathbb{E}_{\bar\nu}[\tilde M_{i,j}\tilde M_{i',j'}]\ \le\ \mathbb{E}_{\bar\nu}[\tilde M_{i,j}]\mathbb{E}_{\bar\nu}[\tilde M_{i',j'}]+\frac{n^2}{L^2}C_{(1,0)}{\delta_{(1,0)}}^{2n/(\log(n))^2}.$$

(f) Let us prove that :

$$\boxed{\sup_{1\le i<i'<j'<j\le L}\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}]=O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right)}. \tag{6}$$

To this end, we will use repeatidly proposition 2.2. Let $1 \le i < i' < j' < j \le L$. Let $\mathcal{L}$ be the set of (k, k', ℓ', ℓ) such that :

$$n_{(i,-)} \le k \le n_{(i,+)},\ n_{(i',-)} \le k' \le n_{(i',+)},\ n_{(j',-)} \le \ell' \le n_{(j',+)},\ n_{(j,-)} \le \ell \le n_{(j,+)}.$$

We have :

$$\boxed{\mathbb{E}_{\bar\nu}\left[M_{i,j}M_{i',j'}\right] = O\left(\frac{n^2\log(\log(n))}{L(\log(n))^4}\right) + \sum_{M\in[-n;n]^2}\ \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \bar\nu(\mathcal{A}_0)} \tag{7}$$

with :

$$\mathcal{A}_0 := \xi_{\lfloor (i-1)n/L\rfloor+1,k} \cap \{S_{k'} - S_k = M\} \cap \zeta_{k',\lfloor i'n/L\rfloor} \cap$$
$$\cap \{S_{\ell'} - S_{k'} = (0,0)\} \cap \xi_{\lfloor (j'-1)n/L\rfloor+1,\ell'} \cap \{S_\ell - S_{\ell'} = -M\} \cap \zeta_{\ell,\lfloor jn/L\rfloor}.$$

Applying proposition 2.2 a first time, we get :

$$\bar\nu(\mathcal{A}_0) \quad\le\quad \tilde K_0 e^{-\frac{1}{2(k'-k)}a\langle M,M\rangle}\left[\frac{\bar\nu(\mathcal{A}_1)}{(k'-k)(\log(n))} + \frac{\bar\nu(\mathcal{A}_1) + \frac{(\bar\nu(\mathcal{A}_1))^{1/p}}{\log(n)}}{(k'-k)^{3/2}}\right] + \frac{\tilde K_0 \bar\nu(\mathcal{A}_1)^{\frac1p}}{(k'-k)^2}$$

with $\mathcal{A}_1 := \zeta_{k',\lfloor i'n/L\rfloor} \cap \{S_{\ell'} - S_{k'} = (0,0)\} \cap \xi_{\lfloor (j'-1)n/L\rfloor+1,\ell'} \cap \{S_\ell - S_{\ell'} = -M\} \cap \zeta_{\ell,\lfloor jn/L\rfloor}$. Using the fact that $(\sum_{m=0}^{k-1}\Phi\circ\bar T^{-m})_{k\ge1}$ has the same distribution as $(-\sum_{m=0}^{k-1}\Phi\circ\bar T^{m})_{k\ge1}$ with respect to $\bar\nu$, we notice that :

$$\bar\nu(\mathcal{A}_1) \quad=\quad \bar\nu\left(\xi_{1,\lfloor jn/L\rfloor-\ell+1} \cap \{S_{\lfloor jn/L\rfloor-\ell'+1} - S_{\lfloor jn/L\rfloor-\ell+1} = M\} \cap \zeta_{\lfloor jn/L\rfloor-\ell'+1,\lfloor jn/L\rfloor-\lfloor (j'-1)n/L\rfloor} \cap\right.$$
$$\left.\cap\{S_{\lfloor jn/L\rfloor-k'+1} - S_{\lfloor jn/L\rfloor-\ell'+1} = (0,0)\} \cap \xi_{\lfloor jn/L\rfloor-\lfloor i'n/L\rfloor+1,\lfloor jn/L\rfloor-k'+1}\right).$$

Hence, applying proposition 2.2 a second time we get :

$$\bar\nu(\mathcal{A}_1) \le \tilde K_0 e^{-\frac{1}{2(\ell-\ell')}a\langle M,M\rangle}\left[\frac{\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)} + \frac{\bar\nu(\mathcal{A}_2) + \frac{(\bar\nu(\mathcal{A}_2))^{1/p}}{\log(n)}}{(\ell-\ell')^{3/2}}\right] + \frac{\tilde K_0 \bar\nu(\mathcal{A}_2)^{\frac1p}}{(\ell-\ell')^2},$$

with $\mathcal{A}_2 := \zeta_{k',\lfloor i'n/L\rfloor} \cap \{S_{\ell'} - S_{k'} = (0,0)\} \cap \xi_{\lfloor (j'-1)n/L\rfloor+1,\ell'}$. In particular we have :

$$\sum_{k',\ell'} \bar\nu(\mathcal{A}_2) \le \mathbb{E}_{\bar\nu}[M_{i',j'}] \quad\text{and}\quad \bar\nu(\mathcal{A}_2) \le \bar\nu(\{S_{\ell'} - S_{k'} = (0,0)\}) \le \frac{\tilde K_0}{\ell'-k'}.$$

Let us notice that, since $\bar\nu(A_1) \le \frac{3K_0}{\ell-\ell'} \le \frac{3K_0(\log(n))^2}{n}$, there exists $K_3 > 0$ such that : $\bar\nu(\mathcal{A}_1) + \frac{6K_0}{\log(n)}(\bar\nu(\mathcal{A}_1))^{1/p} \le \frac{K_3(\bar\nu(\mathcal{A}_1))^{1/p}}{\log(n)}$. Hence we have :

$$\boxed{\bar\nu(\mathcal{A}_0) \le \tilde K_1 e^{-\frac{1}{2(k'-k)}a\langle M,M\rangle}\left[\frac{\bar\nu(\mathcal{A}_1)}{(k'-k)(\log(n))} + \frac{(\bar\nu(\mathcal{A}_1))^{1/p}}{\log(n)(k'-k)^{3/2}}\right] + \frac{\tilde K_1 \bar\nu(\mathcal{A}_1)^{\frac1p}}{(k'-k)^2}.} \tag{8}$$

In the same way, we get :

$$\boxed{\bar\nu(\mathcal{A}_1) \le \tilde K_1 e^{-\frac{1}{2(\ell-\ell')}a\langle M,M\rangle}\left[\frac{\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)} + \frac{(\bar\nu(\mathcal{A}_2))^{1/p}}{\log(n)(\ell-\ell')^{3/2}}\right] + \frac{\tilde K_1 \bar\nu(\mathcal{A}_2)^{\frac1p}}{(\ell-\ell')^2},} \tag{9}$$

Now we enter in the most technical part of the proof. Let us notice that this part could be shortened if we were able to prove that in formulas (8) and (9), the first term is the biggest one. But, unfortunately this is not so evident. Hence, we will estimate each term.

- The following fact will be useful in the sequel : $c=\sup_{d>a}\frac{1}{d}\sum_{M\in\mathbf{Z}^2}e^{-\frac{a}{2d}\langle M,M\rangle}<+\infty$.
- First term in (8) and first term in (9). We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_{M}e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}\langle M,M\rangle}\frac{\bar\nu(\mathcal{A}_2)}{(k'-k)(\ell-\ell')\log(n)^2}.$$

According to the previous point, we have to estimate : $\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{(k'-k)(\ell-\ell')}{(k'-k)+(\ell-\ell')}\frac{\bar\nu(\mathcal{A}_2)}{(k'-k)(\ell-\ell')\log(n)^2}\le$

$$\begin{aligned}
&\le \frac{1}{(\log(n))^2}\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{\bar\nu(\mathcal{A}_2)}{(\ell-n_{(j',+)})+(n_{(i',-)}-k)}\\
&\le \frac{1}{(\log(n))^2}\left(\sum_{k',\ell'}\bar\nu(\mathcal{A}_2)\right)\sum_{k,\ell}\frac{1}{(\ell-n_{(j,-)})+(n_{(i,+)}-k)+\frac{2n}{\log^2(n)}-1}\\
&\le \frac{1}{(\log(n))^2}\mathbb{E}_{\bar\nu}[M_{i',j'}]\sum_{k,\ell}\frac{1}{(\ell-n_{(j,-)})+(n_{(i,+)}-k)+\frac{2n}{\log^2(n)}-1}\\
&\le O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right).
\end{aligned}$$

- First term in (8) and second or third terms in (9). We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_{M}\frac{e^{-\frac{a}{2(k'-k)}\langle M,M\rangle}}{(k'-k)(\log(n))}\frac{(\bar\nu(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^{3/2}}\le$$

$$\begin{aligned}
&\le c\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{1}{\log(n)}\frac{1}{(\ell-\ell')^{3/2}}\frac{1}{(\ell'-k')^{1/p}}\\
&\le c\frac{n^4}{L^4}\frac{1}{\log(n)}\frac{\log^3(n)}{n^{3/2}}\frac{\log^{2/p}(n)}{n^{1/p}}\\
&= O\left(\frac{n^2(\log(\log(n)))^2}{L^2\log^4(n)}\right),
\end{aligned}$$

since $4-3/2-1/p<2$.
- Second or third terms in (8) and first term in (9). We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_{M}\frac{e^{-\left(\frac{a}{2p(\ell-\ell')}\right)\langle M,M\rangle}(\bar\nu(\mathcal{A}_2))^{1/p}}{(k'-k)^{3/2}(\ell-\ell')^{1/p}\log(n)^{1/p}}\le$$

$$\begin{aligned}
&\le c\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{2p(\ell-\ell')^{1-1/p}}{(k'-k)^{3/2}\log(n)^{1/p}(\ell'-k')^{1/p}}\\
&\le c\frac{n^4}{L^4}\frac{2pn^{1-1/p}\log^3(n)\log^{2/p}(n)}{n^{3/2}\log(n)^{1/p}n^{1/p}}=O\left(\frac{n^2(\log(\log(n)))^2}{L^2\log^4(n)}\right),
\end{aligned}$$

since $4+1-1/p-3/2-1/p<2$.
- Second term in (8) and second or third terms in (9). We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M \frac{e^{-\frac{a}{2(k'-k)}\langle M,M\rangle}(\bar\nu(\mathcal{A}_2))^{1/p^2}}{\log(n)(k'-k)^{3/2}(\ell-\ell')^{3/2p}} \leq$$

$$\begin{aligned}
&\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{c}{\log(n)(k'-k)^{1/2}(\ell-\ell')^{3/2p}(\ell'-k')^{1/p^2}}\\
&\leq \frac{n^4}{L^4}\frac{c\log(n)\log^{3/p}(n)\log^{2/p^2}(n)}{\log(n)n^{1/2}(n)^{3/2p}n^{1/p^2}}\\
&= O\left(\frac{n^2(\log(\log(n)))^2}{L^2\log^4(n)}\right),
\end{aligned}$$

since $4-1/2-3/(2p)-1/p^2<2$.

- Third term in (8) and second term in (9). We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M \frac{e^{-\frac{1}{2p(\ell-\ell')}a\langle M,M\rangle}}{\log(n)^{1/p}(k'-k)^2}\frac{(\bar\nu(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{3/(2p)}} \leq$$

$$\begin{aligned}
&\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{cp}{\log(n)^{1/p}(k'-k)^2(\ell-\ell')^{3/(2p)-1}(\ell'-k')^{1/p^2}}\\
&\leq \frac{n^4}{L^4}\frac{cp}{\log(n)^{1/p}}\frac{\log(n)^{2+2/p^2+3/p}}{n^{1+1/p^2+3/(2p)}}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

since $p<3/2$ and $3-1/p^2-3/(2p)<2$.

- Third term in (8) and third term in (9). We have to estimate :

$$\begin{aligned}
\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M \frac{1}{(k'-k)^2}\frac{(\bar\nu(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{2/p}} &\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M \frac{1}{(k'-k)^2(\ell'-k')^{1/p^2}(\ell-\ell')^{2/p}}\\
&\leq \frac{n^4}{L^4}n^2\frac{(\log(n))^{4+2/p^2+4/p}}{n^{2+1/p^2+2/p}}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

since $4-1/p^2-2/p<2$.

(g) In the same way, we can prove that :

$$\boxed{\sup_{1\leq i<i'<j<j'\leq L}\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}] = O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right)}, \tag{10}$$

with the use of the formula :

$$\begin{aligned}
\mathbb{E}_{\bar\nu}\left[M_{i,j}M_{i',j'}\right] &= O\left(\frac{n^2}{\log^4(n)}\right) + \sum_{M\in[-n;n]^2}\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \bar\nu\left(\xi_{\lfloor (i-1)n/L\rfloor+1,k}\cap\{S_{k'}-S_k=M\}\cap\zeta_{k',\lfloor i'n/L\rfloor}\cap\right.\\
&\left.\cap\{S_\ell-S_{k'}=-M\}\cap\xi_{\lfloor (j-1)n/L\rfloor+1,\ell}\cap\{S_{\ell'}-S_\ell=M\}\cap\zeta_{\ell',\lfloor j'n/L\rfloor}\right).
\end{aligned}$$

(h) Using the fact that :

$$\bar\nu\left(M_{i,j}>\tfrac{n\log(\log(n))}{L(\log(n))^{1+\eta}}\text{ and }M_{i',j'}>\tfrac{n\log(\log(n))}{L(\log(n))^{1+\eta}}\right)\le$$
$$\le\bar\nu\left(M_{i,j}M_{i',j'}>\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^{2+2\eta}}\right)\le\frac{\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}]}{\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^{2+2\eta}}}.$$

Hence we have :

$$\sum_{(i,j,i',j')\ :\ \#\{i,j,i',j'\}=4}\bar\nu(C_{i,j}\cap C_{i',j'})=O\left(\frac{L^4}{(\log(n))^{2-2\eta}}\right).$$

(i) Hence, we have :

$$\bar\nu\left(\left\{N_n<(1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)=O\left(\frac{L^2}{(\log(n))^{2/p}}\right)+O\left(\frac{L^4}{(\log(n))^{2-2\eta}}\right).$$

With our choices, we have : $L=\lfloor\log(\log(n))\rfloor$, $2-2\eta>1$ and $2/p>1$. This completes the proof of our result of almost sure convergence. □

Appendix A. Proof of proposition 2.1

We use some results of [7, 6]. We will use the notations of Chernov in [6]. We take k_0 large enough (as in [6]). Let us define $\mathbb{S}:=\bigcup_{k\ge k_0}\{x\in\bar M\ :\ |\varphi_x|=\pi/2-\frac{1}{k^2}\}$, where φ_x is the angular measure of $(\vec n(q),\vec v)$ taken in $[-\pi/2;\pi/2]$ if $x=(q,\vec v)$. Let us consider an integer $m\ge 0$. We denote by ξ^s_m the partition of $\bar M\setminus\left(\bigcup_{p=0}^m\bar T^{-p}(R_0\cup\mathbb{S})\right)$ in connected components. Analogously, we denote by ξ^u_m the partition of $\bar M\setminus\left(\bigcup_{p=0}^m\bar T^{p}(R_0\cup\mathbb{S})\right)$ in connected components. Let us recall the definition of homogeneous stable manifolds and homogeneous unstable manifolds :

- A homogeneous stable manifold is a C^1 curve γ of $\bar M$ contained in $\bar M\setminus\left(\bigcup_{m\ge0}\bar T^{-m}(R_0\cup\mathbb{S})\right)$.
- A homogeneous unstable manifold is a C^1 curve γ of $\bar M$ contained in $\bar M\setminus\left(\bigcup_{m\ge0}\bar T^{m}(R_0\cup\mathbb{S})\right)$.

Let us consider the set Γ^s of homogeneous stable manifolds and the set Γ^u of homogeneous unstable manifolds. We recall that there exist two constants $c_1>0$ and $\delta_1\in(0,1)$ such that, for any integers $n\ge0$ and $m\ge0$:

- let y and z belonging to the same homogeneous unstable manifold. Then, $\bar T^{-n}(y)$ and $\bar T^{-n}(z)$ belong to a same homogeneous unstable manifold and we have : $d(\bar T^{-n}(y),\bar T^{-n}(z))\le c_1{\delta_1}^n$. Moreover, y and z belong to the same atom of ξ^u_n. Moreover, if y and z belong to the same atom of ξ^s_m, then $\bar T^m(y)$ and $\bar T^m(z)$ belong to a same homogeneous unstable manifold.
- let y and z belonging to the same homogeneous stable manifold. Then, $\bar T^n(y)$ and $\bar T^n(z)$ belong to a same homogeneous stable manifold and we have : $d(\bar T^n(y),\bar T^n(z))\le c_1{\delta_1}^n$. Moreover, y and z belong to the same atom of ξ^s_n. Moreover, if y and z belong to the same atom of ξ^u_m, then $\bar T^{-m}(y)$ and $\bar T^{-m}(z)$ belong to a same homogeneous stable manifold.

In [6], for any x,y, Chernov defines : $s_+(x,y):=\min\{n\ge0\ :\ y\notin\xi^s_n(x)\}$ and $s_-(x,y):=\min\{n\ge0\ :\ y\notin\xi^u_n(x)\}$, where $\xi^s_n(x)$ (resp. $\xi^u_n(x)$) is the atom of ξ^s_n (resp. ξ^u_n) containing the point x. Following Chernov in [6] (page 15), let us introduce the following quantities :

$$\tilde{K}_f^{(1)} := \sup_{\gamma^u\in\Gamma^u}\sup_{y,z\in\gamma^u y\neq z}\frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_+(y,z)}}\ \text{ and }\ \tilde{K}_f^{(2)} := \sup_{\gamma^s\in\Gamma^s}\sup_{y,z\in\gamma^s y\neq z}\frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

We observe that we have : $\tilde{K}_f^{(i)} \le 2\|f\|_\infty {\delta_1}^{-\eta m} + K_f^{(i)}$, with :

$$K_f^{(1)} := \sup_{\gamma^u\in\Gamma^u}\sup_{y,z\in\gamma^u; y\neq z; s_+(y,z)\ge m+1}\frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_+(y,z)}}$$

$$\text{and }\ K_f^{(2)} := \sup_{\gamma^s\in\Gamma^s}\sup_{y,z\in\gamma^s; y\neq z; s_-(y,z)\ge m+1}\frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

With these definitions, we have : $K_f^{(1)} \le (c_1)^\eta C_f^{(\eta,u,m)}$ and $K_f^{(2)} \le (c_1)^\eta C_f^{(\eta,s,m)}$. Moreover, Chernov establishes the existence of $c_3>0$ and $\alpha_3\in(0;1)$ such that, for any integer $n\ge 0$ and for any bounded $\mathbb{C}$-valued functions f and g, we have :

$$|Cov_{\bar{\nu}}(f, g\circ\bar{T}^n)| \le c_3\left(\|f\|_\infty\|g\|_\infty + \|f\|_\infty\tilde{K}_g^{(2)} + \|g\|_\infty\tilde{K}_f^{(1)}\right)(\alpha_3)^n$$

(cf. theorem 4.3 of [6] and the remark following this theorem).

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