Extremal dichotomy and rates of convergence for laws of rare events

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Extreme Value Laws

Consider a stationary stochastic process$X_0, X_1, X_2, \ldots$ with marginal d.f. F. Let $\bar{F} = 1 - F$ and $u_F = \sup\{x : F(x) < 1\}$.
We have an *exceedance* of the level $u \in \mathbb{R}$ at time $j \in \mathbb{N}$ if the event $\{X_j > u\}$ occurs. Define a new sequence of random variables (r.v.) $M_1, M_2, \ldots$ given by

$$M_n = \max\{X_0, \ldots, X_{n-1}\}. \tag{1}$$

Definition

We say that we have an EVL for M_n if there is a d.f. $H : \mathbb{R} \to [0, 1]$, with $H(0) = 0$ and, for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$, s.t.

$$n\mathbb{P}(X_0 > u_n) \to \tau, \quad \text{as } n \to \infty, \tag{2}$$

and for which the following holds:

$$\mathbb{P}(M_n \leq u_n) \to \bar{H}(\tau), \quad \text{as } n \to \infty. \tag{3}$$

The independent case

In the case $X_0, X_1, X_2, \dots$ are i.i.d. r.v. then since

$$\mathbb{P}(M_n \le u_n) = (F(u_n))^n$$

we have

$$(1 - \mathbb{P}(X > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to \mathrm{e}^{-\tau}$$

which implies that if (2) holds, then (3) holds with $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and vice versa.

When $X_0, X_1, X_2, \dots$ are not i.i.d. but satisfy some mixing condition $D(u_n)$ introduced by Leadbetter then something can still be said about H.

Condition $D(u_n)$ from Leadbetter

Let $F_{i_1,\dots,i_n}$ denote the joint d.f. of $X_{i_1},\dots,X_{i_n}$, and set $F_{i_1,\dots,i_n}(u) = F_{i_1,\dots,i_n}(u,\dots,u)$.

Condition ($D(u_n)$)

We say that $D(u_n)$ holds for the sequence $X_0, X_1, \dots$ if for any integers $i_1 < \dots < i_p$ and $j_1 < \dots < j_k$ for which $j_1 - i_p > m$, and any large $n \in \mathbb{N}$,

$$\left|F_{i_1,\dots,i_p,j_1,\dots,j_k}(u_n) - F_{i_1,\dots,i_p}(u_n)F_{j_1,\dots,j_k}(u_n)\right| \le \gamma(n,m),$$

where $\gamma(n,m_n) \xrightarrow[n\to\infty]{} 0$, for some sequence $m_n = o(n)$.

Theorem (Leadbetter)

If $D(u_n)$ *holds for* $X_0, X_1, \ldots$ *and the limit* (3) *exists for some* $\tau > 0$ *then there exists* $0 \leq \theta \leq 1$ *such that* $\bar{H}(\tau) = \mathrm{e}^{-\theta\tau}$ *for all* $\tau > 0$

Definition

We say that $X_0, X_1, \ldots$ has an *Extremal Index* (EI) $0 \leq \theta \leq 1$ if we have an EVL for M_n with $\bar{H}(\tau) = \mathrm{e}^{-\theta\tau}$ for all $\tau > 0$.

Absence of clustering

Assuming $D(u_n)$ holds let, $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{4}$$

Condition ($D'(u_n)$)

We say that $D'(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if

$$\limsup_{n\to\infty} n \sum_{j=1}^{[n/k_n]} \mathbb{P}(X_0 > u_n, X_j > u_n) = 0. \tag{5}$$

Theorem (Leadbetter)

Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq u_n) = \mathrm{e}^{-\tau}.$$

The general case

For some $u \in \mathbb{R}$, $q \in \mathbb{N}$, we define the events:

$$U(u) := \{X_0 > u\} \text{ and } A^{(q)}(u) := \{X_0 > u, X_1 \leq u, \ldots, X_q \leq u\}. \quad (6)$$

We also set $A^{(0)}(u) := U(u)$, $U_n := U(u_n)$ and $A_n^{(q)} := A^{(q)}(u_n)$, for all $n \in \mathbb{N}$ and $q \in \mathbb{N}_0$.
Let $B \in \mathcal{B}$ be an event. For some $s, \ell \in \mathbb{N}_0$, we define:

$$\mathscr{W}_{s,\ell}(B) = \bigcap_{i=s}^{s+\ell-1} T^{-i}(B^c). \quad (7)$$

We will write $\mathscr{W}_{s,\ell}^c(B) := (\mathscr{W}_{s,\ell}(B))^c$. Whenever is clear or unimportant which event $B \in \mathcal{B}$ applies, we will drop the B and write just $\mathscr{W}_{s,\ell}$ or $\mathscr{W}_{s,\ell}^c$. Observe that

$$\mathscr{W}_{0,n}(U(u)) = \{M_n \leq u\}.$$

Crucial observation

Proposition

Given an event $B \in \mathcal{B}$, let $q, n \in \mathbb{N}$ be such that $q < n$ and define $A = B \setminus \bigcup_{j=1}^{q} T^{-j}(B)$. Then

$$|\mathbb{P}(\mathscr{W}_{0,n}(B) - \mathbb{P}(\mathscr{W}_{0,n}(A))| \leq \sum_{j=1}^{q} \mathbb{P}\left(\mathscr{W}_{0,n}(A) \cap T^{-n+j}(B \setminus A)\right).$$

New mixing condition

Condition (Д(u_n))

We say that Д(u_n) holds for the sequence $X_0, X_1, \ldots$ if for every $\ell, t, n \in \mathbb{N}$ and $q \in \mathbb{N}_0$,

$$\left| \mathbb{P}\left(A_n^{(q)} \cap \mathscr{W}_{t,\ell}\left(A_n^{(q)}\right)\right) - \mathbb{P}\left(A_n^{(q)}\right) \mathbb{P}\left(\mathscr{W}_{0,\ell}\left(A_n^{(q)}\right)\right) \right| \leq \gamma(q, n, t), \tag{8}$$

where $\gamma(q, n, t)$ is decreasing in t for each q, n and, for every $q \in \mathbb{N}_0$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n = o(n)$ and $n\gamma(q, n, t_n) \to 0$ when $n \to \infty$.

For some fixed $q \in \mathbb{N}_0$, consider the sequence $(t_n)_{n \in \mathbb{N}}$, given by condition Д(u_n) and let $(k_n)_{n \in \mathbb{N}}$ be another sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{9}$$

New general clustering condition

Condition ($Д'_q(u_n)$)

We say that $Д'_q(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ satisfying (9) and such that

$$\lim_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(A_n^{(q)} \cap T^{-j}\left(A_n^{(q)}\right)\right) = 0. \tag{10}$$

Remark

Note that condition $Д'_q(u_n)$ is condition $D^{(q)}(u_n)$ from [CHM91]. Moreover, if $q = 0$ then we get condition $D'(u_n)$ from Leadbetter. Thus the following result, Theorem 4, gives in particular a generalisation of [CHM91, Corollary 1.3] since $Д(u_n)$ is much weaker than the original $D(u_n)$ of Leadbetter. Moreover, as discussed above, condition $Д(u_n)$ follows from sufficiently fast decay of correlations of the underlying stochastic processes.

Theorem (FFT14)

Let $X_0, X_1, \ldots$ be a stationary stochastic process and $(u_n)_{n\in\mathbb{N}}$ a sequence satisfying (2), for some $\tau > 0$. Assume that conditions $Д(u_n)$ and $Д'_q(u_n)$, for some $q \in \mathbb{N}_0$, are satisfied. Then, there exists $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\left|\mathbb{P}(M_n \le u_n) - \mathrm{e}^{-\theta\tau}\right| \le C\Bigg(k_n t_n \frac{\tau}{n} + n\gamma(q,n,t_n) + q\mathbb{P}\left(U_n \setminus A_n^{(q)}\right)$$
$$+ \mathrm{e}^{-\theta\tau}\left(\left|\theta\tau - n\mathbb{P}\left(A_n^{(q)}\right)\right| + \frac{\tau^2}{k_n}\right) + n\sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(A_n^{(q)} \cap T^{-j}\left(A_n^{(q)}\right)\right)\Bigg),$$

where the EI θ is given by equation O'Brien's formula:

$$\theta = \lim_{n\to\infty} \frac{\mathbb{P}\left(A_n^{(q)}\right)}{\mathbb{P}(U_n)}. \tag{11}$$

The setting

Take a system $(\mathcal{X}, \mathcal{B}, \mathbb{P}, f)$, where $\mathcal{X}$ is a Riemannian manifold, $\mathcal{B}$ is the Borel σ-algebra, $f : \mathcal{X} \to \mathcal{X}$ is a measurable map and $\mathbb{P}$ an f-invariant probability measure.
Consider the time series $X_0, X_1, \ldots$ arises from such a system simply by evaluating a given observable $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ along the orbits of the system:

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}. \tag{12}$$

We assume that the r.v. $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ achieves a global maximum at $\zeta \in \mathcal{X}$ (we allow $\varphi(\zeta) = +\infty$). We also assume that φ and $\mathbb{P}$ are sufficiently regular so that:

(R1) for u sufficiently close to $u_F := \varphi(\zeta)$, the event

$$U(u) = \{X_0 > u\} = \{x \in \mathcal{X} : \ \varphi(x) > u\}$$

corresponds to a topological ball centred at ζ. Moreover, the quantity $\mathbb{P}(U(u))$, as a function of u, varies continuously on a neighbourhood of u_F.

(R2) If $\zeta \in \mathcal{X}$ is a repelling periodic point, of prime period $p \in \mathbb{N}$, then we have that the periodicity of ζ implies that for all large u, $\{X_0 > u\} \cap f^{-p}(\{X_0 > u\}) \neq \emptyset$ and the fact that the prime period is p implies that $\{X_0 > u\} \cap f^{-j}(\{X_0 > u\}) = \emptyset$ for all $j = 1, \ldots, p-1$. Moreover, the fact that ζ is repelling means that we have backward contraction implying that there exists $0 < \theta < 1$

$$\mathbb{P}\left(\{X_0 > u\} \cap f^{-p}(\{X_0 > u\})\right) \sim (1-\theta)\mathbb{P}(X_0 > u),$$

for all u sufficiently large.

Hitting Times Statistics

Consider a set $A \in \mathcal{B}$. We define a function that we refer to as *first hitting time function* to A, denoted by $r_A : \mathcal{X} \to \mathbb{N} \cup \{+\infty\}$ where

$$r_A(x) = \min\left\{j \in \mathbb{N} \cup \{+\infty\} :\ f^j(x) \in A\right\}. \tag{13}$$

The restriction of r_A to A is called the *first return time function* to A. We define the *first return time* to A, which we denote by $R(A)$, as the infimum of the return time function to A, *i.e.*,

$$R(A) = \inf_{x \in A} r_A(x). \tag{14}$$

Given a point $\zeta \in X$, by Kac Lemma the expected value of $r_{B_\varepsilon(\zeta)}$ when restricted to the ε-ball $B_\varepsilon(\zeta)$ is $1/\mu(B_\varepsilon(\zeta))$. We say that the system has HTS H for balls around ζ if for each $t \in [0, \infty)$,

$$\lim_{\varepsilon \to 0} \mu\left(\left\{\mu(B_\varepsilon(\zeta)) r_{B_\varepsilon(\zeta)} > t\right\}\right) = H(t)$$

for some d.f. $H : [0, \infty) \to [0, 1]$.

Definition (Decay of correlations)

Let $\mathcal{C}_1, \mathcal{C}_2$ denote Banach spaces of real valued measurable functions defined on $\mathcal{X}$. We denote the *correlation* of non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ w.r.t. a measure $\mathbb{P}$ as

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1}\|\psi\|_{\mathcal{C}_2}} \left| \int \phi\,(\psi \circ f^n)\,\mathrm{d}\mathbb{P} - \int \phi\,\mathrm{d}\mathbb{P} \int \psi\,\mathrm{d}\mathbb{P} \right|.$$

We say that we have *decay of correlations*, w.r.t. the measure $\mathbb{P}$, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if, there exists a rate function $\gamma : \mathbb{N} \to \mathbb{R}$, with $\lim_{n\to\infty} \gamma(n) = 0$, such that, for every $\phi \in \mathcal{C}_1$ and every $\psi \in \mathcal{C}_2$, we have

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) \leq \gamma(n).$$

We say that we have *decay of correlations against L^1 observables* whenever this holds for $\mathcal{C}_2 = L^1(\mathbb{P})$ and $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int |\psi|\,\mathrm{d}\mathbb{P}$.

A dichotomy for the extremal behaviour

Theorem ([AFV13], [FFT13])

Consider a continuous dynamical system $(\mathcal{X}, \mathcal{B}, \mathbb{P}, f)$ *for which there exists a Banach space* $\mathcal{C}$ *of real valued functions such that for all* $\phi \in \mathcal{C}$ *and* $\psi \in L^1(\mathbb{P})$, $Cor_{\mathbb{P}}(\phi, \psi, n) \leq \gamma(n)$, *where* $\sum_{n\geq 0} \gamma(n) < \infty$. *Let* $X_0, X_1, \ldots$ *be given by* (12), *where* φ *has a global maximum at* $\zeta \in \mathcal{X}$

- *If* ζ *is a non periodic point then we have an EVL for* M_n *with EI* $\theta = 1$.
- *If* ζ *is a periodic point of prime period* p, *then we have an EVL for* M_n *with EI* $\theta < 1$ *given by the expansion rate at* ζ *stated in (R2).*

Remark

The dichotomy can be extended to the context of point processes.

Remark

The existence of a such a dichotomy is present in the following (probably incomplete) list of papers: [FFT12, FP12, K12, KR13].

Systems with decay of correlations against L^1 observables:

- Uniformly expanding maps on the circle
- Markov maps
- Piecewise expanding maps of the interval like Rychlik maps
- Higher dimensional piecewise expanding maps like in [S00]

Remark

Observe that decay of correlations against $L^1(\mathbb{P})$ observables is a very strong property. In fact, regardless of the rate, as long as it is summable, one can actually show that the system has exponential decay of correlations of Hölder observables against $L^\infty(\mathbb{P})$. See [AFLV11, Theorem B].

Remark

If we add noise then the dichotomy vanishes and the standard Poisson process always appears in the limit. See [AFV13].

Motivated by the work of

- Abadi [A04]
- Keller [K12],

we build up on the assumption of existences of decay of correlations against L^1 observables to produce sharper error terms.

Theorem

Assume that the system has decay of correlations of observables in a Banach space $\mathcal{C}$ against observables in L^1 with rate function $\gamma : \mathbb{N} \to \mathbb{R}$, where γ is independent of the given observables and there exists $\delta > 0$ such that $n^{2+\delta}\gamma(n) \to 0$, as $n \to \infty$. Let u_n be as in (2). Assume that there exists $q \in \mathbb{N}_0$ such that

$$q := \min\left\{j \in \mathbb{N}_0 : \lim_{n\to\infty} R(A_n^{(j)}) = \infty\right\}.$$

For each $n \in \mathbb{N}$, let $A_n := A_n^{(q)}$, $R_n := R(A_n^{(q)})$, where R is defined as in (14), and let k_n, t_n be integers minimising $\left\{kt\mathbb{P}(A_n) + \frac{n^2}{k}\gamma(t) + \frac{(n\mathbb{P}(A_n))^2}{k}\right\}$. Assume that there exists $M > 0$ such that $\|\mathbf{1}_{A_n}\|_{\mathcal{C}} \le M$ for all $n \in \mathbb{N}$.

Theorem

Then there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$\left|\mathbb{P}(M_n \le u_n) - \mathrm{e}^{-\theta\tau}\right| \le C\mathrm{e}^{-\theta\tau}\left(\left|\theta\tau - n\mathbb{P}(A_n)\right| + k_n t_n \frac{\theta\tau}{n} + \frac{n^2}{k_n}\gamma(t_n) + \frac{(\theta\tau)^2}{k_n} + \theta\tau \sum_{j=R_n}^{\ell_n - 1} \gamma(j)\right),$$

where $\ell_n = \lfloor n/k_n \rfloor - t_n$ and the EI θ is given by equation (11).

Improved error terms for HTS

In what follows $B_\varepsilon(\zeta)$ denotes the open ball of radius ε, around the point $\zeta \in \mathcal{X}$, w.r.t. a given metric on $\mathcal{X}$. Also set set $A_\varepsilon^{(0)}(\zeta) := B_\varepsilon(\zeta)$ and, for each $q \in \mathbb{N}$, let $A_\varepsilon^{(q)}(\zeta) := B_\varepsilon(\zeta) \cap \bigcap_{i=1}^{q} f^{-i}((B_\varepsilon(\zeta))^c)$.

Theorem

Assume that the system has decay of correlations of observables in a Banach space $\mathcal{C}$ against L^1, with $r\gamma : \mathbb{N} \to \mathbb{R}$ independent of the given observables and there exists $\delta > 0$ such that $n^{2+\delta}\gamma(n) \to 0$, as $n \to \infty$. Fix some point $\zeta \in \mathcal{X}$ and assume that there exists $q \in \mathbb{N}_0$ such that

$$q := \min\left\{ j \in \mathbb{N}_0 : \lim_{\varepsilon \to 0} R(A_\varepsilon^{(j)}(\zeta)) = \infty \right\}.$$

For each $\varepsilon > 0$, let $B_\varepsilon := B_\varepsilon(\zeta)$, $A_\varepsilon := A_\varepsilon^{(q)}(\zeta)$, $R_\varepsilon := R(A_\varepsilon^{(q)}(\zeta))$, where R is defined as in (14), and $k_\varepsilon, t_\varepsilon \in \mathbb{N}$ minimize $kt\mathbb{P}(B_\varepsilon) + \frac{\mathbb{P}(B_\varepsilon)^{-2}}{k}\gamma(t) + \frac{1}{k}$.

Theorem

Let $\ell_\varepsilon = \lfloor \mathbb{P}(B_\varepsilon)^{-1}/k_\varepsilon \rfloor - t_\varepsilon$ *and assume that there exists* $M > 0$ *such that* $\|\mathbf{1}_{A_\varepsilon}\|_{\mathcal{C}} \leq M$ *for all* $\varepsilon > 0$.
Then there exists $C > 0$, *depending on* ε *but not on* τ, *such that for all* $\varepsilon > 0$ *and* $\tau > 0$

$$\left|\mathbb{P}\Big(r_{B_\varepsilon(\zeta)} > \frac{\tau}{\mathbb{P}(B_\varepsilon)}\Big) - \mathrm{e}^{-\theta\tau}\right| \leq C\Big(\tau^2\alpha_\varepsilon\Gamma_\varepsilon + \frac{\tau^2}{k_\varepsilon}\Gamma_\varepsilon + \frac{\tau^3}{k_\varepsilon}\alpha_\varepsilon\Gamma_\varepsilon\Big)\mathrm{e}^{-(\theta - k_\varepsilon\Upsilon_{A_\varepsilon})\tau},$$

with $\alpha_\varepsilon = |\theta - \frac{\mathbb{P}(A_\varepsilon)}{\mathbb{P}(B_\varepsilon)} + t_\varepsilon k_\varepsilon \mathbb{P}(A_\varepsilon)|$ *and* Υ_{A_ε} *is s.t.* $k_\varepsilon\Upsilon_{A_\varepsilon} \leq C\Gamma_\varepsilon$, *for some* $C > 0$, *where* $\Gamma_\varepsilon = \Big(k_\varepsilon t_\varepsilon \mathbb{P}(A_\varepsilon) + \frac{\mathbb{P}(B_\varepsilon)^{-2}\gamma(t_\varepsilon)}{k_\varepsilon} + \frac{1}{k_\varepsilon} + \sum_{j=R_\varepsilon}^{\ell_\varepsilon - 1}\gamma(j)\Big)$.

$$\theta = \lim_{\varepsilon\to 0}\frac{\mathbb{P}(A_\varepsilon)}{\mathbb{P}(B_\varepsilon)}. \tag{15}$$

Escape rates in the zero-hole limit

Fix a hole $B_\varepsilon(\zeta)$ around some chosen point ζ and compute the rate of escape of mass through the hole, i.e., find the limit, if it exists,

$$-\lim_{\tau\to\infty}\frac{1}{\tau}\log\mathbb{P}\left(r_{B_\varepsilon(\zeta)}>\tau\right).$$

Moreover, one can consider, as in [KL09, FP12], what happens when the size of the ball goes to zero too.

Corollary

Suppose that the system is as in the previous theorem. Then

$$-\lim_{\varepsilon\to0}\frac{1}{\mathbb{P}(B_\varepsilon(\zeta))}\limsup_{\tau}\frac{1}{\tau}\log\mathbb{P}\left(r_{B_\varepsilon(\zeta)}>\frac{\tau}{\mathbb{P}(B_\varepsilon)}\right)\geq\theta.$$

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