Fluctuation theorems with shrinking intervals

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The Fluctuation Theorem

The Fluctuation Theorem is a simple but remarkable phenomenon in reversible systems, discovered by Cohen and Gallavotti and rigorously proved (for hyperbolic maps) by Gallavotti in the mid 1990s.

Rougly speaking, it says that the time-n average of volume contraction is approximately p with probability e^{np} greater than that it is approximately $-p$. A more precise statement will appear later.

Hyperbolic Diffeomorphisms

$T : M \to M$ diffeomorphism, $X \subset M$ closed, T-invariant.

$T : X \to X$ *hyperbolic* if

1. $T_X M = E^s + E^u$, continuous DT-invariant splitting, such that $\|DT^n|E^s\|, \|DT^{-n}|E^u\| \leq Ce^{-cn}$, for all $n \geq 0$, for some $C, c > 0$;
2. $T : XtoX$ is transitive (i.e. has a dense orbit);
3. periodic orbits are dense in X;
4. there exists an open set $U \supset X$ such that $X = \bigcap_{n=-\infty}^{\infty} T^n U$.

T is *mixing* if, for all non-empty open subsets V, V' of X, there exists $N \geq 0$ such that $T^{-n}V \cap V' \neq \varnothing$ for all $n \geq N$.

Anosov diffeomorphisms

$T : M \to M$ is an *Anosov* diffeomorphism if 1 holds with $X = M$ (in which case, 3 and 4 are automatically satisfied).

Time reversal symmetry

The Fluctuation Theorem concerns hyperbolic diffeomorphisms $T : X \to X$ with *time reversal symmetry*: there exists an involution $i : X \to X$ such that

$$i \circ T \circ i = T^{-1}.$$

For simplicity, we shall also assume our systems are mixing.

Examples

1. Smale horseshoe (conjugate to a full shift).

$$i((x_n)_{n=-\infty}^{\infty}) = (y_n)_{n=-\infty}^{\infty},$$

where $y_n = x_{-n}$.
Then

$$\begin{aligned} i((\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)) &= (\dots, x_2, x_1, x_0, x_{-1}, x_{-2}, \dots) \\ T((\dots, x_2, x_1, x_0, x_{-1}, x_{-2}, \dots)) &= (\dots, x_1, x_0, x_{-1}, x_{-2}, x_{-3}, \dots) \\ i((\dots, x_1, x_0, x_{-1}, x_{-2}, x_{-3}, \dots)) &= (\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1) \\ &= T^{-1}((\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)). \end{aligned}$$

Examples

2. Given an Anosov diffeomorphism $T_0 : M \to M$, define

$$T : M \times M \to M \times M : (x, y) \mapsto (T_0 x, T_0^{-1} y).$$

If $i(x, y) = (y, x)$ then

$$\begin{aligned} i \circ T \circ i(x, y) &= i \circ T(y, x) = i(T_0 y, T_0^{-1} x) \\ &= (T_0^{-1} x, T_0 y) = T^{-1}(x, y). \end{aligned}$$

Examples

3. Define a hyperbolic total automorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x, y) = (y, -x + ay),$$

$a \in \mathbb{Z}$, $|a| > 2$.
Set $i(x, y) = (y, x)$. Then

$$\begin{aligned} i \circ T \circ i(x, y) &= i \circ T(y, x) = i(x, -y + ax) \\ &= (-y + ax, x) = T^{-1}(x, y). \end{aligned}$$

Examples

4. (Not a diffeomorphism!) Let $\phi_t : SM \to SM$ be the geodesic flow over a manifold M.
Defining $i(x, v) = (x, -v)$, for $(x, v) \in S_x M$, we have

$$i \circ \phi_t \circ i = \phi_{-t}.$$

Abstract set up

As before $T : X \to X$ hyperbolic diffeormorphism, $i : X \to X$ time reversing involution.

We consider a Hölder continuous function $\varphi : X \to \mathbb{R}$ and define

$$\psi = \varphi - \varphi \circ i \circ T.$$

Let μ_φ and μ_ψ denote the respective Gibbs measures: the Gibbs measure μ_f for a Hölder function f is uniquely determined by the condition

$$h_{\mu_f}(T) + \int f \, d\mu_f = \sup_{\nu \in \mathcal{M}_T} \left(h_\nu(T) + \int f \, d\nu \right),$$

where $\mathcal{M}_T$ is the set of all T-invariant probability measures on X.

Abstract set up

We suppose that μ_ψ is *not* the measure of maximal entropy μ_0 or, equivalently, that ψ is not cohomologous to a constant (i.e. ψ cannot be written as $u \circ T - u + c$, for u continuous and $c \in \mathbb{R}$.)

Prototypical example

$T : X \to X$ Anosov or, more generally, a C^2 attractor and

$$\varphi = \log |\det DT|E^u|,$$

where E^u is the unstable (expanding) bundle. Then μ_φ is the SRB measure and

$$\psi = \log |\det DT|.$$

This is interesting when T is *not volume preserving*, so ψ is not constant.

Fluctuation Theorem

Write ψ^n for the Birkhoff sum

$$\psi^n = \psi + \psi \circ T + \cdots + \psi \circ T^{n-1}.$$

Theorem (Gallavotti, 1995)

For T, φ, ψ as above, we have

(i) $\int \psi \, d\mu_\varphi > 0$,

(ii) *there exists $p^* > 0$ such that if $|p| < p^*$ then*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{\mu_\varphi\{x : \psi^n(x)/n \in (p - \delta, p + \delta)\}}{\mu_\varphi\{x : \psi^n(x)/n \in (-p - \delta, -p + \delta)\}} \right) = p.$$

Birkhoff averages

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{\psi^n(x)}{n} = \int \psi \, d\mu_\varphi$$

for μ_φ-a.e. $x \in X$, so, if an interval J does not contain $\int \psi \, d\mu_\varphi$, then

$$\lim_{n\to\infty} \mu_\varphi \left\{ x : \frac{\psi^n(x)}{n} \in J \right\} = 0.$$

The Fluctuation Theorem and the Large Deviation Theorem (below) compare the rates in this convergence.

Large Deviations

The Fluctuation Theorem can be understood through the theory of large deviations. Let

$$\mathcal{I}_\psi = \left\{ \int \psi \, d\nu \, : \, \nu \in \mathcal{M}_T \right\}.$$

Theorem (Kifer, 1990; Orey & Pelikan, 1989)

Suppose $\psi : X \to \mathbb{R}$ Hölder continuous and not cohomologous to a constant. Then, for $\alpha \in \operatorname{int}(\mathcal{I}_\psi)$,

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left(\mu_\varphi \left\{ x \, : \, \frac{\psi^n(x)}{n} \in (\alpha - \delta, \alpha + \delta) \right\} \right) = -I(\alpha),$$

where the rate function $I(\alpha) \geq 0$ is given by

$$-I(\alpha) = \inf_{q \in \mathbb{R}} \left(P(\varphi + q\psi) - q\alpha \right).$$

Pressure and the rate function

The pressure function $P(f)$ is defined by

$$\begin{aligned} P(f) &= \sup \left\{ h_\nu(T) + \int f \, d\nu \, : \, \nu \in \mathcal{M}_T \right\} \\ &= \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n x = x} e^{f^n(x)}. \end{aligned}$$

The rate function $I : \mathrm{int}(\mathcal{I}_\psi) \to \mathbb{R}^+$ is real analytic, strictly convex and satisfies

$$I \left(\int \psi \, d\mu_\varphi \right) = 0.$$

Application to the Fluctuation Theorem

The Fluctuation Theorem follows from the Large Deviation Theorem and the following two facts.

1. $\mathcal{I}_\psi = [-p^*, p^*]$, for some $p^* > 0$.
2. For $|p| < p^*$,
$$-I(p) + I(-p) = p.$$

These both follow from the symmetry in the definition of ψ, which gives

$$P(\varphi + t\psi) = P(\varphi - (1+t)\psi),$$

and the definition of I in terms of pressure.

Shrinking intervals

Under a mild condition on ψ, one can show that Fluctuation Theorem holds for *shrinking intervals* $(p - \delta_n, p - \delta_n)$ and $(-p - \delta_n, -p - \delta_n)$, where $\delta_n \to 0$, as $n \to \infty$, at some suitably slow rate.

Diophantine Condition: $T : X \to X$ has three periodic orbits $T^{n_i} x_i = x_i$, $i = 1, 2, 3$, such that

$$\xi := \frac{\psi^{n_3}(x_3) - \psi^{n_1}(x_1)}{\psi^{n_2}(x_2) - \psi^{n_1}(x_1)}$$

is Diophantine, i.e. there exists $c > 0$ and $\beta > 1$ such that

$$|q\xi - p| \geq cq^{-\beta}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Main Theorem

Theorem (Pollicott & Sharp, 2009)

Under the Diophantine Condition, there exists $\kappa > 0$ *such that if* $\delta_n \to 0$ *and* $\delta_n^{-1} = O(n^{1+\kappa})$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\frac{\mu_\varphi\{x : \psi^n(x)/n \in (p - \delta_n, p + \delta_n)\}}{\mu_\varphi\{x : \psi^n(x)/n \in (-p - \delta_n, -p + \delta_n)\}} \right) = p.$$

This follows from a Large Deviations Theorem with similar hypothesis.

Why is it true?

Using symbolic dynamics (Markov partitions), it is sufficient to prove large deviations with shrinking intervals for a *two-sided subshift of finite type* It is more convenient to work with T a *one-sided subshift* and it can be shown that a proof in this case is sufficient.
Suppose we wish to study

$$\mu_\varphi\{x : \psi^n(x)/n \in (\alpha - \delta_n, \alpha + \delta_n)\}.$$

This is equal to

$$\mu_\varphi\{x : \overline{\psi}^n(x) \in (-\epsilon_n, \epsilon_n)\},$$

where $\overline{\psi} = \psi - \alpha$ and $\epsilon_n = n\delta_n$.

Why is it true?

If χ is the indicator function of $(-1, 1)$ and $\chi_n(y) = \chi(y/\epsilon_n)$ then

$$\mu_\varphi\{x : \overline{\psi}^n(x) \in (-\epsilon_n, \epsilon_n)\} = \int \chi_n(\overline{\psi}^n(x))\, d\mu_\varphi(x).$$

It is convenient to modify χ to be a C^k function. (This means $\widehat{\chi}(t) = O(|t|^{-k})$, as $|t| \to \infty$.)

Why is it true?

Then, using the inverse Fourier transform formula,

$$\begin{aligned}\int \chi_n(\overline{\psi}^n(x))\,d\mu_\varphi(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int e^{it\overline{\psi}^n(x)}\,d\mu_\varphi(x)\right)\widehat{\chi}_n(t)\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int e^{(\sigma^*+it)\overline{\psi}^n(x)}\,d\mu_\varphi(x)\right)\widehat{\omega}_n(t)\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int L^n_{\varphi+(\sigma^*+it)\overline{\psi}}1(x)\,d\mu_\varphi(x)\right)\widehat{\omega}_n(t)\,dt,\end{aligned}$$

where

$$\omega_n(y) = e^{-\sigma^* y}\chi_n(y)$$

and $\sigma^* \in \mathbb{R}$ is some specifically chosen value to make the analysis work.

The transfer operator

On the previous slide, $L_{\varphi+(\sigma^*+it)\overline{\psi}}$ denoted the *Ruelle transfer operator* defined by

$$L_{\varphi+(\sigma^*+it)\overline{\psi}}w(x) = \sum_{Tx'=x} e^{\varphi(x')+(\sigma^*+it)\overline{\psi}(x')}w(x').$$

Dolgopyat bounds

Theorem follows from LCLT-type calculations (for $|t|$ small) and Dolgopyat bounds on the transfer operator (for $|t|$ large):

Lemma

Under the Diophantine Condition, given $a > 0$, there exists $\gamma, d, c_1, c_2 >$ such that, for $|t| \geq a$ and $m \geq 1$,

$$\|L^{2Nm}_{\varphi+(\sigma^*+it)\overline{\psi}} 1\|_\infty \leq c_1 e^{-nI(\alpha)} \left(1 - \frac{c_2}{|t|^\gamma}\right)^m,$$

where $N = [d \log |t|]$.

More precisely . . .

More precisely, for $|t|$ small, one can show that

$$\frac{1}{2\pi}\int_{|t|<a}\left(\int L^n_{\varphi+(\sigma^*+it)\overline{\psi}}1(x)\,d\mu_\varphi(x)\right)\widehat{\omega}_n(t)\,dt \sim C\widehat{\chi}(0)\frac{\epsilon_n e^{-nI(\alpha)}}{\sqrt{n}},$$

where $C>0$ is independent of χ, which has exponential growth rate $-I(\alpha)$.

More precisely . . .

Meanwhile, for $|t| \geq a$,

$$\frac{1}{2\pi} \int_{|t|\geq a} \left(\int L^n_{\varphi+(\sigma^*+it)\overline{\psi}} 1(x)\, d\mu_\varphi(x) \right) \widehat{\omega}_n(t)\, dt$$
$$= O\left(e^{-nI(\alpha)} \epsilon_n^{-(k-1)} n^{(1-k)\delta} \right),$$

for some $\delta > 0$, which tends to zero faster than the leading contribution provided we choose $\kappa < \delta$.

Thank you for listening!